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Review: Unitary equivalence to truncated Toeplitz operators

Stephan Ramon Garcia
Pomona College

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Strouse, E. [Strouse, Elizabeth] (F-BORD-IM); **Timotin, D.** (R-AOS);
Zarrabi, M. [Zarrabi, Mohamed] (F-BORD-IM)
Unitary equivalence to truncated Toeplitz operators. (English summary)
Indiana Univ. Math. J. **61** (2012), *no. 2,* 525–538.

A *truncated Toeplitz operator* (TTO) is a compression $A_\varphi^\Theta f = P_\Theta(\varphi f)$ of a multiplication operator to a *model space* $K_\Theta = H^2 \ominus \Theta H^2$, where Θ is an inner function. Here P_Θ denotes the orthogonal projection from the classical Hardy space H^2 onto K_Θ. Although such operators are bounded when the *symbol* φ belongs to H^∞, in contrast to the situation for classical Toeplitz operators, certain unbounded symbols can also give rise to bounded TTOs [A. D. Baranov et al., J. Funct. Anal. **259** (2010), no. 10, 2673–2701; MR2679022; D. E. Sarason, Oper. Matrices **1** (2007), no. 4, 491–526; MR2363975]. For $\Theta = z^n$, the study of TTOs reduces to the study of finite Toeplitz matrices. Much of the recent work on TTOs has been motivated by the 2007 paper of Sarason [op. cit.]. A recent survey of the reviewer and W. T. Ross [in *Blaschke products and their applications*, 275–319, Fields Inst. Commun., 65, Springer, New York, 2013; MR3052299] covered some of the material from the paper under review, and we direct the interested reader there for further details.

TTOs are an important class of *complex symmetric operators* [S. R. Garcia and M. Putinar, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315; MR2187654]. The question of whether every complex symmetric operator can be represented in some fashion using TTOs was raised in the recent paper [Indiana Univ. Math. J. **59** (2010), no. 2, 595–620; MR2648079] of J. A. Cima, Ross, W. R. Wogen and the reviewer. It turns out that the first nontrivial step in this direction was taken in 1965 by Sarason [J. Math. Anal. Appl. **12** (1965), 244–246; MR0192355], who proved that the classical Volterra integration operator on $L^2[0,1]$ is unitarily equivalent to a TTO (although the term *truncated Toeplitz operator* was not yet introduced). In recent years, it has been established that a wide range of complex symmetric operators are unitarily equivalent to TTOs or direct sums of TTOs [see J. A. Cima et al., op. cit.; S. R. Garcia and W. T. Ross, op. cit.]. For instance, B. Lutz, D. Timotin and the reviewer ["Two remarks about nilpotent operators of order two", preprint, arXiv:1206.5523, Proc. Amer. Math. Soc., to appear] recently proved that every operator which is nilpotent of order two is unitarily equivalent to a TTO (that all such operators are complex symmetric was proven in [S. R. Garcia and W. R. Wogen, Trans. Amer. Math. Soc. **362** (2010), no. 11, 6065–6077; MR2661508]).

The paper under review takes these ideas in exciting new directions. The authors study various inflations, tensor products, and direct sums of TTOs. They prove that, under many circumstances, such operators are themselves unitarily equivalent to TTOs. Since there are many results contained in the article, most requiring quite a bit of notation, we survey here what we hope is a representative sample:

(1) One result (Corollary 4.4) tells us that if Θ is an inner function, $\varphi \in L^2$, and B is an inner function with $\dim K_B = k$ for some $k = 1, 2, \ldots, \infty$ such that the operator A_φ^Θ is bounded, then $I_k \otimes A_\varphi^\Theta$ is unitarily equivalent to the truncated Toeplitz operator $A_{\varphi\circ B}^{\Theta\circ B}$. In other words, the authors have discovered yet another class of complex symmetric operators (namely those of the form $I_k \otimes A_\varphi^\Theta$) which can be realized directly as a *single* TTO (i.e., without using direct sums of TTOs).

(2) As another example, the authors show (Proposition 5.4) that if ψ is an analytic

function such that A^B_ψ is bounded, and R is a nonselfadjoint operator of rank one, then the operator $A^B_\psi \otimes R$ is unitarily equivalent to a TTO.

(3) They also investigate circumstances under which $A^\Theta_\varphi \oplus 0$ is unitarily equivalent to a TTO. One such example (Theorem 6.3) states that if $\varphi \in H^\infty$ satisfies $\varphi^2 \in \Theta H^\infty$ and $k = \dim K_\Theta \ominus \ker A^\Theta_\varphi$, then $A^\Theta_\varphi \oplus 0_k$ is unitarily equivalent to a TTO.

The paper under review is also notable for the inclusion of several useful lemmas and propositions, which may be of more general interest. For instance, Proposition 3.1 [see also C. C. Cowen, J. Operator Theory **7** (1982), no. 1, 167–172; MR0650201 (Theorem 1)] states that if B is an inner function, then the formula

$$h \otimes f \mapsto h(f \circ B),$$

defined for $h \in K_B$ and $f \in L^\infty$, can be extended linearly to a unitary operator $\Omega_N\colon K_B \otimes L^2 \to L^2$. The operator Ω_B maps $K_B \otimes H^2$ onto H^2.

Overall, this paper provides a number of welcome additions to the list of complex symmetric operators which are unitarily equivalent to TTOs. In particular, the authors show that a wide variety of operators obtained from TTOs via inflation, tensor products, and direct sums are themselves, up to unitary equivalence, TTOs. *Stephan R. Garcia*

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