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Biquasiles and Dual Graph Diagrams

Deanna Needell
Claremont McKenna College

Sam Nelson
Claremont McKenna College

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Deanna Needell *　　　Sam Nelson†

Abstract

We introduce *dual graph diagrams* representing oriented knots and links. We use these combinatorial structures to define corresponding algebraic structures we call *biquasiles* whose axioms are motivated by dual graph Reidemeister moves, generalizing the Dehn presentation of the knot group analogously to the way quandles and biquandles generalize the Wirtinger presentation. We use these structures to define invariants of oriented knots and links. In particular, we identify an example of a finite biquasile whose counting invariant distinguishes the chiral knot 9_{32} from its mirror image, demonstrating that biquasile counting invariants are distinct from biquandle counting invariants.

KEYWORDS: biquasiles, dual graph diagrams, checkerboard graphs, singular knots and links

2010 MSC: 57M27, 57M25

1 Introduction

The checkerboard colorings of a planar knot complement have long been used in knot theory, going back to papers such as [6]. From the undecorated checkerboard graph, one can reconstruct an unoriented alternating knot or link up to mirror image. In [5], signs are added to edges, enabling reconstruction of not necessarily alternating unoriented knots and links. In this paper, we introduce *dual graph diagrams* for oriented knots and links, a type of diagram using both of the (mutually dual) checkerboard graphs decorated with some edges having signs and others having directions, enabling recovery of arbitrary oriented knots and links. A similar graph without decorations was used in the study of the dimer model of the Alexander and twisted Alexander polynomials in [1].

Analogously to the construction of quandles and biquandles from a coloring scheme for arcs and semiarcs in oriented knot and link diagrams [3], we introduce a coloring scheme for vertices in a dual graph diagram. This coloring scheme motivates a new algebraic structure known as a *biquasile* with axioms determined by the dual graph Reidemeister moves. More precisely, the biquasile axioms are chosen so that biquasile colorings of dual graph diagrams are preserved faithfully by Reidemeister moves. This enables us to define biquasile counting invariants of knots and links and allows the introduction of enhancements of these invariants.

The paper is organized as follows. In Section 2 we introduce dual graph diagrams and the dual graph Reidemeister moves, as well as the reconstruction algorithm; a generic dual graph diagram presents a type of directed bivalent spatial graph, sometimes known as a *magnetic graph* [4, 7, 8]. We introduce a geometric-style oriented link invariant defined from the dual graph representation, the *dual graph component number*, and show that this invariant is bounded above by the braid index. In Section 3 we introduce biquasiles, deriving the biquasile axioms from the dual graph diagram Reidemeister moves and introduce biquasile counting invariants for oriented knots and links. We identify a finite biquasile whose counting invariant detects the difference between the chiral knot 9_{32} and its mirror image, showing that unlike the case with biquandles, the fundamental biquasile of a knot or link need not be isomorphic to that of the mirror image. In Section 4 we turn our focus to the case of Alexander biquasiles, a type of biquasile structure defined on modules over the three-variable Laurent polynomial ring $L = \mathbb{Z}[d^{\pm 1}, s^{\pm 1}, n^{\pm 1}]$. We conclude in Section 5 with some questions for future research.

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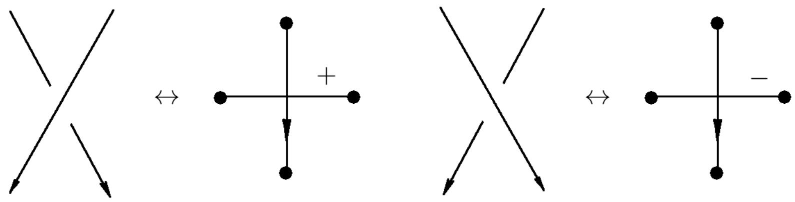
†Email: Sam.Nelson@cmc.edu. Partially supported by Simons Foundation collaboration grant #316709.

2 Dual graph diagrams

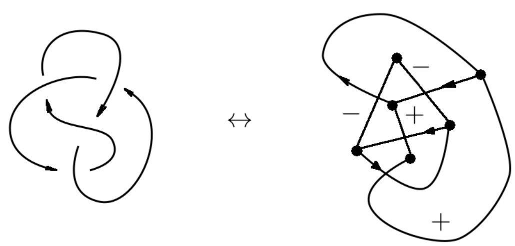
We begin with a definition.

Definition 1. Let D be an oriented knot or link diagram. The *dual graph diagram* G associated to D has a vertex associated to each region of the planar knot complement and edges joining vertices whose regions are opposite at crossings. The edges are given directions or $+/-$ signs as pictured:

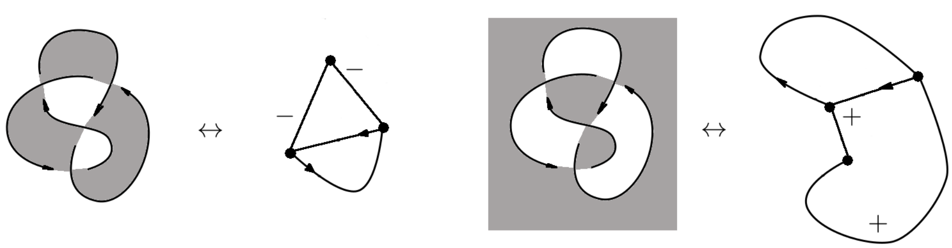


Conversely, given a pair $G \cup G'$ of dual planar graphs (i.e., such that each region of $S^2 \setminus G$ contains a unique vertex of G' with adjacent regions in $S^2 \setminus G$ corresponding to adjacent vertices in G'), $G \cup G'$ becomes a dual graph diagram when we assign either a direction or $+$ or $-$ sign to each edge such that each pair of crossed edges has one signed edge and one directed edge.

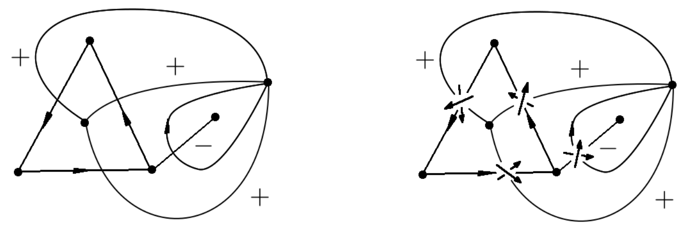
Example 1. The oriented knot diagram below has the corresponding dual graph diagram below.



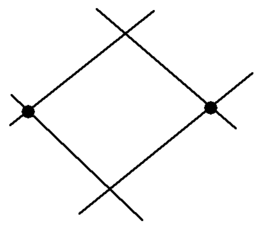
We can understand the dual graph diagram as the result of superimposing the two checkerboard graphs associated to the knot or link diagram and decorating the edges to indicate orientation and crossing information.



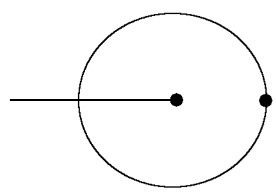
A natural question is which dual graph diagrams present oriented knots or links. Let us consider the reconstruction algorithm for obtaining the original oriented knot or link diagram from its dual graph diagram. First, we note that each crossing of edges in the dual graph represents a crossing in the original link diagram, so we can start by putting a crossing at each edge-crossing with crossing information as determined by the direction and sign decorations carried by the edges:



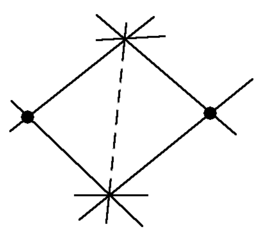
We then observe that a dual graph diagram tiles the sphere S^2 with quadrilaterals with two corners given by vertices and two corners given by edge crossings as depicted:



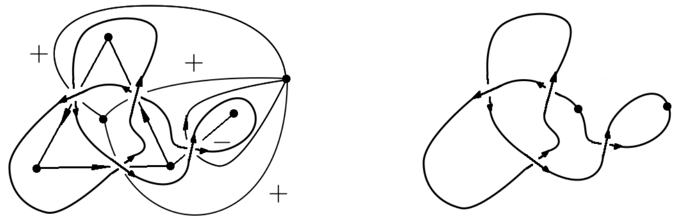
Some such quadrilaterals may be degenerate, with boundary formed by a leaf and a loop:



Within each quadrilateral there is a unique path (up to planar isotopy) connecting the ends of the crossings; drawing these paths completes the diagram:



We note that for some dual graph diagrams, the orientations at the ends of a strand may disagree; in these cases, we can include bivalent vertices in the interior of such an arc, obtaining a bivalent spatial graph with source-sink orientations; such diagrams are known as *magnetic graphs* in [4]:

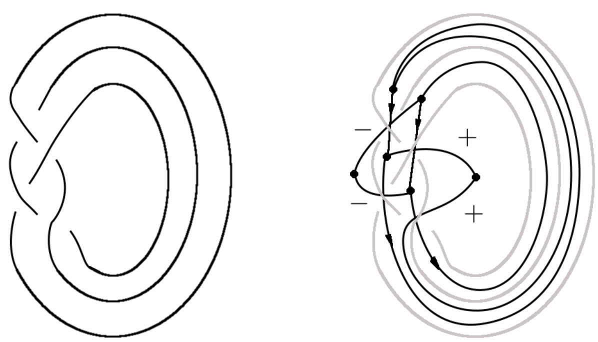


We have the following:

Proposition 1. *Every oriented knot or link has a dual graph diagram with directed edges consisting of disjoint cycles.*

Proof. Simply put the knot or link diagram L in closed braid form; the dual graph diagram then consists of n disjoint directed cycles (where the braid being closed to form L has $n+1$ strands) running vertically between the strands of the braid overlaid by locally horizontal signed edges. □

Example 2. The figure eight knot 4_1 has closed braid presentation with corresponding dual graph diagram as depicted:

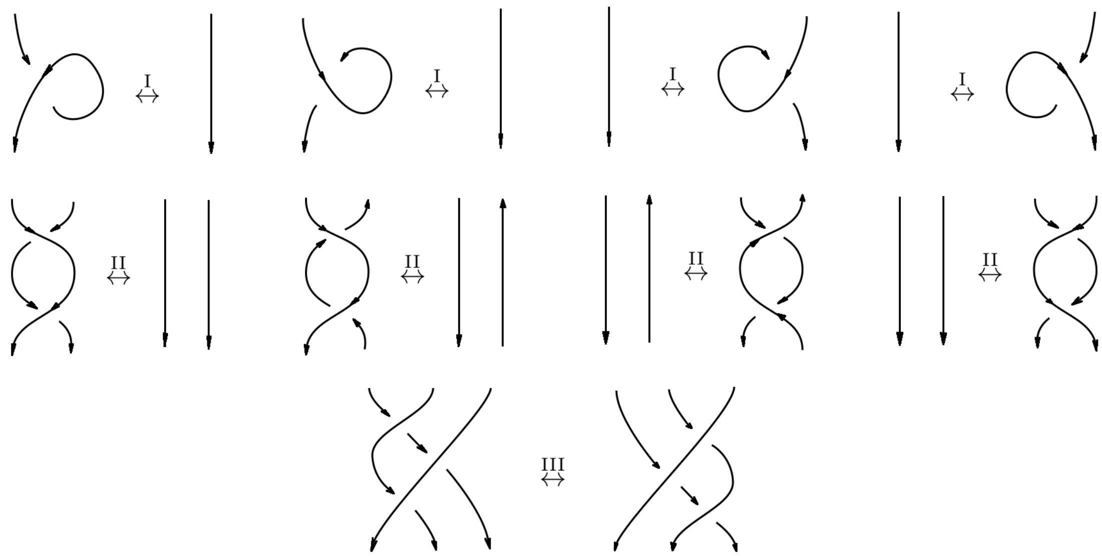


Definition 2. The *dual graph component number* of a knot or link L is the minimal number of directed components in the graph obtained from a dual graph diagram representing L by deleting the signed edges, taken over the set of all dual graph diagrams representing L.

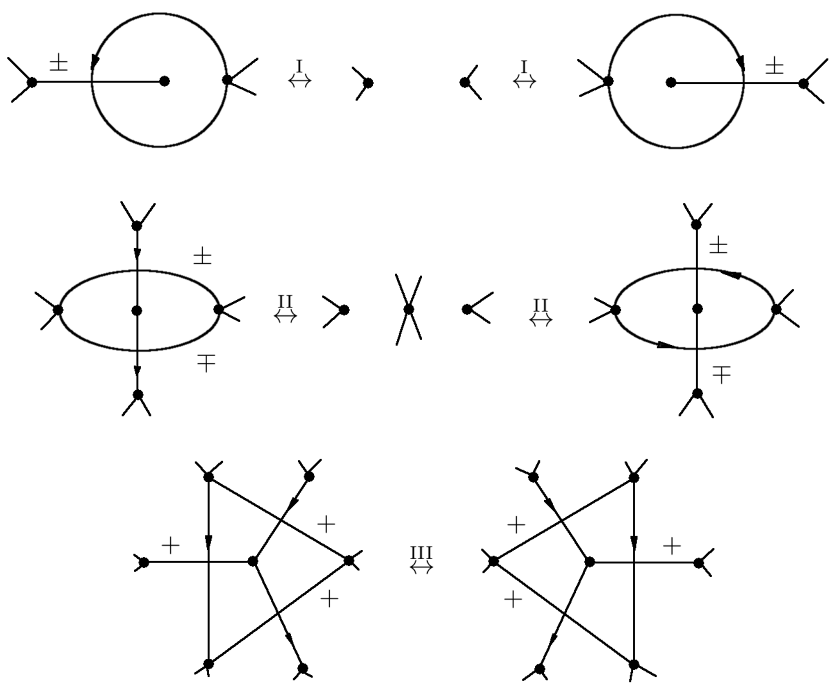
In light of the proof of Proposition 1, we have the following easy observation:

Proposition 2. *The dual graph component number of an oriented link is bounded above by the braid index.*

Recall that two oriented knot or link diagrams represent ambient isotopic knots or links if and only if they are related by a sequence of *oriented Reidemeister moves*:



Translating these Reidemeister moves into dual-graph format, we obtain the following moves:



Note that while dual graph Reidemeister moves do allow local vertex-introducing and vertex-removing moves, they do not allow strands of the knot or link to move past bivalent vertices; thus, the larger category of dual graph diagrams modulo dual graph Reidemeister moves is a slightly different category from the usual case of directed bivalent spatial graphs. In particular, classical knots and links form Reidemeister equivalence classes of the subset of dual graph diagrams whose reconstructions do not require bivalent vertices, with moves restricted to forbid local orientation-reversing moves.

3 Biquasiles

Let X be a set. We would like to define an algebraic structure on X with operations and axioms motivated by the Reidemeister moves on dual graph diagrams in order to define knot and link invariants. Let us impose on X two binary operations, $* : X \times X \to X$ and $\cdot : X \times X \to X$ each defining a quasigroup structure on X, i.e., such that each operation has both a right and left inverse operation (not necessarily equal), which we will denote respectively by $/^*, /, \backslash^*$ and $\backslash$. More precisely, we have the following definition:

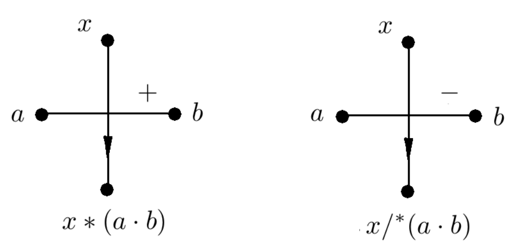
Definition 3. Let X be a set with binary operations $*, \cdot, \backslash^*, /^*, \backslash, / : X \times X \to X$ satisfying

$$\begin{array}{rcccl} y\backslash^*(y*x) & = & x & = & (x*y)/^*y \\ y\backslash(y\cdot x) & = & x & = & (x\cdot y)/y. \end{array}$$

Then we say X is a *biquasile* if for all $a, b, x, y \in X$ we have

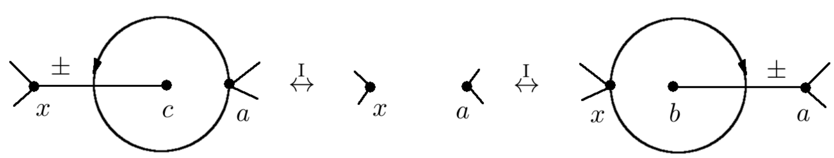
$$\begin{array}{rcll} a*(x\cdot[y*(a\cdot b)]) & = & (a*[x\cdot y])*(x\cdot[y*([a*(x\cdot y)]\cdot b)]) & (i) \\ y*([a*(x\cdot y)]\cdot b) & = & (y*[a\cdot b])*([a*(x\cdot[y*(a\cdot b)])]\cdot b) & (ii). \end{array}$$

We will interpret these operations as the following vertex coloring rules at crossings in a dual graph diagram:



We want to establish axioms for our algebraic structure to ensure that for each valid coloring on one side of the move, there is a unique valid coloring on the other side.

Consider the two positive Reidemeister I moves:



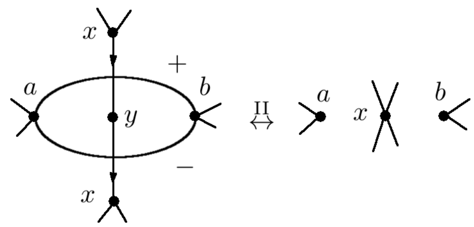
The condition we need is that for all $x, a \in X$ there exist unique elements $b, c \in X$ such that

$$\begin{array}{rcl} a & = & a*(x\cdot c) \\ x & = & x*(b\cdot a). \end{array}$$

That is, we must be able to solve the equations $a = a*(x\cdot c)$ and $x = x*(b\cdot a)$ for c and b respectively; this requires that $*$ has a left inverse operation $\backslash^*$ and that $\cdot$ has both a left and right inverse operation $\backslash$ and $/$; provided that X is a quasigroup under both $*$ and $\cdot$, these conditions are satisfied, with

$$\begin{array}{rcl} x\backslash(a\backslash^*a) & = & c \\ (x\backslash x)/a & = & b. \end{array}$$

Taking the direct Reidemeister II move



we get the requirement that for every $x, a, b \in X$, there exists a unique $y \in X$ such that

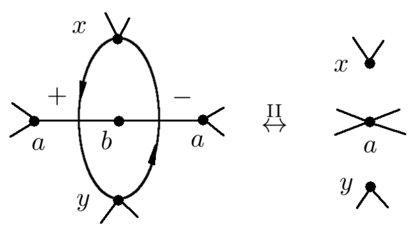
$$\begin{aligned} y &= x * (a \cdot b) \quad (dii.i) \\ x &= y /^* (a \cdot b) \quad (dii.ii); \end{aligned}$$

then we have

$$[x * (a \cdot b)] /^* (a \cdot b) = x$$

as required by definition of the operations $*$ and $/^*$. The other direct II move is similar.

The reverse Reidemeister II move



requires that for every $x, y, a \in X$ there exists a unique b such that

$$\begin{aligned} y &= x * (a \cdot b) \quad (rii.i) \\ x &= y /^* (a \cdot b) \quad (rii.ii). \end{aligned}$$

The existence of the right and left inverse operations for $*$ and $\cdot$ means we can solve (rii.i) for b, obtaining

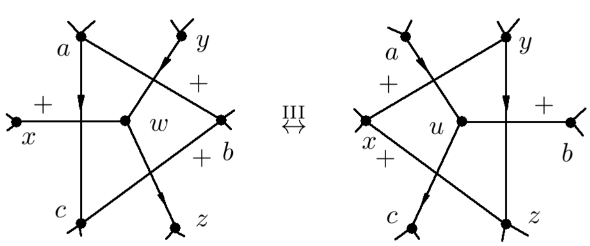
$$b = a \backslash (x \backslash^* y).$$

Substituting in (rii.ii), we then have

$$\begin{aligned} x &= y /^* (a \cdot b) \\ x &= y /^* (a \cdot [a \backslash (x \backslash^* y)]) \\ x &= y /^* (x \backslash^* y) \\ x * (x \backslash^* y) &= y \\ y &= y \end{aligned}$$

and equation (rii.ii) is satisfied.

The third Reidemeister move



yields the conditions

$$\begin{array}{rcl} w & = & y*(a\cdot b) \\ c & = & a*(x\cdot w) \\ z & = & w*(c\cdot b) \end{array} \quad \text{and} \quad \begin{array}{rcl} u & = & a*(x\cdot y) \\ c & = & u*(x\cdot z) \\ z & = & y*(u\cdot b) \end{array}$$

so we have

$$\begin{array}{rcl} c & = & a*(x\cdot[y*(a\cdot b)]) \\ z & = & (y*(a\cdot b))*(c\cdot b) \end{array} \quad \text{and} \quad \begin{array}{rcl} c & = & (a*(x\cdot y))*(x\cdot z) \\ z & = & y*([a*(x\cdot y)]\cdot b) \end{array}$$

whence

$$\begin{array}{rcll} a*(x\cdot[y*(a\cdot b)]) & = & (a*[x\cdot y])*(x\cdot[y*([a*(x\cdot y)]\cdot b)]) & (i) \\ y*([a*(x\cdot y)]\cdot b) & = & (y*[a\cdot b])*([a*(x\cdot[y*(a\cdot b)])]\cdot b) & (ii) \end{array}$$

must be satisfied for all $a, b, x, y \in X$. We can consider these to be somewhat complicated analogues of the distributive law. We can reformulate these slightly by defining functions

$$f_{a,b}(x,y) = x*(a\cdot[b*(x\cdot y)]) \quad \text{and} \quad g_{a,b}(x,y) = y*([a*(x\cdot y)]\cdot b); \tag{1}$$

then (i) and (ii) are the requirements that

$$f_{a,b}(x,y) = f_{a,b}(x*(a\cdot b),y) \quad \text{and} \quad g_{a,b}(x,y) = g_{a,b}(x,y*(a\cdot b)). \tag{2}$$

Example 3. (Dehn Biquasile of an abelian group) Let A be any abelian group; then A is a biquasile under the operations

$$a\cdot b = a+b \quad \text{and} \quad x\cdot y = y-x,$$

as we can easily verify:

$$a/b = a-b, \quad a\backslash b = b-a, \quad x/^*y = y-x, \quad \text{and} \quad x\backslash^*y = x+y$$

and

$$\begin{array}{rcl} a*(x\cdot[y*(a\cdot b)]) & = & x+b-y \\ & = & (a*[x\cdot y])*(x\cdot[y*([a*(x\cdot y)]\cdot b)]) \\ y*([a*(x\cdot y)]\cdot b) & = & x-a+b \\ & = & (y*[a\cdot b])*([a*(x\cdot[y*(a\cdot b)])]\cdot b). \end{array}$$

Since the Dehn presentation relation $ax^{-1}by^{-1} = 1$ abelianizes to $y = a+b-x = x*(a\cdot b)$, this type of biquasile can be understood as a generalization of the Dehn presentation of the knot group.

As with other algebraic structures, we can specify a biquasile structure on a finite set $X = \{x_1, \dots, x_n\}$ with a pair of matrices encoding the operation tables of the $*$ and $\cdot$ operations. More precisely, the *biquasile matrix* of a biquasile $(X, *, \cdot)$ of cardinality n is the $n \times 2n$ block matrix with (j,k) entry $m \in \{1, 2, \dots, n\}$ where

$$x_m = \begin{cases} x_j * x_k & 1 \le k \le n \\ x_j \cdot x_k & n+1 \le k \le 2n. \end{cases}$$

Example 4. Our `python` computations reveal 72 biquasile structures on the set $X = \{x_1, x_2, x_3\}$ of three elements, including for instance

$*$	x_1	x_2	x_3
x_1	x_3	x_2	x_1
x_2	x_2	x_1	x_3
x_3	x_1	x_3	x_2

$\cdot$	x_1	x_2	x_3
x_1	x_3	x_1	x_2
x_2	x_1	x_2	x_3
x_3	x_2	x_3	x_1

or more compactly

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 1 & 2 & 3 \\ 1 & 3 & 2 & 2 & 3 & 1 \end{array}\right].$$

As with other algebraic categories, we have the following standard definitions.

Definition 4. A *biquasile homomorphism* is a map $f : X \to Y$ between biquasiles such that $f(x * x') = f(x) * f(x')$ and $f(x \cdot x') = f(x) \cdot f(x')$ for all $x, x' \in X$. A bijective homomorphism is an *isomorphism*, and an isomorphism $f : X \to X$ is an *automorphism*.

Example 5. There are four biquasile structures on the set $X = \{x_1, x_2\}$, given by

$$X_1 = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{array}\right],\ X_2 = \left[\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{array}\right],\ X_3 = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{array}\right] \text{ and } X_4 = \left[\begin{array}{cc|cc} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{array}\right].$$

Of these, there are two isomorphism classes, $\{X_1, X_4\}$ and $\{X_2, X_3\}$. The 72 biquasiles of order three break down into 19 isomorphism classes, and our computations reveal 6912 biquasiles of order four comprising 361 isomorphism classes.

Definition 5. A *sub-biquasile* of X is a subset $S \subset X$ which is itself a biquasile under the operations of X; for $S \subset X$ to be a sub-biquasile we need closure of S under $*, \cdot$ and the right and left division operations of both. Say a biquasile is *simple* if it has no nontrivial sub-biquasiles.

Example 6. The biquasile structure on $X = \{x_1, x_2, x_3\}$ with operation matrix

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 1 & 2 & 3 \\ 1 & 3 & 2 & 2 & 3 & 1 \end{array}\right].$$

in Example 4 has one nontrivial sub-biquasile, the singleton set $\{3\}$. The biquasile with operation matrix

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 2 & 1 & 3 \\ 3 & 2 & 1 & 1 & 3 & 2 \\ 2 & 1 & 3 & 3 & 2 & 1 \end{array}\right]$$

is simple since we have $1 \cdot 1 = 2$, $2 \cdot 2 = 3$ and $3 \cdot 3 = 1$, so the closure of every nonempty subset of $X = \{x_1, x_2, x_3\}$ under the biquasile operations is all of X.

Definition 6. Let X be a set and $W(X)$ the set defined recursively by the following rules:

(i) $x \in X$ implies $x \in W(X)$, and

(ii) $x, y \in W(X)$ implies $x * y, x \cdot y, x /^* y, x \backslash^* y, x/y$ and $x \backslash y \in W(X)$.

We call the elements of $W(X)$ *biquasile words* in the *generators* X. Then we define the *free biquasile on* X to be the set of equivalence classes of biquasile words in X modulo the relations determined by the biquasile axioms, e.g. $(x*y)/^*y \sim x$, $y*([a*(x\cdot y)]\cdot b) \sim (y*[a\cdot b])*([a*(x\cdot[y*(a\cdot b)])]\cdot b)$, etc. More generally, given a set of generators X and a set of *relations* R, i.e., equations of biquasile words, the *biquasile presented by* $\langle X \mid R\rangle$ is the set of equivalence classes of biquasile words in X modulo the equivalence relation generated by the biquasile axioms together with the relations in R.

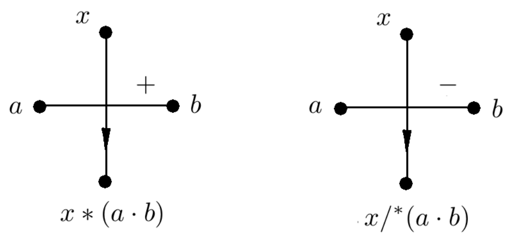
As in other universal algebraic systems, biquasiles presented by presentations related by the following *Tietze moves* are isomorphic:

(i) Adding or removing a generator x and relation of the form $x = W$ where W is a word in the other generators not involving x, and

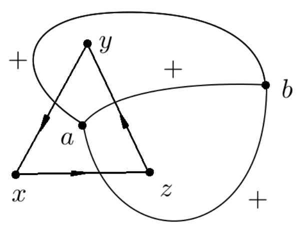
(ii) Adding or removing a relation which is a consequence of the other relations.

An important example is the *fundamental biquasile* of an oriented knot or link, defined as follows:

Definition 7. Let L be a dual graph diagram and X its set of vertices. Then the *fundamental biquasile* of L is the biquasile with presentation $\langle X \mid R \rangle$ where at each edge crossing in the dual graph diagram we have a relation as pictured.



Example 7. The dual graph diagram D below has the fundamental biquasile presentation listed.



$$\begin{aligned}&\langle x, y, z, a, b \mid x * (b \cdot a) = z, z * (b \cdot a) = y, y * (b \cdot a) = x\rangle \\ &= \langle x, y, a, b \mid [x * (b \cdot a)] * (b \cdot a) = y, y * (b \cdot a) = x\rangle \\ &= \langle y, a, b \mid [(y * [b \cdot a]) * (b \cdot a)] * (b \cdot a) = y\rangle.\end{aligned}$$

By construction, we have the following:

Proposition 3. *The isomorphism class of the fundamental biquasile of an oriented link is a link invariant.*

Definition 8. Let X be a biquasile and D a dual graph diagram. The *biquasile counting invariant* of D, denoted $\Phi_X^{\mathbb{Z}}(D)$, is the cardinality of the set of X-colorings of D. We can interpret colorings of D by X as homomorphisms from the fundamental biquasile of D to X.

By construction, we have the following:

Theorem 4. *If $|X|$ is finite, then $\Phi_X^{\mathbb{Z}}(D) \leq |X|^{|V|}$ where V is the set of vertices in D. If D and D' are related by Reidemeister moves, then $\Phi_X^{\mathbb{Z}}(D) = \Phi_X^{\mathbb{Z}}(D)'$ and hence $\Phi_X^{\mathbb{Z}}(D)$ is an oriented link invariant.*

Example 8. The dual graph diagram D in Example 7 has nine colorings by the biquasile X from example 4, as we can find by trying all assignments of element of X to the generators y, a, b in the presentation of the fundamental biquasile of D and checking which such assignments satisfy the relation R given by $[(y * [b \cdot a]) * (b \cdot a)] * (b \cdot a) = y$:

y	a	b	R?	y	a	b	R?	y	a	b	R?
1	1	1	✓	2	1	1		3	1	1	
1	1	2		2	1	2	✓	3	1	2	
1	1	3		2	1	3		3	1	3	✓
1	2	1		2	2	1	✓	3	2	1	
1	2	2		2	2	2		3	2	2	✓
1	2	3	✓	2	2	3		3	2	3	
1	3	1	✓	2	3	1		3	3	1	
1	3	2		2	3	2		3	3	2	✓
1	3	3		2	3	3	✓	3	3	3	

Example 9. We selected a biquasile of order 4,

$$X = \left[\begin{array}{cccc|cccc} 3 & 4 & 2 & 1 & 4 & 2 & 3 & 1 \\ 2 & 1 & 3 & 4 & 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 & 1 & 3 & 4 & 2 \\ 1 & 3 & 4 & 2 & 2 & 4 & 1 & 3 \end{array}\right]$$

and computed the counting invariant for all prime knots with up to eight crossings and prime links with up to seven crossings; the results are collected below.

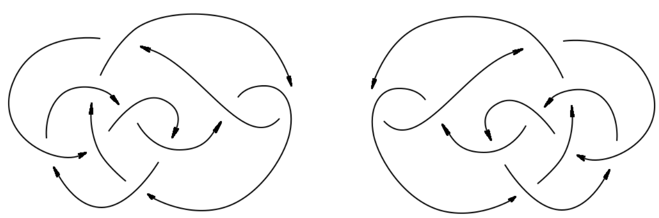
$\Phi_X^{\mathbb{Z}}(K)$	K
16	Unknot, $5_1, 5_2, 7_1, 8_{12}$
18	8_2
20	$6_2, 8_{11}$
22	8_6
24	$8_1, 6_1, 6_3, 7_2, 7_3, 7_6, 8_4, 8_8, 8_{14}, 8_{17}$
28	$7_7, 8_{13}, 8_{21}$
30	$8_7, 8_{16}$
32	$7_5, 8_1, 8_3, 8_{15}, 8_{20}$
34	8_{19}
40	$3_1, 8_9, 8_{10}$
48	8_5
56	8_{18}

$\Phi_X^{\mathbb{Z}}(L)$	L
16	$L6a4$
20	$L7a6$
22	$L5a1$
24	$L2a1, L6a2$
26	$L7n1$
28	$L6n1$
32	$L4a1, L6a1, L7a4$
34	$L7a3$
36	$L7a1, L7a5, L7a7, L7n2$
40	$L6a5, L7a2$
48	$L6a3.$

Example 10. For our final example in this section, we computed the biquasile counting invariant with respect to the biquasile $X = \{x_1, x_2, x_3, x_4\}$ given by

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 2 & 4 & 3 \\ 2 & 4 & 1 & 3 & 3 & 1 & 2 & 4 \\ 4 & 3 & 2 & 1 & 2 & 4 & 3 & 1 \\ 3 & 1 & 4 & 2 & 4 & 3 & 1 & 2 \end{array}\right]$$

for the chiral knot 9_{32} and its mirror image $\overline{9_{32}}$. Since these values are distinct, the knots are distinguished by the invariant:



$\Phi_X^{\mathbb{Z}}(9_{32}) = 22$ $\qquad$ $\Phi_X^{\mathbb{Z}}(\overline{9_{32}}) = 26$

In particular, since a knot and its mirror image have isomorphic fundamental biquandles (see Theorem 1 in [2] for instance), this example demonstrates that *biquasile counting invariants are not determined by the fundamental biquandle of a knot.*

4 Alexander Biquasiles

As with previous knot-coloring structures, we can consider the case of biquasile structures with linear operations, which we call *Alexander biquasiles.* We can think of Alexander biquasiles as generalizations of Dehn biquasiles.

Proposition 5. *Let $L = \mathbb{Z}[d^{\pm 1}, n^{\pm 1}, s^{\pm 1}]$. An L-module X is a linear biquasile, also called an Alexander biquasile, under the operations*

$$x * y = (-dsn^2)x + ny \quad \text{and} \quad x \cdot y = dx + sy. \tag{3}$$

Proof. First, we note that the invertibility of the variables d, s and n makes $*$ and $\cdot$ quasigroup operations. Instate the notation of (1), where the operations are now given by (3). We seek to verify the relations in (2). One readily computes that

$$f_{a,b}(x,y) = (dn)a + (-n^3s^2d)b + (s^2n^2)y = f_{a,b}(x * (a \cdot b), y).$$

Similarly, we have

$$g_{a,b}(x,y) = (-n^3d^2s)a + (s)b + (d^2n^2)y = g_{a,b}(x, y * (a \cdot b)).$$

This verifies (2) and completes the claim. □

Example 11. *(Alexander biquasile structures on $\mathbb{Z}_m$)* As a special case, one can consider groups $\mathbb{Z}_m$ and allow $d, n, s \in \mathbb{Z}_m^\times$. For example, in $\mathbb{Z}_3$ there are seven unique (non-isomorphic) possibilities for assigning d, n, s that satisfy (2):

d	**n**	**s**	$-n^2ds$
1	1	2	1
2	1	1	1
2	2	1	1
1	1	1	2
1	2	1	2
2	1	2	2
2	2	2	2

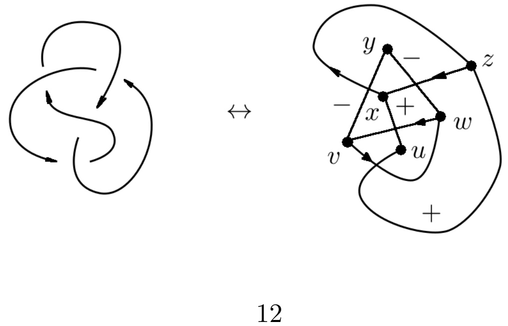
We may also be interested in the number of configurations of (d, n, s) that satisfy the conditions (2) (as well as forming a quasigroup), and how many of those configurations create non-isomorphic structures:

m	**# configurations**	**# non-isomorphic**
2	1	1
3	8	7
4	8	7
5	64	34
6	8	7
7	216	137
8	64	33
9	216	152
10	64	34

As an extension to this example, we consider a dual graph diagram K over $\mathbb{Z}_3$ and compute the number of colorings, $\Phi_{\mathbb{Z}_3}^{\mathbb{Z}}(K)$. Consider the operations

$$x * y = x + y, \quad x /^* y = x + 2y, \quad x \cdot y = x + 2y,$$

which correspond to the selection of $d = 1, n = 1$, and $s = 2$. Consider the following knot and corresponding dual graph diagram K, where we have labeled the nodes for reference.



This dual graph diagram yields the following four equations (along with their equivalent forms):

$$\begin{array}{rcl} z/^*(y\cdot w)=x & \leftrightarrow & z+2y+w=x \\ x/^*(y\cdot v)=z & \leftrightarrow & x+2y+v=z \\ w*(x\cdot u)=v & \leftrightarrow & w+x+2u=v \\ v*(z\cdot u)=w & \leftrightarrow & v+z+2u=w. \end{array}$$

Rewriting as homogeneous equations and putting in matrix form (with respect to the vector $[u, v, w, x, y, z]$), these give:

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \underset{\leftrightarrow}{\text{Row moves over } \mathbb{Z}_3} \begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

The solution space is thus two-dimensional, and so we have $\Phi_{\mathbb{Z}_3}^{\mathbb{Z}}(K) = 3^2 = 9$.

Example 12. As in the case of quandles and biquandles [3], we can define a module-valued Alexander invariant of oriented knots and links by considering the Alexandrization of the fundamental biquasile of an oriented knot or link, i.e. the fundamental biquasile written as an Alexander biquasile. More precisely, the kernel of the coefficient matrix of the homogeneous system of linear equations over L is an L-module valued invariant of oriented knots and links analogous to the classical Alexander invariant; from it, we can derive polynomial-valued invariants via the Gröbner basis construction described in [2].

For instance, the knot 4_1 in example 11 has Alexander biquasile given by the kernel of the matrix below with entries in L:

$$\begin{bmatrix} 0 & 0 & s & -1 & d & -dsn^2 \\ 0 & s & 0 & -dsn^2 & d & -1 \\ s & -1 & -dsn^2 & d & 0 & 0 \\ s & -dsn^2 & -1 & 0 & 0 & d \end{bmatrix}.$$

These invariants will be the subject of another paper.

5 Questions

We end with a few collected questions for future research.

- We've seen in Example 10 that biquasile invariants are not secretly biquandle invariants. What, if anything, is the relationship between biquasiles and biquandles?
- What enhancements of the biquasile counting invariants can be defined?
- What kinds of categorifications of biquasiles and their invariants are possible?

References

[1] M. Cohen, O. T. Dasbach, and H. M. Russell. A twisted dimer model for knots. *Fund. Math.*, 225(1):57–74, 2014.

[2] A. S. Crans, A. Henrich, and S. Nelson. Polynomial knot and link invariants from the virtual biquandle. *J. Knot Theory Ramifications*, 22(4):134004, 15, 2013.

[3] M. Elhamdadi and S. Nelson. *Quandles—an introduction to the algebra of knots*, volume 74 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2015.

[4] N. Kamada and Y. Miyazawa. A 2-variable polynomial invariant for a virtual link derived from magnetic graphs. *Hiroshima Math. J.*, 35(2):309–326, 2005.

[5] L. H. Kauffman. A Tutte polynomial for signed graphs. *Discrete Appl. Math.*, 25(1-2):105–127, 1989. Combinatorics and complexity (Chicago, IL, 1987).

[6] C. N. Little. *KNOTS, WITH A CENSUS FOR ORDER TEN*. ProQuest LLC, Ann Arbor, MI, 1885. Thesis (Ph.D.)–Yale University.

[7] Y. Miyazawa. Magnetic graphs and an invariant for virtual links. *J. Knot Theory Ramifications*, 15(10):1319–1334, 2006.

[8] Y. Miyazawa. Link polynomials derived from magnetic graphs. *Topology Appl.*, 157(1):228–246, 2010.

Department of Mathematical Sciences
Claremont McKenna College
850 Columbia Ave.
Claremont, CA 91711