

Finding Beauty: A Case Study in Insights from Teaching Developmental Mathematics

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Recommended Citation

Piercey, V. and Aly, G. "Finding Beauty: A Case Study in Insights from Teaching Developmental Mathematics," *Journal of Humanistic Mathematics*, Volume 9 Issue 2 (July 2019), pages 130-148. DOI: 10.5642/jhummath.201902.09 . Available at: <https://scholarship.claremont.edu/jhm/vol9/iss2/9>

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Finding Beauty: A Case Study in Insights from Teaching Developmental Mathematics

Cover Page Footnote

The authors thank Anil Venkatesh, whose idea it was to consider Lie groups and algebras as a setting to generalize exponents. The authors would also like to thank Gizem Karaali and the referees for their thoughtful comments, encouragement, and discussion.

Finding Beauty: A Case Study in Insights from Teaching Developmental Mathematics

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Synopsis

As mathematicians, we often fail to appreciate the opportunities open to us when we teach developmental mathematics. One such opportunity is that we may deepen our understanding of mathematics that we have taken for granted. This paper contains a brief case study concerning what we have learned about operations, inverses, and exponents in the process of teaching beginning algebra. Our inquiry takes us from student questions about signed numbers, through the category of rings, to the world of Lie groups and Lie algebras.

Suppose you just finished your Ph.D. in pure mathematics and landed a tenure-track job. You are assigned to teach one section of Complex Analysis (your favorite undergraduate course!) and two sections of developmental Beginning Algebra. Wait, algebra? Not abstract algebra with groups and rings and other fun things, but basic high school algebra?

Unless you teach at an elite institution, it is very likely that you will be teaching developmental or lower-level mathematics courses. Due to the changing demands of the 21st century workforce, more faculty are teaching developmental mathematics. Sixty-five percent of jobs in the United States by 2020 will require some education beyond high school [3].

Many college students are not appropriately prepared for college-level work; as of 2009, studies estimated that approximately 40% of students in college require developmental courses [2], a figure which rises to around 60% if we look only at community colleges [1]. There is no indication that these findings have changed since. Eliminating developmental courses and requiring co-requisite supplemental classes or the completion of computer-assisted modules are other trends in supporting under-prepared students (see, e.g., [14], [17]). Even if such courses are eliminated, it is left to the mathematics professor to prepare students who are not yet ready for college-level classes so they can advance and earn their undergraduate degrees.

Now suppose you have been teaching at the same institution for 25 years. You have taught thousands of students, mostly in developmental mathematics. The students seem to be getting worse, and you have lost touch with the love of mathematics that inspired you to go to graduate school many years ago. You dread the beginning of the semester because teaching such classes pulls you away from the work you find more interesting and engaging.

Even if these scenarios do not resonate with you, perhaps these vignettes remind you of a colleague, former professor, or acquaintance. It is not unusual to find faculty who seem bored or frustrated teaching lower-level mathematics classes. How can such faculty, who see these courses as mundane, spark curiosity and wonder in students? These perspectives may be why Uri Treisman has called developmental mathematics “the burial ground for the aspirations of students” [13].

There are many faculty who care deeply about the mission of developmental mathematics and developmental students. For example, we approach teaching developmental mathematics as an opportunity to eliminate barriers by making mathematics accessible to a hitherto marginalized population of learners. For us, this is a social justice mission and reflects our perspective on teaching and learning (see, e.g., [15]). We are far from unique in this respect. A group of faculty at Westfield State University have created the *Discovering the Art of Mathematics* series for precisely this purpose [8]. Faculty at Augsburg University, led by Su Doreé, have created a course entitled Just Enough Algebra with the same goal in mind [7]. Reform efforts and quantitative literacy movements have sprung up across the country (see, e.g., [5], [4], [9], [16]). Despite pockets of growth, the predominant attitude we have encountered toward teaching developmental mathematics seems negative.

We suggest that one of many ways to find excitement in teaching developmental mathematics is to use student questions as opportunities to look deeper into basic algebra and make connections with more sophisticated mathematics. The story we outline below provides an example of what this kind of inquiry looked like for us, starting from students' questions about signed numbers and ending in the world of Lie groups and Lie algebras. While the original student questions themselves may be considered "trivial" for anyone teaching mathematics at the college level, they frustrated our students for being a part of the seemingly endless set of random and arbitrary rules in mathematics [12]. In our quest to find accessible answers that could be understood by and satisfy our students, we also discovered something unexpected: we learned new things about mathematics.

Even if you work at an elite institution, or have a course-load where you are not teaching developmental mathematics, you will likely instruct mathematics majors, preservice teachers, and other students who will eventually enter teaching. The vignette we offer below could help such students connect content in their graduate or undergraduate mathematics courses with the more elementary content they will teach. Building bridges between developmental and more sophisticated content helps bring a deeper richness to the discussion for struggling learners, while motivating the more advanced student.

We expect that the following ideas are not new. However, they have never been articulated to us. We suppose we may have understood what follows implicitly, but now we are making them explicit. In this paper, we share some of these thoughts. We do not claim to be breaking any new mathematical ground. Rather, we are describing an intellectual journey that started in teaching developmental algebra. We hope the reader finds something exciting and maybe even inspiring. If we dare dream, by looking at Beginning Algebra with an inquisitive perspective, we can start finding beauty in developmental mathematics.

1. Reflections, translations, and negative numbers

There are two questions for which we have searched for simple, "elevator-speech" responses:

1. Why is it that when we multiply or divide both sides of an inequality by a negative number, the direction of the inequality flips, but this does not happen with addition or subtraction?
2. Why is the product of two negative numbers positive?

To address the first question, an instructor can use an example or an algebraic explanation. But we feel that a geometric explanation is richer. Such a geometric approach also provides a simple reason for the rules behind the multiplication of signed numbers that comes up in the second question.

Geometrically, multiplying any real number by -1 results in a **reflection** over the origin on the number line. Reflections flip the relative positions of points on the number line and are order-reversing. Inequalities, which indicate relative order, must be reversed. On the other hand, adding any real number (positive, zero, or negative) results in a **translation** which is an order-preserving transformation. This explains why the direction of an inequality remains the same when we add or subtract a number whereas the direction changes when we multiply or divide by a negative number.

We can use this response to the first question to answer the second question on why the product of two negative numbers is positive. Start with $(-1) \times (-1)$. If we consider our initial position on the number line to be $+1$, then this multiplication is the composition of two reflections: first we reflect from the positive side of the number line, over the origin, to the negative side. Then we reflect from the negative side of the number line back to the positive side. Hence the resulting product is $+1$, a positive number. In other words, multiplying two negatives is the same as reflecting from one side of the number line to the other, and then back again. The same reasoning applies to a product such as $(-3) \times (-4)$, where we also apply a dilation / scaling along the number line resulting from multiplying 3×4 .

These explanations provide students with a visual understanding of how to manipulate signed numbers, and they emphasize reason over arbitrary rules in the developmental classroom. Students can further appreciate these explanations because they are familiar with a direct application: shifting (translating) or reflecting parts of an image using photo manipulating software. This analogy is readily understood by anyone who is familiar with social media and smart phone cameras.

On the other hand, for a mathematician, viewing multiplication with signed numbers as reflection and dilation is exactly how multiplication is described in the complex plane: a dilation by the product of the magnitudes and a rotation by the sum of the arguments.

That could have been the end of the story. We have answers for students that are simple and can be demonstrated visually. However, as mathematicians, we wanted to continue to look deeper. We noticed that while a number like -3 is an **additive** inverse element, it can be obtained by a reflection of 3 over the origin. This sounds like our description of **multiplication** of signed numbers. Is there a pattern here? Could this pattern be extended to exponents? We wanted to further understand these questions by looking closer, untangling the role of operations and inverse elements.

2. Inverses, operations, and -1

To untangle the role of operations and inverse elements and expand our initial explanations behind multiplying signed numbers, we start by comparing operations. Consider addition, multiplication, and exponentiation on natural numbers as a hierarchy of operations, which comes from thinking of them in the following way: multiplication is repeated addition and exponentiation is repeated multiplication.¹ This hierarchy is displayed in Figure 1.

Given $x \in \mathbb{R}$, its inverse element under addition is obtained by multiplying it by -1 . In other words, we apply the next operation in the hierarchy, **multiplication**, to our number and -1 . The result on the number line is a reflection over the **additive** identity 0 . This is depicted in Figure 2.

Now suppose $x \neq 0$ and consider its inverse under multiplication. This is obtained by raising x to the power of -1 .² In other words, we apply the next operation in the hierarchy, **exponentiation**, to our number and -1 .

¹ Later in our story, we explain how we realized that this is a problematic way of thinking about exponentiation. Also see a list of links to a long thread of blog posts by Keith Devlin on the nature of multiplication here: [6].

² While it is true that $x^{-1} = 1/x$ is how we **define** negative exponents, this definition is not arbitrary. If we look at whole-number exponents as designating repeated multiplication, this definition is forced on us. In order to increase the exponent n on x^n by 1 , we have to multiply by x . Hence in order to decrease the exponent by 1 , we

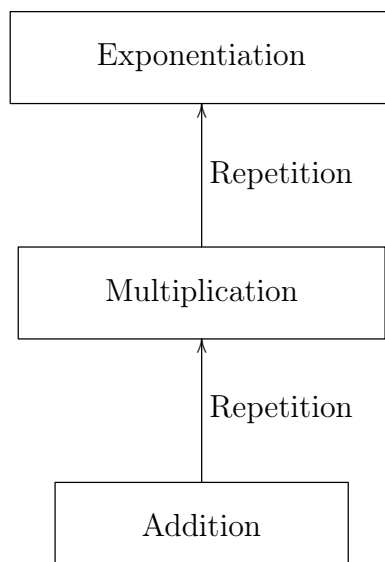


Figure 1: A tentative hierarchy of operations.

The resulting transformation on the number line is also a type of reflection. This time we have a reflection over the **multiplicative** identity $+1$. See Figure 3. This is a reflection exactly in the sense of reflecting over the unit circle in the complex plane. In fact, these constructions of the additive and multiplicative inverses are often generalized to the complex plane using linear fractional transformations.

As mathematicians, we were hard-pressed to see this repetition as a coincidence. We explored the possibility that there exists a deeper framework for understanding inverse elements from an algebraic perspective. This is a simple idea: to obtain the inverse of a mathematical object under an operation, apply the “next operation” (whatever that may mean) to our original object and -1 . The geometric result will be a reflection over an “identity” of some sort. We call this a “general inversion framework”, see Figure 4.

have to multiply by $1/x$. It follows that $x^0 = x^1 \cdot 1/x = 1$ and $x^{-1} = x^0 \cdot 1/x = 1/x$. Does this mean we are describing something intrinsic to the expression x^{-1} ? As the reader will see below, fitting exponentiation into our framework gave us more trouble than we expected and some skepticism is warranted.

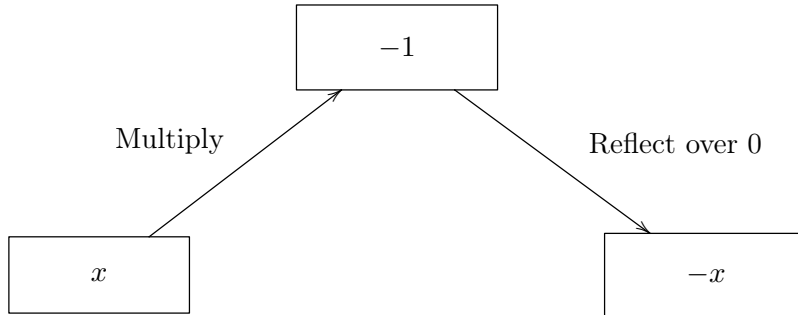


Figure 2: Inverting under addition.

Our next question was whether we can generalize our framework further. Since we are dealing with operations, we thought that the appropriate setting for generalization was rings. Let R be a ring with a unit 1 . The ring of integers \mathbb{Z} is an initial object in the category of rings, which means that there is a unique ring homomorphism $\varphi : \mathbb{Z} \rightarrow R$. The ring homomorphism definition forces $\varphi(0) = 0$, $\varphi(1) = 1$, and $\varphi(-1) = -1$. Then we find for $n > 0$, $\varphi(n) = 1 + 1 + \cdots + 1$, while $n < 0$ yields $\varphi(n) = -1 + -1 + \cdots + -1$. This defines an action of \mathbb{Z} on R given by “multiplication”, where $n \in \mathbb{Z}$ is identified with $\varphi(n)$ in R . Then for any $x \in R$, $nx = (1 + 1 + \cdots + 1)x = x + x + \cdots x$. Hence we recover multiplication by n as repeated addition.

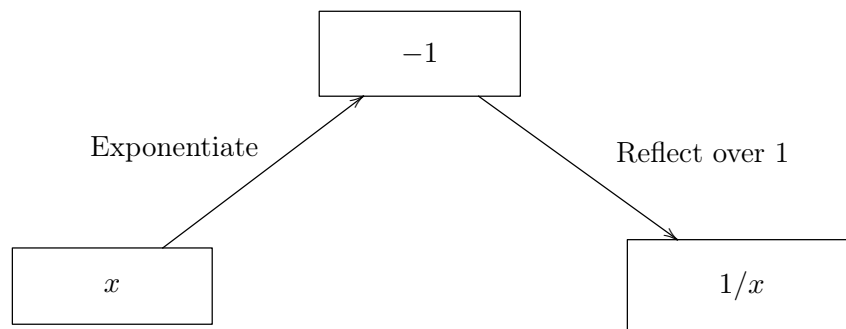


Figure 3: Inverting under multiplication.

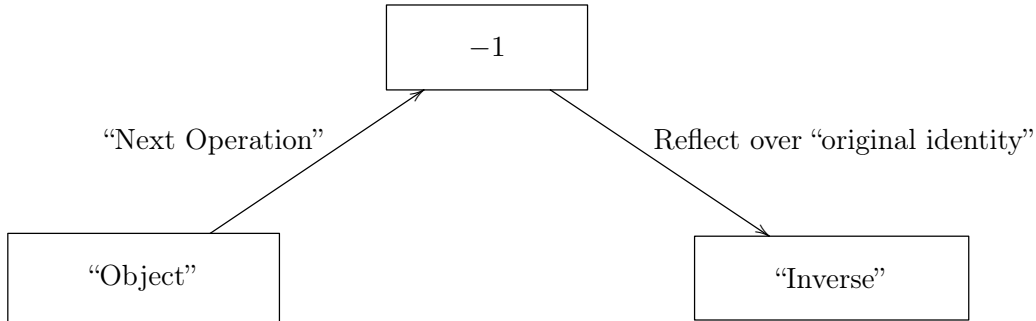


Figure 4: General Inversion Framework

The inverse of x is $-x$, which is $-1 \cdot x$, a fact verified in abstract algebra courses.³

Multiplication is part of the structure of a ring; we just checked that the multiplicative structure was consistent with repeated addition when restricted to appropriate elements. So what about exponentiation? For exponentiation, we would have to reverse this process: start with repeated multiplication and extend to other pairs of elements.

In a ring R , we can define x^n to be x multiplied by itself n times, where $n \in \mathbb{Z}$ and $n > 0$. We can also define $x^0 = 1$. If $x \in R^\times$, we can define negative exponents using the multiplicative inverse. In this case, the multiplicative inverse is denoted x^{-1} .⁴

As we thought about extending repeated multiplication in a ring, we started to doubt whether we were in the right category for this question. For example in the ring $k[x]$, where k is a field, there is no element b^x (where $b \in k$).

³ Note that while we have recovered most of the structure we identified above, we have not addressed the reflection. In order to address the question of reflections, we would have to consider the meaning of reflection in an arbitrary ring. The typical topology associated with a ring comes from the prime spectrum $\text{Spec}(R)$ with its usual Zariski topology. See Chapter II, Section 2 of [11]. We haven't looked into this yet, as questions about exponentiation more urgently demanded our attention.

⁴ As remarked in a footnote above, the same argument can be used to explain why this definition of x^{-1} makes sense, but we continue to question whether this is truly something structural.

Even in \mathbb{R} , we noticed some anomalies. First of all, it is obvious that exponentiation is noncommutative. After all, $2^3 \neq 3^2$. This isn't terribly concerning; matrix multiplication is also noncommutative. More anomalous is that exponentiation is not associative, for example $(2^3)^4 \neq 2^{(3^4)}$.

Another issue with exponentiation in \mathbb{R} has to do with identities. Due to the lack of commutativity and associativity, it's not surprising that there might be a left identity and a right identity that are different. The right identity is a value x so that $b^x = b$ for all b . That is simple enough: the right identity is 1. A left identity is trickier. We need a base b for which $b^x = x$ for all possible x . There is no such value. If we choose $x = 1$, we would have $b = 1$. But then $b^2 \neq 2$.

There are other issues as well. Working in \mathbb{R} , unlike multiplication and addition, we cannot apply exponentiation to any pair of real numbers. For example, we cannot have $b = -1$ and $x = 1/2$ without including complex numbers. Within the complex numbers, we would have to deal with multi-valued functions and branching. This is quite different from multiplication and addition, where we can grab any two real (or complex) numbers, apply the operation, and uniquely obtain another real number.

All of these observations made us question whether or not exponentiation really is a binary operation: a map of sets $S \times S \rightarrow S$. It seems tempting to treat exponentiation as a binary operation, however restricted, as we can take any positive base b and any real exponent x and produce a real number b^x . However, the problems we noticed made us skeptical, leading us to question whether our "general inversion framework" was valid.

3. A Clearer Picture of Exponents: Lie Groups and Lie Algebras

In order to further examine the validity of our proposed "general inversion framework," we needed to better understand exponentiation and subject the claim that exponentiation is a binary operation to further scrutiny. First we had to figure out the right category for generalizing exponentiation. It turns out a more natural, abstract setting to discuss exponentiation is in the category of Lie groups and algebras.⁵ This setting is also appropriate

⁵This section is inspired by conversations with Anil Venkatesh (Adelphi University) who reminded us of the role exponentiation plays in Lie groups and algebras.

as we are looking for a “larger” mathematical space in which to answer our question, similar to how mathematicians are typically introduced to groups, then rings, then integral domains, and then fields in the classroom in order to solve different types of algebraic “equations”.

For the reader unfamiliar with Lie theory, a few basic facts and examples both serve as background and help us show more clearly why this is a more natural setting for exponentiation. Hall [10] provides a very readable introduction to Lie groups and algebras. The properties below can be found in Chapters 1 and 2, and while Hall is limiting himself to matrix groups, the facts can be generalized.

1. A Lie group G is a group that is also a smooth manifold. As part of the definition, the operation \cdot in G is a smooth map $\cdot : G \times G \rightarrow G$ and inversion is a smooth map $G \rightarrow G$ given by $x \mapsto x^{-1}$, where x^{-1} denotes the inverse element of x under the operation \cdot in G .

Some examples include $\mathbb{R} \setminus \{0\}$ with multiplication, the unit circle \mathbb{S}^1 with rotation, and the general linear group of n by n matrices over a field k (denoted $\text{GL}(n, k)$) with multiplication.

2. Given a Lie group G , there is a corresponding Lie algebra \mathfrak{g} , which is the tangent space to G at the identity I ($\mathfrak{g} = T_I G$). As with smooth manifolds, the topological manifold structure of G is modeled on a field k and the Lie algebra is a vector space over k (the same field).⁶

For $\mathbb{R} \setminus \{0\}$, the Lie algebra is \mathbb{R} as a vector space over itself. For $\text{GL}(n, k)$, the Lie algebra is the set of all $n \times n$ matrices over k (denoted $\text{M}(n, k)$) which is isomorphic to $k^{(n^2)}$ as a vector space.

3. There is a map called $\exp : \mathfrak{g} \rightarrow G$. This is defined in multiple ways, the simplest is probably using vector fields. Given an element $v \in \mathfrak{g}$, there is a smooth curve $\gamma : k \rightarrow G$ generating a one-parameter subgroup of G such that $\gamma(0) = I$ and $\gamma'(0) = v$. Then $\exp(v) = \gamma(1)$. Part of this definition includes a proof that this is well-defined. It can also be shown that $\exp(0) = I$ by setting γ to be the constant map $\gamma(t) = I$ for all $t \in k$.

⁶ If $k \neq \mathbb{R}$ and $k \neq \mathbb{C}$, algebraic geometry may be required to define continuity and smoothness. See Chapter III, section 10 of [11]

For matrix groups such as $GL(n, k)$, the map \exp is given by the power series expansion. Specifically, for any matrix $A \in M(n, k)$ (the Lie algebra for $GL(n, k)$):

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!},$$

where the exponents in the expansion denote repeated matrix multiplication. In $\mathbb{R} \setminus \{0\}$, which is $GL(1, \mathbb{R})$, for any $x \in \mathbb{R}$ (the Lie algebra), we recover $\exp(x) = e^x$.

4. The map $\exp : \mathfrak{g} \rightarrow G$ can be shown to satisfy the property

$$\exp((s+t)v) = \exp(sv) \cdot \exp(tv),$$

where $s, t \in k$ and $v \in \mathfrak{g}$.

This needs a little unpacking. On the left side, $s+t$ is addition in the field k . Then multiplication by v is scalar multiplication in \mathfrak{g} as a vector space over k . This produces an element $(s+t)v \in \mathfrak{g}$ that we can apply \exp to and obtain an element downstairs in G . On the right, sv and tv are scalar multiplication in \mathfrak{g} and the multiplication of their respective images in G under the \exp map is the operation in G .

This can be expressed with a commutative diagram:

$$\begin{array}{ccccc}
 \mathfrak{g} \times k \times k & \xrightarrow{(v,s,t) \mapsto (v,s+t)} & \mathfrak{g} \times k & \xrightarrow{(v,s+t) \mapsto (s+t)v} & \mathfrak{g} \\
 \downarrow (v,s,t) \mapsto (sv,tv) & & & & \downarrow (s+t)v \mapsto \exp((s+t)v) \\
 \mathfrak{g} \times \mathfrak{g} & \xrightarrow{(sv,tv) \mapsto (\exp(sv), \exp(tv))} & G \times G & \xrightarrow{(\exp(sv), \exp(tv)) \mapsto \exp(sv) \cdot \exp(tv)} & G
 \end{array}$$

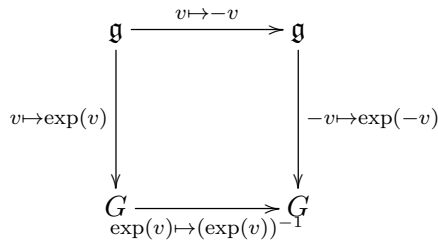
Note that for $v, w \in \mathfrak{g}$, it is not generally the case that $\exp(v+w) = \exp(v) \cdot \exp(w)$ (an additional hypothesis in v and w is required to make this true). But note the difference between $\exp(v+w)$ and $\exp((s+t)v)$: in the first expression, addition is in \mathfrak{g} while in the second expression the addition is in k .

5. The map $\exp : \mathfrak{g} \rightarrow G$ can be shown to satisfy the property

$$\exp(-v) = (\exp(v))^{-1},$$

where the inverse on the right side is the inverse element of $\exp(v)$ under the operation in G .

This can also be expressed with a commutative diagram:



6. There is an open subgroup $G^* \leq G$ for which the map $\exp : \mathfrak{g} \rightarrow G$ is invertible with inverse map $\ln : G^* \rightarrow \mathfrak{g}$. Since $\exp(0) = I$, it follows that $\ln(I) = 0$. For $G = \mathbb{R} \setminus \{0\}$ we have $G^* = (0, \infty)$, as we would expect.

The image in Figure 5 illustrates the relationship between a Lie group, its Lie algebra, and the exponentiation map. The Lie group in the figure is the circle \mathbb{S}^1 with rotation as the operation, choosing for the identity the point $(1, 0)$ (using the usual embedding $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2$). The Lie algebra in this case can be viewed as a copy of \mathbb{R} embedded into \mathbb{R}^2 as tangent to the circle at our identity point. The \exp map is $\exp(x) = e^{ix}$ and the \ln map is a branch of the argument.

Within the setting of Lie groups and Lie algebras, let us try to do what we couldn't do in the category of rings: extend repeated "multiplication". Here, multiplication refers to the operation \cdot in G . Since \cdot is associative, we can simply examine $b \cdot b \cdots b$, where we perform the operation n times.⁷ We can denote this as b^n . Let us call the result of repeated multiplication a **power**, so we can use our usual description of b^n as the n th power of b (and for simplicity call n the exponent). We will restrict the term **exponentiation** to the use of the \exp map.

⁷ Note that the underlying ring structure of k means $n \in k$ and satisfies $n = 1+1+\cdots+1$. Note also that not every element in the field k has this form.

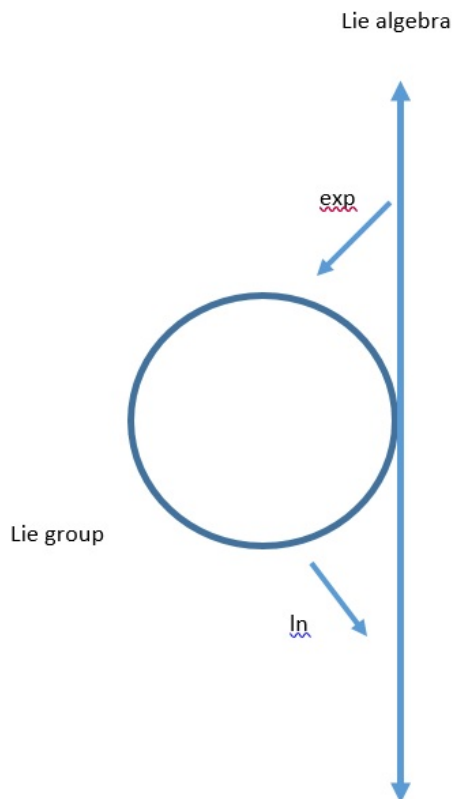


Figure 5: A Lie group, its Lie algebra, the exp map, and the ln map.

If we want to extend repeated multiplication to something resembling exponentiation or some other “operation”, we should ask whether there is a relationship between powers and the exp map that can extend the definition of b^n .

We start with relating powers to exponents. Since exp requires elements in \mathfrak{g} , let us restrict attention to $b \in G^*$. This means we have $\ln b \in \mathfrak{g}$ and we can use exp. Note that $b = \exp(\ln b)$. Observe that:

$$b^n = b \cdot b \cdots b = \exp(\ln b) \cdot \exp(\ln b) \cdots \exp(\ln b).$$

To further simplify, we note that for $s, t \in k$ and $v \in \mathfrak{g}$:

$$\begin{aligned} \exp(n \ln b) &= \exp(\ln b + \ln b + \cdots + \ln b) \\ &= \exp((1 + 1 + \cdots + 1) \ln b) \\ &= \exp(\ln b) \cdot \exp(\ln b) \cdots \exp(\ln b) \end{aligned}$$

since $\exp((s + t)v) = \exp(sv) \cdot \exp(tv)$ (and by associativity we can extend this to any sum of scalars in k). It now follows that

$$b^n = \exp(n \ln b),$$

giving us a connection between powers (as repeated multiplication) and exponentiation.

The relationship we identified can be described by saying that powers (of elements in G^*) factor through exponentiation. Notice that the exponents in powers come from the field k . This means that we can extend b^n , repeated multiplication, to any exponent $x \in k$. Thus, powers are maps $G^* \times k \rightarrow G$ defined by $b^x = \exp(x \ln b)$. This can be made into a diagram:

$$G^* \times k \xrightarrow{(b,x) \mapsto x \ln(b)} \mathfrak{g} \xrightarrow{x \ln(b) \mapsto \exp(x \ln(b))} G$$

Given these observations, we can say that neither powers nor exponentiation should be considered a binary operation. As we have seen, power is a map $G^* \times k \rightarrow G$ while exponentiation is a map $\mathfrak{g} \rightarrow G$. Powers can't be binary operations since G^* , k , and G are all different. Exponentiation can't be a binary operation since \mathfrak{g} and G aren't the same and the domain doesn't consist of pairs. Thus, questions about identities, commutativity, and associativity do not make sense. For example, we mentioned above that we could not define a left identity, a base b for which $b^x = x$ for all x . That makes sense now, since b^x lives in G while x lives in k , hence they cannot be equal. The reason it is so tempting to want to view powers or exponentiation as binary operations is because for the Lie group $G = \mathbb{R} \setminus \{0\}$, the field k is also \mathbb{R} , and the Lie algebra $\mathfrak{g} \cong \mathbb{R}$, so it is easy to confuse the field on which our Lie group is modeled (and over which our Lie algebra is a vector space) with the underlying Lie group itself.

After all of this, what we have learned is that exponentiation is a more subtle process than we thought. In particular, it isn't a binary operation at all, so it is qualitatively different from addition and multiplication.

Indeed, it wasn't lost on us that once we passed from rings to Lie groups, we only had one operation (which we called multiplication). Our "general inversion framework" was more or less lost at that point.

Although we did not succeed in our quest to generalize the relationship between objects and inverses using our hypothesized "general inversion framework", we gained a deeper insight concerning intrinsic properties of exponents using Lie groups and algebras. In our exploration, we generalized repeated multiplication in a way we found to be quite beautiful. However, as Devlin notes in his posts, (c.f. [6]), we also saw that this can sometimes be a limited and limiting metaphor in certain ways.

4. Concluding thoughts

Our students typically think of algebra as a bunch of unrelated procedures and rules that they have to memorize and use to solve problems on an exam. Some mathematicians may think of algebra and developmental mathematics as so basic that there is no value in exploring it deeply.

When we learned algebra ourselves, of course, we were not sufficiently sophisticated in our mathematical thinking to explore these fundamental concepts in depth. By the time we have sufficient sophistication, many of us lack the motivation to go back and explore these foundational issues, or perhaps it does not occur to us to do so. However, as well-trained mathematicians teaching developmental mathematics courses, we took the opportunity to revisit such "basic" topics to provide a better learning experience for our students. In exploring areas where students are often confused, have questions, or have made consistent errors, we have helped support our students' learning and found areas that have stimulated our own intellect and kept us engaged and interested in these foundation topics (beyond our intrinsic commitment to developmental students).

We started our intellectual journey with some simple questions about signed numbers. Our geometrically-inspired response led us to observe a pattern relating operations, inverses, and -1 within a framework of a hierarchy of operations. In attempting to generalize this pattern to the category of rings, we found some problems with exponentiation. These problems led us to examine a new category: Lie groups and Lie algebras.

Our investigation in this new context led us to conclude that exponentiation should not be considered a binary operation at all. The reason it was tempting to think about exponentiation as a binary operation was due to the specific example of the Lie group $\mathbb{R} \setminus \{0\}$ whose Lie algebra is identified with \mathbb{R} .

As with any line of inquiry, there were other directions we could have gone. One concrete example we looked at without much progress was functions (of a real variable). For functions, we have the operation of composition with identity element $f(x) = x$. For one-to-one functions, the inverse function under composition has a graph that is a reflection over the graph of the identify function.

There are also other student questions that are related to our investigation here which we didn't address. For example, students may ask why, when we solve an equation involving an exponential function, do we use a root to solve for the base but a logarithm to solve for the exponent. Roots and logarithms appear vastly different to students in developmental mathematics. We expect this has to do with the distinction made between powers and exponents in the discussion above. In the case of real numbers, this relationship can be observed by noting that any base $b \in (0, \infty)$ can be written as $b = e^r$ for suitable r , and therefore $b^x = e^{rx}$. Hence solving for the base means solving for r and solving for the exponent is solving for x . We had trouble describing this in the context of inverses and as such it didn't fit into the discussion of our inquiry.

One impact that our work has had on us is that we have become more empathetic teachers. For one thing, we have learned to identify our own basic assumptions. In this case, we had always assumed exponentiation was a binary operation. What underlying assumptions persist in our teaching? What do we assume about our students? About ourselves as instructors? By identifying and scrutinizing our assumptions, we can learn about our students' struggles and how they vary from one individual to another. We have also learned more about failure. The "negative result" which we arrived at is actually instructive, and the same happens when our students learn. Finally, and more specifically, we have learned that exponentiation is very subtle, which helps us understand why developmental students struggle with exponents, roots, and logarithms.

The value of this paper is the proposition that we should be motivated to look beneath the hood of developmental mathematics. Kicking the tires and investigating basic algebra with a more sophisticated eye uncovers deeper structures waiting to be explored. The ideas described above serve as an example of how we can be intellectually engaged in teaching this subject while developing lessons based on students' curiosities. For instructors and mathematicians, we have found beauty in the mundane.

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