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Incorporating Philosophy, Theology, and the History of Mathematics in an Introduction to Proof Course

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Synopsis

In this article I describe a project activity for an undergraduate introduction to proof course aimed at mathematics and computer science majors that combines logic and philosophy with a significant dimension of writing. Pedagogically, the project involves a broader range of critical thinking skills than is usual in such courses. Undergraduate students analyze Anselm of Canterbury's and Kurt Gödel's proofs of the existence of God using modal logic.

> *I write because I don't know what I think until I read what I say.*
> -Flannery O'Connor

1. Introduction

In this article I describe a pedagogical initiative I undertook in an undergraduate introduction to mathematical proof course during the spring 2017 semester. Symbolic logic is a crucial topic in such courses, and in my course I help students connect it to ontological arguments in philosophy and theology.

I had three aims in undertaking this initiative. The first aim was to expose students to a subject that reinforces the common origins of mathematics and computer science. My second aim was to excite students about logic and the analysis of mathematical proofs by incorporating a novel application not typically included in these kinds of courses. Indeed, locating resources in the textbook literature[1] that give an explication of Gödel's ontological proof at the undergraduate level and in English seem few and far between. But this very dearth of material also affords the opportunity of immersing students in a more meaningful research enterprise than simply relying on Google or Wikipedia. My third goal was to provide an on-ramp for mathematicians who are non-experts in modal logic but who would enjoy learning about it in a historically interesting context.

The reader might wonder why I chose to focus on modal logic. One reason is that modal logic provides an interesting extension of propositional and predicate logic; indeed it can be used to excite both mathematics and computer science majors. The history of computer science and information technology may be broadly construed as part of the history of mathematics, and logic is a way of impressing on students the common origins of mathematics and computer science. In a recent article [5], Chris Dixon describes how the history of computers is best understood as the history of ideas that had its incarnation as mathematical logic and that had its ultimate origins in the logic of Aristotle, (384-322 BCE).

But a more important reason is the fact that Kurt Gödel devised an ontological proof of the existence of God. Gödel's towering stature in the history of mathematics makes his proof captivating in its own right.

This article describes the logistics of the project and provides an exposition of both Anselm of Canterbury's[2] and Gödel's ontological proofs accessible to undergraduates. The logic is completely self-contained and written for nonspecialists. Thus, I hope this work will be useful information for other instructors wanting to explore the same topic with their classes.

2. Background

Math 180: *Foundations of the Language of Mathematics* is a lower division transitions/introduction to proof course usually taken in the second or third but sometimes the fourth semester. It is required for all students in our Applied Mathematics and Computer Science majors at the University of Wisconsin-Stout. During the spring 2017 semester there were 28 students

[1] There are papers in the journal literature.
[2] Anselm inaugurated the tradition of ontological arguments.

enrolled. The class prerequisite is one semester of calculus. Topics covered include the usual fare of logic and proofs (including both propositional and first-order), set theory, relations and partitions, functions, and cardinality.

There was a good number of both mathematics and computer science students in the class. Very often these two populations seem to be of different academic cultures. Though there are of course exceptions, it could be argued that computer programming[3] and mathematical proof have different epistemological[4] traditions. For a mathematical proof to be correct it must satisfy a strict canon of deductive logic. For a computer program to be correct, merely testing that it produce correct output for a finite set of input test data may be used in lieu of theoretical proofs of program correctness.

In developing the project described in this article, I had in mind a number of desiderata. The project should be both interesting and novel. It should make interdisciplinary connections. It should broach applications of logic. It should increase appreciation for the common origins of mathematics and computer science. Finally, it should sharpen students' critical thinking, writing, and research skills. I believe the project described here mostly lived up to my expectations. Certainly it was novel. The logical analysis of ontological arguments might be more commonplace in philosophy courses. To my knowledge it is nonexistent in the mathematics curriculum.

3. Pedagogy

In this section I describe how I implemented the project. There were four major pillars. Students had to learn LaTeX, learn modal logic, learn Anselm's and Gödel's ontological proofs, and write a research paper.

3.1. Logistics

During the first weeks, students were introduced to LaTeX by way of the free online platform *Overleaf*.[5] This involved one to two class sessions.

[3] Computer science and computer programming of course are not the same thing. I will not belabor this point however.

[4] Ways of knowing.

[5] *Overleaf, Online LaTeX Editor*, available at https://www.overleaf.com, last accessed on January 29, 2019.

Overleaf contains an excellent online text editor that enables compilation of files without having to download and locally install a version of LaTeX. *Overleaf* also contains a generous collection of detailed LaTeX templates. Students could choose either a research paper template or a book template. Most students were not familiar with LaTeX or any other variant of TeX. This was not a serious barrier as the mathematical symbols required (such as $\rightarrow, \Box, \diamond, \vee, \wedge$ etc.) constitute a very small subset of TeX's typesetting capabilities.

As students worked on developing TeX skills, they were given an overview of how to write a research paper using library databases such as EBSCO and MathSciNet, tracking down references in papers, talking to people, internet research using Google, Google-Scholar, Google-Books, JSTOR as well as foreign language Google and Wikipedia domains such as Google-Germany and Wikipedia Germany[6]. Students were also introduced to some of the important web resources in philosophy such as the *Stanford Encyclopedia of Philosophy*, written by and for professional philosophers, and the *Internet Encyclopedia of Philosophy*, considered a little less advanced in some circles.

Although each student was required to complete an individual research paper, in class we approached the project as a group effort; thus the instructor would model how to "find things out" and a mediator would scaffold their processing of the material uncovered. Although verbal versions of Gödel's ontological argument are given in a number of places online, students needed to find a precise modal logic formulation, along with proofs. We eventually settled on Andrè Fuhrmann's approach given in [6]. A description of the nuts and bolts of Fuhrmann's formulation of Gödel's proof is given in Section 7 below.

3.2. Writing Across the Curriculum

In the preface of his celebrated 1789 treatise *The Elements of Chemistry*, Antoine Laurent Lavoisier once wrote:

> "We think only through the medium of words. Languages are true analytical methods. Algebra, which is adapted to its purpose in every species of expression, in the most simple, most exact, and best manner possible, is at the same time a language and an

[6] Google USA generally outperforms Google Germany, but the latter will occasionally list more German language hits first. Foreign language Wikipedia pages are not always mere translations and can be significantly different.

> analytical method. The art of reasoning is nothing more than a language well arranged."

Lavoisier was not the first to make this observation. Indeed, the idea that language is the vehicle for human thought and reason was observed even in ancient times. The Greeks called this *logos*.[7] Learning to read, write and understand mathematical proofs involves firstly the very important step of learning the language of mathematics and its conventions. This creates a natural nexus and opportunity to make connections between a first proofs course and writing across the curriculum efforts. For some recent articles on writing across the curriculum in the context of collegiate mathematics, I recommend the special issue of PRIMUS [9] as well as the article [4].

3.3. Parameters

The format requirements given to the class were as follows. Note in particular that the final bullet point emphasizes that the writing is intended as an individual project and not a group project.

The paper must be written in LaTeX and contain at least the following parts:

- *An introduction that gives the reader a preview of what lies ahead.*
- *A section that introduces and describes the basic ideas of modal logic. (Including what we covered in class would be sufficient.)*
- *A section on Anselm's ontological argument. This should include a bit of history, who Anselm was, as well as the intellectual backdrop of the medieval scholastics. It should include both a verbal description of his ontological argument and a rendering of it in modal logic. (You may base this on Suber's treatment but do not just copy and paste.)*
- *A section on Gödel's modal logic ontological argument. As with the section on Anselm, this should include a bit of history, who Kurt Gödel was and a statement, with your own commentary, of Gödel's argument.*
- *A section in which you explain each logical step by filling in gaps (if any) in the proof.*

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[7] The Greek noun for "word".

- *A conclusion that briefly summarizes and sums up the most significant parts of the paper.*
- *A professionally written bibliography.*

The technical requirements were as follows:

- *The paper should be professionally written in LaTeX.*
- *Then paper should be about ten single-spaced pages long or more. However conciseness can sometimes be appropriate and empty verbiage should be avoided.*
- *An appointment should be made to submit a rough draft to the instructor at least three weeks before the due date. This is to provide a check on everyone's progress and support if needed.*[8]
- *When finished, both the LaTeX source file and a pdf of the paper should be submitted electronically.*
- **Warning:** *You must do your own work. Absolutely no plagiarism is permitted. Handing in something which is verbatim the same as someone else's will result in both people receiving a grade of zero on the project. Likewise, copying and pasting any web pages in your paper will be considered plagiarism and result in a zero. You may discuss the project with others but absolutely no plagiarism is allowed. All sources should be properly attributed and cited in the bibliography.*

4. Modal Logic

Modal logic is an extension of propositional logic and first-order logic. In addition to the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$ and quantifiers $\forall$, $\exists$, modal logic adds two unary operators. $\Box$ is a unary connective that models the ontological idea of necessity. $\Diamond$ is a unary connective that models the ontological idea of contingency or possibility.[9]

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[8] This milestone was checked by email.

[9] There is a distinction between contingency and possibility, but I choose to deemphasize this.

For example,

$$\Box p = \text{ p is necessarily true}$$

$$\Diamond p = \text{p is possibly true}$$

$$p = \text{p is actually true}$$

Conceptually speaking, we may understand necessity and possibility using a "many worlds" interpretation, a notion that had its origins in the philosophy of G.W. Leibniz[10], later formalized by Saul Kripke and others. Something is necessarily true if it is true in every possible world.[11] Something is possibly true if it is true in at least one possible world. Sometimes the term "contingent" is used as well. Something is contingently true if it is true in some worlds but false in other worlds. It is also worth noting that from this perspective $\Box$, "for all worlds", is a modal counterpart of $\forall$, "for all", and $\Diamond$, "in at least one world", is a modal counterpart of $\exists$, "there exists at least one". A standard and quite accessible textbook on modal logic is [2].

To put this into context, here is an illustrative example. Jerry weighs 174 pounds. If 174 pounds is greater than 150 pounds then Jerry's weight is greater than 150 pounds. So, in the actual world Jerry's weight is greater than 150 pounds. It is necessary that 175 pounds is greater than 150 pounds (given that we don't equivocate on the meaning of "pound"). However, it is not true that it is necessary that if 174 pounds is greater than 150 pounds then Jerry's weight is greater than 150 pounds. This is because there are possible worlds in which Jerry's weight is less than 150 pounds. Even though P is necessary and P implies Q in the actual world, the implication relationship between P and Q need not be necessary, i.e., hold in all possible worlds.

4.1. Axioms of Modal Logic

Different systems have different axioms. We employ the system known as S5. This makes use of the following axioms. Here p and q are propositions.

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[10] Leibniz, *Theodicy*, 1710.

[11] More precisely, this is defined in terms of *accessibility relations*, the set of all worlds *accessible* from the actual world.

$$\Diamond p \longleftrightarrow \neg\Box\neg p \tag{1}$$
$$\Box(p \longrightarrow q) \longrightarrow (\Box p \longrightarrow \Box q) \tag{2}$$
$$\Box p \rightarrow p \tag{3}$$

After replacing p by $\neg p$ in axiom (1), and negating, we arrive at the companion statement

$$\Box p \longleftrightarrow \neg\Diamond\neg p \tag{4}$$

that we'll also refer to as an axiom.

Modal Modus Tollens:

$$(p \longrightarrow q) \longrightarrow (\Box\neg q \longrightarrow \Box\neg p) \tag{5}$$

Becker's Axioms:[12]

$$\Box p \longrightarrow \Box\Box p \tag{6}$$
$$\Diamond p \longrightarrow \Box\Diamond p \tag{7}$$

5. Ontological Proofs

In philosophy, metaphysics is the study of the nature of reality. Ontology is a subfield of metaphysics that is concerned with the study of *being* or *existence*. A classical example of an ontological question is whether God exists. An *ontological proof* is an argument for the existence of God based on logic alone. One can approach an ontological proof from a philosophical / theological stance; that is, what is its intellectual merit as an ontological argument? But we can also approach an ontological proof from a mathematical stance; that is, what is the logical structure of the argument? Is the argument valid? Is the argument sound? While the idea of a philosophical proof of the existence

[12] Named for Oscar Becker, 1889-1964.

of God from logic alone seems most compelling, my main interest for this course was in the logical structure of the argument. As means of reassuring students this was not an attempt to indoctrinate them with any particular religious commitment, I made an effort to discuss some of the weaknesses of such arguments by pointing out some of the weaknesses that trace back to their origins in medieval scholastic thought.

The medieval philosophers and theologians known as scholastics had an almost fetishistic preoccupation with the notion of the perfect or perfection. This idea had its roots in the earlier teachings of Augustine of Hippo (354-430 CE) and their antecedents, Neoplatonism, and Platonism itself.[13] After the works of Aristotle were reintroduced to western Europe, philosophical inquiry took a renewed interest in the material world. But the scientific method, as we understand it today, was centuries away. Scholastics adopted the Aristotelian emphasis on derivation from first principles, but where Aristotle's first principles came from philosophy, the scholastics substituted the Bible. Scholastics became associated with an overconfidence in the power of human reason, combined with appeal to scripture. A colorful, if not comical description of this was Francis Bacon's recounting[14] of scholastics attempting to determine the number of teeth in a horse's mouth through reason and scriptural reference alone.

> *In the year of our Lord 1432, there arose a grievous quarrel among the brethren over the number of teeth in the mouth of a horse. For thirteen days the disputation raged without ceasing. All the ancient books and chronicles were fetched out, and wonderful and ponderous erudition such as was never before heard of in this region was made manifest. At the beginning of the fourteenth day, a youthful friar of goodly bearing asked his learned superiors for permission to add a word, and straightway, to the wonderment of the disputants, whose deep wisdom he sore vexed, he beseeched them to unbend in a manner coarse and unheard-of and to look in*

[13] Plato's philosophy emphasized perfect forms or ideas; Neoplatonism was a late antiquity resurgence of Platonism which influenced early Christian theology.

[14] Likely an apocryphal caricature of the ethos of the time, generally believed to be misattributed to Francis Bacon, its original source is obscure but appeared as early as 1901 in the rather strange book [12] published by Charles Carrington, a publisher of erotica.

> *the open mouth of a horse and find answer to their questionings. At this, their dignity being grievously hurt, they waxed exceeding wroth; and, joining in a mighty uproar, they flew upon him and smote him, hip and thigh, and cast him out forthwith. For, said they, surely Satan hath tempted this bold neophyte to declare unholy and unheard-of ways of finding truth, contrary to all the teachings of the fathers. After many days more of grievous strife, the dove of peace sat on the assembly, and they as one man declaring the problem to be an everlasting mystery because of a grievous dearth of historical and theological evidence thereof, so ordered the same writ down.*

My point of sharing this story was not to discredit the intellectual merit of ontological arguments but rather to impress on students that our interest would lie in the logical structure of the argument and not on any religious dogma. It was against this intellectual backdrop that Saint Anselm of Canterbury (1033-1109), in his famous work, the *Proslogion* [1], gave the first ontological proof[15] for the existence of God. Although there have been numerous interpretations and formulations of Anselm's argument, the gist of it is that the idea of perfection implies the existence of perfection. And perfection is identified with God. [16]

6. Anselm's Ontological Proof

Presenting Anselm's Ontological proof before Gödel's ontological proof provided students with a scaffolded introduction to Gödel's more advanced argument. To get a sense of how complicated Gödel's ontological proof is for students, see the student comments in Section 9.1 below.

Students were first introduced to Hartshorne's modal logic formulation [8] of Anselm's argument, restated by Peter Suber [11]. This was helpful because there are some similarities between Gödel's argument and that of Anselm, and also because Anselm's argument is simpler.

[15] The term "ontological" was first used by Kant in his *Critique of Pure Reason*.

[16] The self-referential character of the proof has long been a source of uneasiness by critics, especially those familiar with Russell's paradox or the Gödel Incompleteness Theorem.

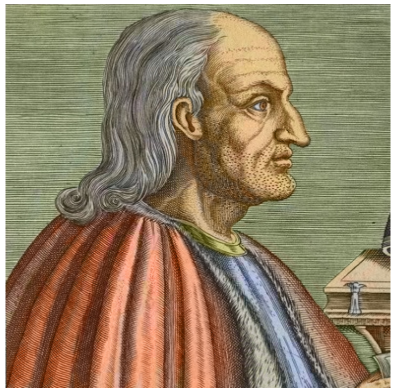

Figure 1: Anselm of Canterbury. Public domain image from `Wikimedia Commons`.

In the argument below, $P(x)$ is the property that "x is perfect". Hence $(\exists x)P(x)$ represents "something perfect exists". The symbol "p" represents an arbitrary proposition. Some of the steps in the proof apply to arbitrary propositions p while others apply to the specific proposition $(\exists x)P(x)$.

Anselm Axiom: *Perfection cannot exist contingently.*

$$(\exists x)P(x) \to \Box(\exists x)P(x) \tag{8}$$

Anselm Axiom : Perfection is possible (expressed as being not impossible).

$$\neg\Box\neg(\exists x)P(x) \tag{9}$$

Modal Axiom (3):

$$\Box p \longrightarrow p \tag{10}$$

Excluded Middle:

$$\Box p \vee \neg\Box p \tag{11}$$

Becker's Postulate:

$$\neg\Box p \longrightarrow \Box\neg\Box p \tag{12}$$

This is actually (7) applied to $\neg\Box p$.

Inference Step:

$$\Box p \vee \Box\neg\Box p \tag{13}$$

This follows from (11) and (12).

Inference Step:

$$\Box\neg\Box(\exists x)P(x) \longrightarrow \Box\neg(\exists x)P(x) \tag{14}$$

This follows from (8) and (5).
Inference Step:

$$\Box(\exists x)P(x) \vee \Box\neg(\exists x)P(x) \tag{15}$$

This follows from (13) and (14).
Inference Step:

$$\Box(\exists x)P(x) \tag{16}$$

This follows from (9) and (15) and disjunctive syllogism.
Conclusion that God Exists:

$$(\exists x)P(x)$$

which follows from (16) and (10) and modus ponens.

7. Gödel's Ontological Proof

Kurt Gödel (1906-1978) is well known for his celebrated incompleteness theorem. He first composed an ontological argument around 1941 while still in

Figure 2: Kurt Gödel, as a student in Vienna. Public domain image from https://en.wikiquote.org/.

his thirties. He did not not make this public until 1970 when he discussed it with Dana Scott and Oskar Morgenstern at a time he though he might be dying. Gödel never published it during his life time, though Dana Scott presented it in a seminar at Princeton in 1970. Gödel's interest was not theological but rather in the logical structure of such an argument. Although Anselm's ontological proof was presented because of its relative simplicity, other ontological proofs had been given by both Descartes and Leibnitz and Gödel's proof is most closely related to Leibnitz's.

The general outline and reconstructions of Gödel's ontological proof appears in several places in the published literature, for example, in the essays by Jordan Howard Sobel [10] and Robert Merrihew Adams [7]. Both give a lot of interesting history, context and further references. The class made use of the reconstruction of the proof given by André Fuhrmann in [6]. This was because Fuhrmann's reconstruction uses Becker's axioms in an explicit way that the students will have had some exposure to in their preliminary study of Anselm's proof above. The account below is largely the same except for some minor notational changes. At the time of this writing, there appear to be very few expositions of the formalized modal logic version of Gödel's ontological proof accessible to undergraduates. In what follows, the uppercase letters (e.g. X) are typically properties and the lower case letters (e.g. x) are typically individuals.

Axiom 1: *Every property is either positive or negative.* More formally,

$$(\forall X) P(\neg X) \longrightarrow \neg P(X)$$
$$(\forall X) \neg P(X) \longrightarrow P(\neg X)$$

This is a kind of excluded middle law for the condition of being positive. Either X or $\neg X$ must be positive, but not both.

Axiom 2: *Properties necessarily implied by positive properties are themselves positive.* More formally:

$$(\forall X)(\forall Y)\,[(P(X) \wedge ((\Box(\forall x)(X(x) \longrightarrow Y(x))) \longrightarrow P(Y))]$$

Definition 1: *An individual x is said to be God-like, $G(x)$, if it possesses all and only positive properties.* More formally,

$$(\forall x)\,[G(x) \longleftrightarrow (\forall X)(P(X) \longrightarrow X(x))]$$

Axiom 3: *God-likeness is a positive property.* That is,

$$P(G)$$

Theorem 1: *Positive Properties are consistent.* To wit,

$$(\forall X)(P(X) \longrightarrow \Diamond(\exists x)X(x))$$

The proof is by contradiction. Assume the contrary that X is a positive property, $P(X)$, but $\neg\Diamond(\exists x)X(x)$. By modal axiom (1) we rewrite this as $\Box\neg(\exists x)X(x)$. By quantifier duality[17] this gives $\Box\forall x\neg X(x)$. Let $Q_x(y)$ denote the relation " y is identical to x". To more conveniently express the sought after contradiction, note that the statement

$$\Box\forall x(X(x) \longrightarrow \neg Q_x(x))$$

follows from our assumption X is not consistent. Now, Gödel axiom 2 states $P(X) \wedge \Box\forall x(X(x) \longrightarrow Y(x)) \longrightarrow P(Y)$. Replacing Y by $\neg Q_x$ in this gives, of course, $P(X)\wedge\Box\forall x(X(x) \longrightarrow \neg Q_x(x)) \longrightarrow P(\neg Q_x)$. Another application of modus ponens now gives $P(\neg Q_x)$. By Gödel axiom 1, $P(\neg X) \longrightarrow \neg P(X)$, and after a substitution and modus ponens yields $\neg P(Q_x)$. Next consider the tautology $\Box\forall x(X(x) \longrightarrow Q_x(x))$. Substituting Q_x for Y in Godel axiom 2 finally gives $P(X) \wedge \Box\forall x(X(x) \longrightarrow Q_x(x)) \longrightarrow P(Q_x)$ and using modus ponens we conclude $P(Q_x)$, a contradiction.

Corollary 1: *It is possible that a God-like being exists.* That is,

$$\diamond(\exists x)G(x)$$

The proof is immediate. By Gödel axiom 3, G is a positive property so by Theorem 1, is consistent.[18]

Axiom 4: *Positive properties are necessarily positive.* More formally,

$$(\forall X)\,[P(X) \longrightarrow \Box P(X)]$$

Definition 2: *A property X is an essential property of an individual x if all other properties of x necessarily follow from X.*

[17] $\neg\exists \equiv \forall\neg$, i.e. De Morgan's law in predicate logic.
[18] Impossibility of existence would be inconsistent.

To formalize "X is an essential property of x", a relation Ess(X,x), whose first argument is a property and whose second argument is an individual, is introduced. That is, Ess is a property of the pair (X,x).

$$(\forall x)(\forall X)\,[Ess(X,x) \longleftrightarrow X(x) \wedge (\forall Y)(Y(x) \longrightarrow \Box(\forall y)(X(y) \longrightarrow Y(y)))]$$

Theorem 2: *God-likeness is an essential property of any and all individuals.* More formally:

$$(\forall x)[G(x) \longrightarrow Ess(G,x)]$$

To prove this assume $G(x)$ and $Y(x)$. By Definition 2, it is sufficient to show $\Box(\forall y)(G(y) \longrightarrow Y(y))$. Now Y will be either a positive or negative property. Suppose Y not positive, i.e. $\neg P(Y)$. By Gödel axiom 1, $P(\neg Y)$. Thus by definition 1, $\neg Y(x)$. But that would contradict our assumption that $Y(x)$. On the other hand if Y is positive, $P(Y)$, then by Gödel axiom 4, $\Box P(Y)$. Now by definition 1, $P(Y) \longrightarrow (\forall y)(G(y) \longrightarrow Y(y))$. Moreover, if y is God-like, it is so in every possible world. If Y is positive, it is so in every possible world. Therefore, the conditional is true in every possible world[19] and thus

$$\Box(P(Y) \longrightarrow (\forall y)(G(y) \longrightarrow Y(y))).$$

By modal axiom (2) , this implies

$$\Box P(Y) \longrightarrow \Box(\forall y)(G(y) \longrightarrow Y(y)).$$

Since $\Box P(Y)$, by modus ponens

$$\Box(\forall y)(G(y) \longrightarrow Y(y))$$

which completes the proof.

Corollary 2: *If two individuals are God-like, then they are necessarily identical.*

$$(\forall x)[G(x) \longrightarrow \Box(\forall y)(G(y) \longrightarrow (x = y))]$$

This is an assertion of "monotheism". If there are two God-like individuals, they must be one and the same.

[19] This is called strong implication, sometimes written as $\supset$.

Proof: Suppose $G(x)$. Set $X = G$ and $Y = Q_x$ in definition 2. This follows from theorem 2 because G is an essential property.

Definition 3: *An individual necessarily exists, E(x), if all its essential properties hold true necessarily.* More formally,

$$(\forall x)\,[E(x) \longleftrightarrow (\forall X)(Ess(X,x) \longrightarrow \Box(\exists y)X(y))]$$

Notice that the $\Box$ symbol doesn't appear because it is already "contained" in E.

Axiom 5: *Necessary existence is a positive property.*

$$P(E)$$

Theorem 3: (Anselm's principle) *If the existence of a God-like individual is possible, then it is necessary.*

$$\Diamond(\exists x)G(x) \longrightarrow \Box(\exists x)G(x)$$

I refer the reader to the Hartshorne-Suber proof given above.[20]

Corollary 3: *It is necessary that at least one God-like individual exists and if there are more, they are all identical.*

$$\Box(\exists x)(G(x) \wedge [(\forall y)(G(y) \longrightarrow (x = y))])$$

Proof: This follows from Anselm's principle and Corollary 1 above.

8. Ways of Integrating and Using Modal Logic in a Course

Here I list a few ways instructors might be able to use modal logic in the context of Gödel's ontological proof. Many of these ideas will also be applicable to the modal logic version of Anselm's ontological argument:

- Modal logic in this context can be used to enhance the development of proof construction skills. Students can be asked to fill in details of proofs or supply proofs themselves.

[20] Actually, I am cheating here a bit in the interest of brevity. An important difference between Anselm and Gödel's proofs is that this is an assumption in Anselm but a theorem, which Gödel proves in his argument. For the details see [6].

- It can be used as an add-on to propositional logic and first-order predicate logic for honors courses.
- Gödel's ontological proof gives a nice example and a natural on-ramp into second-order logic.
- Because of its relative novelty in undergraduate literature, this modal logic path to Gödel's proof offers a novel venue in which to push student originality as they cannot model their work on others.
- It provides a venue for teaching mathematical writing and writing across the curriculum.
- It gives a nice context, involving only a small set of math symbols, for teaching undergraduates skills like LaTeX.
- It offers a way of connecting mathematical logic as it is taught in a math course with philosophy courses in logic and presents an opportunity for team teaching.
- It affords a way of including humanistic themes like philosophy with mathematics and provides opportunities to engage with some celebrated figures in the history of mathematics, such as Kurt Gödel, Gottfried Leibnitz, René Descartes and Anselm of Canterbury.
- It furnishes material that can help bridge the gulf that sometimes exists in the perceptions and attitudes of mathematics and computer science (or other) majors by providing a common historical link.
- In our case it also provided an opportunity to show students the value of knowing a foreign language and that there are valuable things published in languages other than English.

9. Student Outcomes

Students gained skills in LaTeX, research, written communication, critical thought, and logic. Most of them had not seen LaTeX before and many struggled[21] but felt it was an important skill to acquire for future math classes.

[21] As perhaps many of us have ourselves in learning any of the dialects of TeX.

The research component was the most challenging. Perhaps this is not surprising given that it involved writing about a topic for which there were only fragmented sources of uneven quality accessible at the undergraduate level, namely web pages, not to mention the fact that truly original research in this field may require a graduate-level maturity. Student reactions varied from panic at not being able to find anything to frustration that there wasn't more information available online.

So the class needed some special scaffolding here. The fact that Fuhrmann's article [6] is in German presented a teachable moment. In our often English-centric speaking culture, students learned that there are worthwhile texts that are not in English. This also afforded them the opportunity to see some of the shortcomings of tools like Google-Translate and that such computer translation utilities still do not supplant the importance of learning a foreign language.

Student experience, as gauged by comments was somewhat varied. Some students felt the project was either too complex and tedious or too much busy work. Other students felt the argument was difficult to understand. Still others expressed dismay at the dearth of online resources on the topic. Another sentiment expressed was that the project took us too far afield of and detracted too much from the standard course material. A few students felt that the apparent religious undertones of the topic should be avoided simply because religion tends to be controversial or out of place in a mathematics class.

9.1. Student Perceptions and Feedback

Students were invited to and offered extra credit for submitting their reflections and feedback on the project. While there were no scathing reviews, there were both a number of positive as well as a number of more mixed reviews. The following student comments are a sampling; each is unedited and included in its entirety.

First here are some of the more positive comments:

- I felt this project was one of the most interesting projects I have done in college. This may stem from my interest on the topic as a Christian, but overall I felt it was enjoyable. It helped me to understand modal logic and proofs more strongly. Overall I must say the proofs were very difficult however. The topic was interesting but balancing the work with other classes was quite hard. I can see how it may be difficult and not enjoyable for someone who is not interested in the topic. Thankfully I am very intrigued by the topic.

- Overall, I liked doing this project. This fits as a good overview of the class, combining different types of logic. It was also a very interesting topic. Few professors would even touch something like this as it relates to religion. However, regardless of your religious stance, this project was a very good learning experience, especially for me. I have not written in Latex before and I think this was a good introductory piece on how to. The paper was easy to write to up even little knowledge of Latex writing. This should help you in future classes/jobs. I like doing things that will expand my knowledge on a new subject. I believe anyone with this mindset would enjoy this as well. I really don't have any negative feedback on this project. Again, it was a very good project and hope you continue this or something like it in the future.

- I would like to give you my opinion of the Ontological Argument by Gödel, inspired by St. Anselm of Canterbury. It was hands down the hardest project of my semester (over any class), but the most viable for my growth as an individual and definitely as a Mathematician. I learned the techniques and rules to writing proof based mathematics and some of the intricacies of the relating math to vocabulary. I learned a greater deal of the art of logical fallacies, the concept that something can be perfectly logical, yet completely un-sound. Today where the majority of information we receive from people, news, and other resources are complete false, or skewed to support some agenda. What I learned from this paper was more of a life skill, for that I thank you kindly. To also state: In my opinion any individual whom viewed this project as taboo due to its reference of God; is critically incorrect in their thinking, and severely closed minded to the point of losing their ability to critically think. This project does as much for the theology of religion as it does against it. I would have to assume that anyone that argues that this project is prejudice or "triggering" did nothing of actual work towards the project, and made the greatest mistake of their college career by not being able to step out of the boundaries of their own mind and the inability to attempt perceiving something that is beyond their comprehension, due to their own bias and ignorance. In closing I would like to say that I found this project extremely challenging, yet very intuitive for my development as a critical thinker and scholar.

- In its entirety, I thought the project was a great and interesting addition to the study of mathematical language and its practice. It was nice to see a real-world example where these concepts and ideals were translated to modal logic and related to a debate that has been a part of not only our nations, but of our worlds history as well. It was interesting to research the history of the ontological argument, and how it played a part in the world as its creation. While I thought, the project was a great addition, I think a couple things that would make it more efficient for following years would be to spend some class time going over the history of the language of math. These concepts include Anselm and other historic figures rather than simply leaving it to single handed research. I think supplementing what you currently do, which is a great teaching method, and delving more into the history itself in class, would make this project even better. Ultimately, the ontological argument was fun to research and I think it would be great to continue teaching it to future classes, while maybe also teaching a few other famous arguments in addition.

Here are some of the more mixed reviews:

- Hey sorry for submitting this a little bit later from when I submitted the project. These are my thoughts on the Ontological Proof assignment that was assigned to us as you've asked us to submit to you so we can get those extra credit points. I personally thought it was a pretty cool proof actually and once you explained it in class I thought it was interesting but unfortunately I also think that it was too high of a level proof to try to fully explain accurately on our own. I found myself looking back at my notes and based off what we've learned all semester I really had no idea how to prove that argument. Also learning LaTex was a little rocky but I think it's going to be good skill to have if we'll actually need to know how to use it in the future. For recommendations for the future I think you should explain this Godel proof first in the modal logic unit instead of having us prove it by ourselves, then have the students explain the first modal logic argument we first did as a class this semester (sorry I can't remember what the argument is called). Other than that I thought it was a good experience, like I said just switch the order of the two modal logic arguments you introduced to us and that should make things more understandable.

- I believe that the Gödel paper had very good intentions but not so perfect execution. The topic of Gödel has very few resources and it was very difficult to do the paper without significant guidance. Latex was also new to the majority of students and that was as added element to an already difficult concept. With that said, going through the basic symbols and set up of Latex papers in class was very helpful. Going through the proofs in class also made the project significantly easier and I personally have a much better finished project because of the class time spent on the proofs. The amount of time taken however could have been better spent on the book material because I have often struggled to grasp the concepts in class and could've benefitted from extra time on the concepts in class. In hindsight, I would've preferred a project that used more of the concepts in class, other than just modal logic, as a way to practice the material. The project's concept was very reasonable but due to the lack of information, it went from a research project about Gödel's argument into a research project on Latex and the backgrounds of Gödel and Anselm. In the future, I would suggest revising the project into something that requires less guidance in order to allow for more class time spent on the book material. However, with the project as it is, I feel like you gave the proper amount of guidance to allow students to complete this project.

- From a student standpoint, this project was one of the more tedious and complex projects I have had to do during my school career. One of the main reasons that I found this project difficult was because of the lack of available sources for the project. While there were several different interpretations Anselm's proof, it was hard to find any information of Gödel's proof for a variety of reasons. For one, Gödel was better known for his mathematical proofs and less for his philosophical ideas. As a result, there are much fewer people who have written and published work on him. Second, the main papers that have been published about Gödel's proof are in a foreign language that most of the class does not speak fluently, making translations difficult at time. The biographies I found on Gödel often neglected to mention much about his proof at all, instead focusing on his completeness and incompleteness theorems as some of his best achievements.

One thing I did like about this project was the fact that time was spent working through the proof of the argument in class so that we understood how the argument was being made. This certainly helped me out when I was trying to explain the proof in my paper later. I also liked that we spent time learning latex in class as I had never worked with it before and would have been totally lost trying to format my paper had we not gone over some of the things in class. A recommendation I would have for if you ever did a latex project again would be to provide the students with a variety of resources on where to go to find symbols in latex or answers to other formatting questions, as I found that to be one of the most challenging parts of starting this paper. I did not know any of the symbols or formatting, and so it was quite complex to use. Thankful, Overleaf has a rich text beta editor which made typing the written portion of the paper much easier. This allowed me to learn Latex in a more friendly environment.

Overall, I would recommend that if this project were done again use a subject material that is easier to research and treat the project as more of a chance to teach Latex, as this is one of the main things I took from the project. I think that introducing Latex in this course is excellent if it is used in higher level mathematics as it will be something that we will encounter as students later in our college careers.

- Overall, I thought this project was worthwhile in the fact that I gained a better understanding for different axioms and how to prove stuff. One thing that I felt that didn't work so well was that we had to try to learn Latex on our own. I think it would have been very helpful if you would have went over more about how to create the different symbols and explained how to do what not and importing packages. I felt like how you explained the proofs to the class really helped me understand how to do a proof better, because I still didn't exactly understand how to do a proof until you went over them. A future recommendation that I would give is to pick a different topic. I felt as though explaining a proof about the existence of a god or perfect being was a little out of place in a math class, and there was definitely a more appropriate math related proof that we could have done instead.

- I suppose the best feedback I have is to ensure that this concept is not super overwhelming. At first, I thought this paper would be impossible. Once I got through it, the subject matter made a lot of sense. For example, try to emphasize that Godel's proof boils down to the 6 theorems and corollaries. Whatever you can do to make this project not sound as complicated as it did for me would help future students.

- I found the project interesting. At first it seemed very daunting and didn't seem like I could understand anything about the argument. But after going over some of the proof in class it made more sense. Using LaTex was also interesting but not having much previous use of LaTex made it hard and more stressful when going about the project. I see how its helpful but just to use it to do a main project that's worth a lot of points made it a little more worrying. If I have used LaTex before for something like a homework assigenemnt I could see not having as much as a problem. I also found it hard understanding the logic behind the proof. I could never argue this proof as to me it seems very flawed. This made it a little harder to complete the project as it was kinda painful to have to try and understand then give my own wording of it.

- I believe this project was beneficial in understanding modal logic. I understand and can explain the symbols more in depth than I could before completing the project. On the other hand, I don't think this project was necessary. It felt like busy work to me at times rather than being something to benefit the class overall. In class days talking about the project helped to further my progress on it, but it took away time from the book topics, which seem to be the purpose of the class and was also what I was struggling on. The basis of a project similar to this is a good idea, however I would recommend a different topic (religious topic seems to be controversial) as well as spending more time on how to start a proof and conduct one rather than providing them for us. Overall, I did learn some things from this project, so I hope you don't take this feedback the wrong way.

9.2. Instructor Reflections

In the spirit of the scholarship of teaching and learning, and lesson study, the following observations seem warranted. From student comments, the most concerning issue was the degree to which the project detracted from in-class time spent on homework problems from the textbook and more traditional topics for an introduction to proof class. Part of this was probably due to the fact that I was learning the material myself for the first time along with the students. This can hopefully be remedied in the future through better time management on how the project is introduced and developed. A second concern was student's perceptions of the relevance of the topic in the context of a course of this type. This can hopefully be remedied by placing more emphasis on the separation between theological interest and mathematical interest in the logical structure of the argument.

As mentioned above, there are very few places that exposit the Gödel's ontological proof in either the literature or online.[22] It could be asked whether something like the modal logic version of Gödel's ontological proof is simply too advanced for undergraduate students. However a better question might be whether the *open exposition problem*[23] of rendering the proof accessible to undergraduate is possible. This paper springs in part from the belief that this is possible.

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[22] At the time of this writing; the web is always in flux.

[23] In [3], Timothy Y. Chow has proposed the concept of an open exposition problem. He writes "All mathematicians are familiar with the concept of an open research problem. I propose the less familiar concept of an open exposition problem. Solving an open exposition problem means explaining a mathematical subject in a way that renders it totally perspicuous. Every step should be motivated and clear; ideally, students should feel that they could have arrived at the results themselves. The proofs should be 'natural'..."

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