1-1-2015

Spherical 2-Designs and Lattices from Abelian Groups

Albrecht Böttcher
Technische Universitat Chemnitz

Lenny Fukshansky
Claremont McKenna College

Stephan Ramon Garcia
Pomona College

Hiren Maharaj
Pomona College

Spherical 2-designs and lattices from Abelian groups

Albrecht Böttcher, Lenny Fukshansky,
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Abstract. We consider lattices generated by finite Abelian groups. The main result says that such a lattice is strongly eutactic, which means the normalized minimal vectors of the lattice form a spherical 2-design, if and only if the group is of odd order or if it is a power of the group of order 2. This result also yields a criterion for the appropriately normalized minimal vectors to constitute a uniform normalized tight frame.

1 Introduction and main result

A collection of points $\boldsymbol{y}_1, \dots, \boldsymbol{y}_m$ on the unit sphere $\mathbb{S}_{n-1}$ in $\mathbb{R}^n$ is called a *spherical t-design* for some integer $t \geq 1$ if for every polynomial $f(\boldsymbol{X}) = f(X_1, \dots, X_n)$ with real coefficients of degree $\leq t$ the equality

$$\int_{\mathbb{S}_{n-1}} f(\boldsymbol{X}) \, d\mu(\boldsymbol{X}) = \frac{1}{m} \sum_{i=1}^{m} f(\boldsymbol{y}_i) \tag{1}$$

holds, where μ is the surface measure normalized so that $\mu(\mathbb{S}_{n-1}) = 1$. Spherical designs were introduced in the celebrated 1977 paper [2] of Delsarte, Goethals, and Seidel and have been studied extensively ever since for their remarkable properties and many applications within and outside of mathematics. The strong connection between spherical designs and lattices was first observed by B. B. Venkov. See in particular [9] (also surveyed in [6], Chapter 16) and the nice survey of Venkov's fundamental work on this subject written by Nebe [7].

Let $\Lambda \subset \mathbb{R}^n$ be a lattice of full rank. The *minimal norm* of Λ is defined as

$$|\Lambda| = \min \{ \|\boldsymbol{x}\| : \boldsymbol{x} \in \Lambda \setminus \{\boldsymbol{0}\} \},$$

where $\| \ \|$ denotes the Euclidean norm, and we let

$$S(\Lambda) = \{ \boldsymbol{x} \in \Lambda : \|\boldsymbol{x}\| = |\Lambda| \}$$

MSC 2010: Primary 11H06, 05B30, 52C17; Secondary 11H31, 11H50, 42C15
Keywords: strongly eutactic lattice, spherical 2-design, uniform normalized tight frame
Fukshansky acknowledges support by NSA grant H98230-1510051, Garcia acknowledges support by NSF grant DMS-1265973.

denote the *set of minimal vectors* of Λ. Clearly $S(\Lambda)$ is a symmetric set, and the set

$$S'(\Lambda) := \frac{1}{|\Lambda|} S(\Lambda) = \left\{ \frac{1}{|\Lambda|} \boldsymbol{x} : \boldsymbol{x} \in \Lambda, \|\boldsymbol{x}\| = |\Lambda| \right\}$$

is a finite subset of the unit sphere $\mathbb{S}_{n-1}$. The lattice Λ is called *strongly eutactic* if $S'(\Lambda)$ is a spherical 2-design.

We remark that the original definition of strongly eutactic lattices is different. However, it is equivalent to the one given here, that is, to the property that $S'(\Lambda)$ is a spherical 2-design. We refer to [6] (Section 3.2 and Chapter 16, especially Corollary 16.1.3) and [7] for further information on this.

Here is an an easy to use criterion for spherical 2-designs, which is actually equivalent to Venkov's criterion as in Proposition 16.1.2 and Theorem 16.1.4 of [6] and which can be deduced from a more general result of [3]. We write vectors in $\mathbb{R}^n$ as columns and let $\overline{\mathbf{e}}_i$ denote the i-th standard basis vector in $\mathbb{R}^n$. Then $\overline{\mathbf{e}}_i^\top \boldsymbol{y}$ is just the i-th component of $\boldsymbol{y}$. The criterion is as follows.

Proposition 1.1 (Theorem 4.1 of [3]). *The set of points $S'(\Lambda) = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m\}$ is a spherical 2-design if and only if*

$$\sum_{i=1}^{m} (\overline{\mathbf{e}}_j^\top \boldsymbol{y}_i)(\overline{\mathbf{e}}_k^\top \boldsymbol{y}_i) = \begin{cases} m/n & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

for all $1 \leq j, k \leq n$.

In other words, the lattice Λ is strongly eutactic if and only if the n rows of the $n \times m$ matrix Y constituted by the columns $\sqrt{n/m}\,\boldsymbol{y}_1, \ldots, \sqrt{n/m}\,\boldsymbol{y}_m$ form an orthonormal system in the Euclidean space $\mathbb{R}^m$, or, equivalently, if $YY^\top = I$. Since the surface measure μ is invariant under orthogonal transformations, definition (1) implies that if V is an orthogonal operator, then $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ form a spherical t-design if and and only if so do $V\boldsymbol{y}_1, \ldots, V\boldsymbol{y}_m$. For spherical 2-designs this also follows from Proposition 1.1, because $YY^\top = I$ if and only if $VY(VY)^\top = I$.

Here are two simple examples. Every one-dimensional full-rank lattice is strongly eutactic, because the two points 1 and -1 are a spherical 2-design (even a spherical t-design) in $\mathbb{R}^1$, provided the integral in (1) is interpreted as $f(-1)/2 + f(1)/2$. In this case the matrix Y is the row

$$\frac{1}{\sqrt{2}}(-1\ 1). \tag{2}$$

The hexagonal lattice is strongly eutactic in $\mathbb{R}^2$, since the six vertices of a regular hexagon inscribed in $\mathbb{S}_1$ form a spherical 2-design in $\mathbb{R}^2$. Here the matrix Y is

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1/2 & -1/2 & -1 & -1/2 & 1/2 \\ 0 & \sqrt{3}/2 & \sqrt{3}/2 & 0 & -\sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix}. \tag{3}$$

Strongly eutactic lattices are important in lattice theory due to their central role in discrete optimization problems, especially sphere packing. For instance, A. Schürmann recently proved [8] that all perfect strongly eutactic lattices are periodic extreme, i.e., these lattices cannot be "locally modified" to yield a better periodic packing. In fact, it has been proved by Voronoi around 1900 that a lattice is extreme (i.e., is a local maximum of the packing density function on the space of lattices in a fixed dimension) if and only if it is perfect and *eutactic*, a condition being weaker than strong eutaxy; we refer to the reader to Martinet's book [6] and Nebe's paper [7] for definitions and further information. While many of the standard lattices, such as indecomposable root lattices, are known to be strongly eutactic, a full classification of strongly eutactic lattices is only known in small dimensions. This makes constructions of strongly eutactic lattices in arbitrary dimensions particularly interesting.

In this paper, we revisit the family of lattices generated by finite Abelian groups that we studied previously in [1], showing that many of them are strongly eutactic.

Here is our main result.

Theorem 1.2. *Let $G = \{g_1 := 0, g_2, \dots, g_n\}$ be a finite (additively written) Abelian group of order $n \geq 2$, let L_G be the sublattice of the root lattice*

$$A_{n-1} = \left\{ \boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 0 \right\},$$

which is defined by

$$L_G = \left\{ \boldsymbol{x} = (x_1, \dots, x_n) \in A_{n-1} : \sum_{j=2}^{n} x_j g_j = 0 \right\}, \tag{4}$$

and let $B : \operatorname{span}_{\mathbb{R}} A_{n-1} \to \mathbb{R}^{n-1}$ be an arbitrary linear isometry. Then $B(L_G)$ is strongly eutactic in $\mathbb{R}^{n-1}$ if and only if G has odd order or is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\nu$ for some $\nu \geq 1$.

We finally want to mention the connection between spherical 2-designs and frames. Let $m \geq n$ and let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in \mathbb{R}^n$ be a collection of vectors of norm $\sqrt{n/m}$ such that

$$\mathbb{R}^n = \operatorname{span}_{\mathbb{R}}\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_m\} \text{ and } \sum_{i=1}^{m} |\boldsymbol{x}_i^\top \boldsymbol{y}|^2 = \|\boldsymbol{y}\|^2 \text{ for each } \boldsymbol{y} \in \mathbb{R}^n.$$

Such a collection of vectors is called a *uniform normalized tight (UNT) (m,n)-frame*. It is well-known (see, for instance Proposition 1.2 of [5]) that a finite subset $\boldsymbol{y}_1, \dots, \boldsymbol{y}_m$ of the unit sphere $\mathbb{S}_{n-1}$ in $\mathbb{R}^n$ is a spherical 2-design if and only if $\sqrt{n/m}\ \boldsymbol{y}_1, \dots, \sqrt{n/m}\ \boldsymbol{y}_m$ is a UNT (m,n)-frame and $\sum_{i=1}^{m} \boldsymbol{y}_i = \boldsymbol{0}$. In fact, this observation was actually made earlier by B. B. Venkov [9]: it is a special case of a more general criterion, which works for all t (see Proposition 16.1.2 of [6]). Since sets of minimal vectors of lattices are $\boldsymbol{0}$-symmetric, the condition $\sum_{i=1}^{m} \boldsymbol{y}_i = \boldsymbol{0}$ is automatically satisfied, and so our lattices also provide a family of UNT (m,n)-frames, where m is the number of minimal vectors in the corresponding lattice, as given by (5) below.

In Section 2, we recall the formula for the number of minimal vectors in the lattice L_G and give the proof of Theorem 1.2. This proof is based on four computational lemmas. The proofs of these lemmas are shifted to Section 3.

2 Proof of the main result

For convenience, when the context is clear, we will refer to the lattice point $(x_1, \ldots, x_n)$ by using the formal sum $x_1 g_1 + x_2 g_2 + \ldots + x_n g_n$ in the group ring $\mathbb{Z}[G]$. The following result is proved in [4] in the special case of lattices from elliptic curves. Subsequently, in [1] we pointed out that this theorem is valid for the lattices L_G with virtually no change to the proof. As that the proof is important for our present purposes, we give it in detail here anew.

Proposition 2.1. *Assume that $n \geq 4$ and let κ denote the order of the subgroup $G_2 := \{x \in G : 2x = 0\}$ of G. Then the number of minimal vectors in L_G is*

$$\frac{n}{\kappa} \cdot \frac{(n-\kappa)(n-\kappa-2)}{4} + \left(n - \frac{n}{\kappa}\right) \cdot \frac{n(n-2)}{4}. \tag{5}$$

Proof. As shown in [1, 4], every minimal vector of L_G is of the form $p + q - r - s$ where $p, q, r, s \in G$ are distinct and $p + q = r + s$. Consider the homomorphism $\tau : G \to G$ defined by $\tau(p) = 2p$. The kernel of τ is the subgroup G_2 of G and the image $\mathrm{Im}(\tau)$ of τ has n/κ points. Fix an element z of G. First we count the number of solutions to the equation $p + q = z$ where p, q are distinct elements of G. Observe that $p = q$ if and only if $z \in \mathrm{Im}(\tau)$.

If $z \in \mathrm{Im}(\tau)$, there are κ solutions p to $2p = z$. Thus there are $n - \kappa$ possible p such that $q := z - p \neq p$, and so there are $(n-\kappa)/2$ pairs p, q such that $p + q = z$ and $p \neq q$. Hence the number of pairs r, s disjoint from $\{p, q\}$ and such that $r + s = z$ is $(n - \kappa - 2)/2$. In total, there are $(n-\kappa)/2 \cdot (n-\kappa-2)/2 = (n-\kappa)(n-\kappa-2)/4$ possible minimal vectors $p + q - r - s$ such that $p + q = z = r + s$. The size of the image of τ is $\frac{n}{\kappa}$ so the total number of possible minimal vectors $p + q - r - s$ such that $p + q = z = r + s$ with $z \in \mathrm{Im}(\tau)$ is $\frac{n}{\kappa} \cdot \frac{(n-\kappa)(n-\kappa-2)}{4}$.

If $z \notin \mathrm{Im}(\tau)$, there are no solutions p to $2p = z$. A similar reasoning as above shows that there are $(n - \frac{n}{\kappa}) \cdot \frac{n(n-2)}{4}$ minimal vectors $p + q - r - s$ with $p + q \notin \mathrm{Im}(\tau)$. Thus, by the above argument, the number of minimal vectors of L_G is given by (5). □

We remark that $\kappa = 1$ if and only if and only if n is an odd number while $\kappa = n$ if and only if $G = (\mathbb{Z}/2\mathbb{Z})^\nu$ for some $\nu \geq 2$.

Let $S'(L_G) = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be the set of (normalized) minimal vectors of L_G written as $n \times 1$ columns. The lattice L_G is not a full-rank lattice in the column space $\mathbb{R}^n$. We rather think of L_G as a full-rank lattice in the $(n-1)$-dimensional Euclidean space that is spanned by the real linear combinations of the vectors in the root lattice A_{n-1}. To be more precise, we consider the lattice $B(L_G)$ where $B : \mathrm{span}_{\mathbb{R}} A_{n-1} \to \mathbb{R}^{n-1}$ is the linear isometry from Theorem 1.2. We want to understand whether $B(L_G)$ is strongly eutactic in $\mathbb{R}^{n-1}$.

First of all, the answer to the question whether $B(L_G)$ is strongly eutactic does not depend on the concrete choice of B. Indeed, if B and C are two such linear isometries, then CB^{-1} is a linear isometry of $\mathbb{R}^{n-1}$ onto itself, and hence is given by an orthogonal matrix V. We therefore may write $C = VB$. Consequently, $C(S'(L_G)) = VB(S'(L_G))$, and as pointed out in Section 1, this implies that $C(S'(L_G))$ is a spherical 2-design if and only if so is $B(S'(L_G))$. Thus, it suffices to prove Theorem 1.2 for a single specified linear isometry B.

Our specification of B is as follows. Let A be the $(n-1)\times n$ matrix whose j-th row is

$$\frac{1}{\sqrt{j^2+j}}(\underbrace{1,1,\ldots,1}_{j\text{-ones}},-j,0,0,\ldots,0).$$

We are working with column spaces and may therefore identify matrices with the linear operators they induce. Thus, we think of A as a linear operator of $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$. It is not difficult to check that A maps $\operatorname{span}_{\mathbb{R}} A_{n-1}$ isometrically onto $\mathbb{R}^{n-1}$. The orthogonal complement of $\operatorname{span}_{\mathbb{R}} A_{n-1}$ in $\mathbb{R}^n$ is spanned by the vector $(1,\ldots,1)^\top$, and A is the zero operator on this orthogonal complement. We specify B to be the restriction of A to $\operatorname{span}_{\mathbb{R}} A_{n-1}$.

Let $A_{i,a}$ denote the (i,a)-entry of the matrix A. It is immediately seen that A has the following properties.

(P1) The rows of A form an orthonormal set, that is, $\sum_{a=1}^n A_{j,a}A_{k,a}=\delta_{j,k}$.

(P2) The rows of A are orthogonal to the $1\times n$ vector $(1,1,\ldots,1)$, that is, $\sum_{a=1}^n A_{j,a}=0$.

We have $B(S'(L_G))=\{A\mathbf{u}_1,A\mathbf{u}_2,\ldots,A\mathbf{u}_m\}$, where $\mathbf{u}_1,\ldots,\mathbf{u}_m$ are the normalized minimal vectors of L_G. The set $B(S'(L_G))$ is a subset of $\mathbb{R}^{n-1}$. Thus, when applying Proposition 1.1 to $\{A\mathbf{u}_1,A\mathbf{u}_2,\ldots,A\mathbf{u}_m\}$, we have to replace n by $n-1$. Incidentally, the same change from n to $n-1$ must also be done when considering UNT frames in this context. To draw an interim balance, what we have to prove is the following.

Theorem 2.2. *Let $n\geq 2$. With B as specified above, the set $B(S'(L_G))$ is a 2-design if and only if $\kappa=1$ or $\kappa=n$.*

The cases $n=2$ and $n=3$ can be disposed of by straightforward inspection: for $n=2$, $B(S'(L_G))$ is the doubleton $\{-1,1\}$, and we get the matrix (2), while for $n=3$, the lattice L_G is a hexagonal lattice whose minimal vectors are the vertices of a regular hexagon and after normalization lead to the matrix (3). Therefore we henceforth suppose that $n\geq 4$.

The j-th entry of $A\mathbf{u}_i$ is $\mathbf{e}_j^\top A\mathbf{u}_i$ where $\mathbf{e}_j$ is the $(n-1)\times 1$ column vector whose only nonzero entry is 1 in the jth position. For $1\leq a\leq n$, let $\overline{\mathbf{e}}_a$ be the $n\times 1$ vector whose only nonzero entry is 1 in the a-th position. We denote the (j,a)-th entry of A by $A_{j,a}$. Note that $A_{j,a}=\mathbf{e}_j^\top A\overline{\mathbf{e}}_a$.

As already mentioned in the proof of Proposition 2.1, it was shown in [1, 4] that all minimal vectors of L_G are of the form $\overline{\mathbf{e}}_a+\overline{\mathbf{e}}_b-\overline{\mathbf{e}}_c-\overline{\mathbf{e}}_d$ such that $g_a+g_b=g_c+g_d$ and g_a,g_b,g_c,g_d are all distinct elements of G. Now suppose that $\mathbf{u}=\frac{1}{2}(\overline{\mathbf{e}}_a+\overline{\mathbf{e}}_b-\overline{\mathbf{e}}_c-\overline{\mathbf{e}}_d)$ (the factor $\frac{1}{2}$ is a normalization factor). Then

$$\mathbf{e}_j^\top A\mathbf{u}=\frac{1}{2}(A_{j,a}+A_{j,b}-A_{j,c}-A_{j,d}).$$

For once and for all we fix integers j,k such that $1\leq j,k\leq n-1$ and put

$$f(a,b,c,d):=(A_{j,a}+A_{j,b}-A_{j,c}-A_{j,d})(A_{k,a}+A_{k,b}-A_{k,c}-A_{k,d}).$$

Then, since

$$(\mathbf{e}_j^\top A\mathbf{u})(\mathbf{e}_k^\top A\mathbf{u}) = \frac{1}{4}f(a,b,c,d),$$

the sum

$$\sum_{\mathbf{u}\in S'(L_G)} (\mathbf{e}_j^\top A\mathbf{u})(\mathbf{e}_k^\top A\mathbf{u}) \tag{6}$$

is the same as the sum of $\frac{1}{4}f(a,b,c,d)$ over all (minimal) vectors $\overline{\mathbf{e}}_a + \overline{\mathbf{e}}_b - \overline{\mathbf{e}}_c - \overline{\mathbf{e}}_d$ such that $g_a + g_b = g_c + g_d$ and g_a, g_b, g_c, g_d are all distinct elements of G.

For a fixed $z \in G$, we let R_z be the set of all ordered pairs (a,b) such that $g_a + g_b = z$ and $a \neq b$. For a fixed $(a,b) \in R_z$, we let $R_z(a,b)$ denote the set of all ordered pairs (c,d) such that $g_c + g_d = z$, $c \neq d$ and $\{c,d\} \cap \{a,b\} = \emptyset$. Notice that $R_z(a,b) = R_z \setminus \{(a,b),(b,a)\}$. From the proof of Proposition 2.1 we infer that the sum (6) is the same as

$$\sum_{z\in \mathrm{Im}(\tau)} \tfrac{1}{2} \sum_{(a,b)\in R_z} \tfrac{1}{2} \sum_{(c,d)\in R_z(a,b)} \tfrac{1}{4} f(a,b,c,d) + \sum_{z\notin \mathrm{Im}(\tau)} \tfrac{1}{2} \sum_{(a,b)\in R_z} \tfrac{1}{2} \sum_{(c,d)\in R_z(a,b)} \tfrac{1}{4} f(a,b,c,d). \tag{7}$$

The summation over $z \notin \mathrm{Im}(\tau)$ is over all $z \in G \setminus \mathrm{Im}(\tau)$. The number of minimal vectors is given by

$$\sum_{z\in \mathrm{Im}(\tau)} \tfrac{1}{2} \sum_{(a,b)\in R_z} \tfrac{1}{2} \sum_{(c,d)\in R_z(a,b)} 1 + \sum_{z\notin \mathrm{Im}(\tau)} \tfrac{1}{2} \sum_{(a,b)\in R_z} \tfrac{1}{2} \sum_{(c,d)\in R_z(a,b)} 1.$$

We will compute

$$\Phi f := \sum_{z\in \mathrm{Im}(\tau)} \sum_{(a,b)\in R_z} \sum_{(c,d)\in R_z(a,b)} f(a,b,c,d) + \sum_{z\notin \mathrm{Im}(\tau)} \sum_{(a,b)\in R_z} \sum_{(c,d)\in R_z(a,b)} f(a,b,c,d). \tag{8}$$

The $\frac{1}{2}$ and $\frac{1}{4}$ factors from (7) are suppressed for convenience.

Our goal is to prove the following Theorem 2.3. We need some notation before proceeding. For a fixed $z \in G$, let S_z be the set of all ordered pairs (a,b) such that $g_a + g_b = z$ and let T_z be the set of all a such that $2g_a = z$. Note that $|S_z| = n$ for all $z \in G$. Furthermore, $|T_z|$ is κ if $z \in \mathrm{Im}(\tau)$ and 0 otherwise.

Theorem 2.3. *The quantity Φf equals*

$$(4n^2 - 12n + 8\kappa)\delta_{j,k} - 4\kappa \sum_{z\in \mathrm{Im}(\tau)} \sum_{(a,b)\in S_z} A_{j,a}A_{k,b} - 8 \sum_{z\in \mathrm{Im}(\tau)} \left(\sum_{a\in T_z} A_{j,a}\right)\left(\sum_{c\in T_z} A_{k,c}\right). \tag{9}$$

Formula (9) follows immediately from the four Lemmas 3.1, 3.2, 3.3, 3.4 we will prove in the next section. So suppose (9) is established. To prove Theorem 2.2 we need one more auxiliary result.

Lemma 2.4. *Denote the distinct cosets of $H := \mathrm{Im}(\tau)$ in G by*

$$H_1 = g_{a_1} + H, \quad H_2 = g_{a_2} + H, \quad \ldots, \quad H_\kappa = g_{a_\kappa} + H.$$

Then

$$\frac{\Phi f}{16} - \frac{m}{n-1}\delta_{j,k} = \left(\frac{\kappa+1}{4} - \frac{\kappa-1}{4(n-1)}\right)\delta_{j,k} - \frac{\kappa}{4}\sum_{\ell=1}^{\kappa}\left(\sum_{g_a\in H_\ell} A_{j,a}\right)\left(\sum_{g_b\in H_\ell} A_{k,b}\right)$$
$$-\frac{1}{2}\sum_{z\in \mathrm{Im}(\tau)}\left(\sum_{a\in T_z} A_{j,a}\right)\left(\sum_{c\in T_z} A_{k,c}\right). \quad (10)$$

Proof. A straightforward computation using (9) gives

$$\frac{\Phi f}{16} - \frac{m}{n-1}\delta_{j,k} = \left(\frac{\kappa+1}{4} - \frac{\kappa-1}{4(n-1)}\right)\delta_{j,k} - \frac{\kappa}{4}\sum_{z\in \mathrm{Im}(\tau)}\sum_{(a,b)\in S_z} A_{j,a}A_{k,b}$$
$$-\frac{1}{2}\sum_{z\in \mathrm{Im}(\tau)}\left(\sum_{a\in T_z} A_{j,a}\right)\left(\sum_{c\in T_z} A_{k,c}\right). \quad (11)$$

For each a, we have $g_a + g_b \in \mathrm{Im}(\tau)$ if and only if $g_b \in -g_a + \mathrm{Im}(\tau) = g_a + \mathrm{Im}(\tau)$. Hence

$$\sum_{z\in \mathrm{Im}(\tau)}\sum_{(a,b)\in S_z} A_{j,a}A_{k,b} = \sum_{a=1}^{n}\sum_{g_b\in -g_a+\mathrm{Im}(\tau)} A_{j,a}A_{k,b} = \sum_{a=1}^{n} A_{j,a}\sum_{g_b\in g_a+\mathrm{Im}(\tau)} A_{k,b}$$
$$= \sum_{\ell=1}^{\kappa}\sum_{g_a\in g_{a_\ell}+H} A_{j,a}\sum_{g_b\in g_{a_\ell}+H} A_{k,b} = \sum_{\ell=1}^{\kappa}\sum_{g_a\in H_\ell} A_{j,a}\sum_{g_b\in H_\ell} A_{k,b}$$
$$= \sum_{\ell=1}^{\kappa}\left(\sum_{g_a\in H_\ell} A_{j,a}\right)\left(\sum_{g_b\in H_\ell} A_{k,b}\right).$$

Inserting this in (11) we arrive at (10). □

Proposition 2.5. *If $\kappa = 1$ or $\kappa = n$, then $\frac{1}{16}\Phi f = \frac{m}{n-1}\delta_{j,k}$ and consequently, the set $B(S'(L_G))$ is a spherical 2-design.*

Proof. Let first $\kappa = 1$. Then $H = \mathrm{Im}(\tau) = G$. Consequently, the first sum on the right of (10) equals

$$-\frac{\kappa}{4}\sum_{\ell=1}^{\kappa}\left(\sum_{g_a\in H_\ell} A_{j,a}\right)\left(\sum_{g_b\in H_\ell} A_{k,b}\right) = -\frac{1}{4}\left(\sum_{a=1}^{n} A_{j,a}\right)\left(\sum_{b=1}^{n} A_{k,b}\right),$$

and this is zero because of property (P2). Furthermore, since each set T_z is a singleton, the second sum in (10) is

$$-\frac{1}{2}\sum_{z\in \mathrm{Im}(\tau)}\left(\sum_{a\in T_z} A_{j,a}\right)\left(\sum_{c\in T_z} A_{k,c}\right) = -\frac{1}{2}\sum_{a=1}^{n} A_{j,a}A_{k,a},$$

which equals $-\frac{1}{2}\delta_{j,k}$ due to property (P1). In summary, (10) becomes

$$\frac{1+1}{4}\delta_{j,k} - \frac{1}{2}\delta_{j,k} = 0,$$

as asserted. Now suppose $\kappa = n$. Then $H = \text{Im}(\tau) = \{0\}$ and hence the first sum on the right of (10) is equal to

$$-\frac{\kappa}{4}\sum_{\ell=1}^{\kappa}\left(\sum_{g_a\in H_\ell} A_{j,a}\right)\left(\sum_{g_b\in H_\ell} A_{k,b}\right) = -\frac{n}{4}\sum_{a=1}^{n} A_{j,a}A_{k,b},$$

which is $-\frac{n}{4}\delta_{j,k}$ by virtue of property (P1). Since $T_0 = G$, we see that the second sum on the right of (10) is

$$-\frac{1}{2}\sum_{z\in\text{Im}(\tau)}\left(\sum_{a\in T_z} A_{j,a}\right)\left(\sum_{c\in T_z} A_{k,c}\right) = -\frac{1}{2}\left(\sum_{a=1}^{n} A_{j,a}\right)\left(\sum_{c=1}^{n} A_{k,c}\right),$$

and from property (P2) we infer that this is zero. Thus, in this case (10) equals

$$\left(\frac{n+1}{4} - \frac{1}{4}\right)\delta_{j,k} - \frac{n}{4}\delta_{j,k} = 0,$$

again as asserted. □

Proposition 2.6. *If $1 < \kappa < n$, then $\frac{1}{16}\Phi f \neq \frac{m}{n-1}\delta_{j,k}$ and consequently, the set $B(S'(L_G))$ is not a spherical 2-design.*

Proof. Assume the contrary, that is, assume that we do have a spherical 2-design. Then the right side of (10) must be 0. In particular, for $j = k = 1$ we have

$$\frac{\kappa+1}{4} - \frac{\kappa-1}{4(n-1)} = \frac{\kappa}{4}\sum_{\ell=1}^{\kappa}\left(\sum_{g_a\in H_\ell} A_{1,a}\right)^2 - \frac{1}{2}\sum_{z\in\text{Im}(\tau)}\left(\sum_{a\in T_z} A_{1,a}\right)^2 \tag{12}$$

Now the first row of A has one $1/\sqrt{2}$, one $-1/\sqrt{2}$ and the remaining entries are all 0. Thus implies that

$$\frac{\kappa+1}{4} - \frac{\kappa-1}{4(n-1)} = \frac{\kappa}{4}\lambda_1 - \frac{1}{2}\lambda_2$$

where λ_1, λ_2 are either 0 or 1. Multiplying both sides by 4 we get

$$\kappa + 1 - \frac{\kappa-1}{(n-1)} = \kappa\lambda_1 - 2\lambda_2$$

Since $1 < \kappa < n$, the left-hand side is strictly between κ and $\kappa+1$ while the right-hand side is at most κ. This is the desired contradiction. □

Combining Propositions 2.5 and 2.6, we arrive at Theorem 2.2 and thus at Theorem 1.2.

3 The remaining four lemmas

Recall that Φf is defined by (8).

Lemma 3.1. *The quantity* Φf *equals*

$$\sum_{z\in G}\sum_{(a,b)\in S_z}\sum_{(c,d)\in S_z} f(a,b,c,d) - 2\sum_{z\in \mathrm{Im}(\tau)}\sum_{(a,b)\in S_z}\sum_{c\in T_z} f(a,b,c,c) + \sum_{z\in \mathrm{Im}(\tau)}\sum_{a\in T_z}\sum_{c\in T_z} f(a,a,c,c).$$

Proof. Fix $z \in \mathrm{Im}(\tau)$ and $(a,b) \in R_z$. We then have

$$\sum_{(c,d)\in R_z(a,b)} f(a,b,c,d) = \sum_{(c,d)\in R_z\setminus\{(a,b),(b,a)\}} f(a,b,c,d)$$
$$= \sum_{(c,d)\in R_z} f(a,b,c,d) - f(a,b,a,b) - f(a,b,b,a) = \sum_{(c,d)\in R_z} f(a,b,c,d)$$

because $f(a,b,a,b) = f(a,b,b,a) = 0$. Now

$$\sum_{(c,d)\in R_z} f(a,b,c,d) = \sum_{(c,d)\in S_z} f(a,b,c,d) - \sum_{c\in T_z} f(a,b,c,c).$$

So

$$\sum_{(c,d)\in R_z(a,b)} f(a,b,c,d) = \sum_{(c,d)\in S_z} f(a,b,c,d) - \sum_{c\in T_z} f(a,b,c,c).$$

It follows that

$$\sum_{(a,b)\in R_z}\sum_{(c,d)\in R_z(a,b)} f(a,b,c,d)$$
$$= \sum_{(a,b)\in R_z}\left(\sum_{(c,d)\in S_z} f(a,b,c,d) - \sum_{c\in T_z} f(a,b,c,c)\right)$$
$$= \sum_{(a,b)\in S_z}\left(\sum_{(c,d)\in S_z} f(a,b,c,d) - \sum_{c\in T_z} f(a,b,c,c)\right)$$
$$- \sum_{a\in T_z}\left(\sum_{(c,d)\in S_z} f(a,a,c,d) - \sum_{c\in T_z} f(a,a,c,c)\right)$$
$$= \sum_{(a,b)\in S_z}\sum_{(c,d)\in S_z} f(a,b,c,d) - \sum_{(a,b)\in S_z}\sum_{c\in T_z} f(a,b,c,c) - \sum_{a\in T_z}\sum_{(c,d)\in S_z} f(a,a,c,d)$$
$$+ \sum_{a\in T_z}\sum_{c\in T_z} f(a,a,c,c).$$

We claim that

$$\sum_{(a,b)\in S_z}\sum_{c\in T_z} f(a,b,c,c) = \sum_{a\in T_z}\sum_{(c,d)\in S_z} f(a,a,c,d). \tag{13}$$

This can be seen as follows. Since

$$\begin{aligned} f(a,b,c,c) &= (A_{j,a}+A_{j,b}-2A_{j,c})(A_{k,a}+A_{k,b}-2A_{k,c}) \\ &= (2A_{j,c}-A_{j,a}-A_{j,b})(2A_{k,c}-A_{k,a}-A_{k,b}) = f(c,c,a,b), \end{aligned}$$

we get

$$\sum_{(a,b)\in S_z}\sum_{c\in T_z} f(a,b,c,c) = \sum_{(a,b)\in S_z}\sum_{c\in T_z} f(c,c,a,b) = \sum_{a'\in T_z}\sum_{\{c',d'\}\in S_z} f(a',a',c',d'),$$

where we make the change of variable $c' := a, d' := b, a' := c$. This gives (13). Thus, we have

$$\begin{aligned} &\sum_{(a,b)\in R_z}\sum_{(c,d)\in R_z(a,b)} f(a,b,c,d) \\ &= \sum_{(a,b)\in S_z}\sum_{(c,d)\in S_z} f(a,b,c,d) - 2\sum_{(a,b)\in S_z}\sum_{c\in T_z} f(a,b,c,c) + \sum_{a\in T_z}\sum_{c\in T_z} f(a,a,c,c). \end{aligned} \tag{14}$$

For a fixed $z \notin \mathrm{Im}(\tau)$, using a similar analysis as above, we get

$$\sum_{(a,b)\in R_z}\sum_{(c,d)\in R_z(a,b)} f(a,b,c,d) = \sum_{(a,b)\in S_z}\sum_{(c,d)\in S_z} f(a,b,c,d) \tag{15}$$

From (8), (14) and (15) the asserted formula for Φf follows. □

In the following three lemmas we compute the three sums occurring in Lemma 3.1.

Lemma 3.2. *We have*

$$\sum_{z\in G}\sum_{(a,b)\in S_z}\sum_{(c,d)\in S_z} f(a,b,c,d) = 4n^2\delta_{j,k}.$$

Proof. We begin with observing that

$$\begin{aligned} f(a,b,c,d) &= (A_{j,a}+A_{j,b}-A_{j,c}-A_{j,d})(A_{k,a}+A_{k,b}-A_{k,c}-A_{k,d}) \\ &= (A_{j,a}+A_{j,b})(A_{k,a}+A_{k,b}) + (A_{j,c}+A_{j,d})(A_{k,c}+A_{k,d}) \\ &\quad -(A_{j,a}+A_{j,b})(A_{k,c}+A_{k,d}) - (A_{j,c}+A_{j,d})(A_{k,a}+A_{k,b}). \end{aligned} \tag{16}$$

This is $h_1 + h_2 - h_3 - h_4$ with

$$\begin{aligned} h_1(a,b,c,d) &= (A_{j,a}+A_{j,b})(A_{k,a}+A_{k,b}), \quad h_2(a,b,c,d) = (A_{j,c}+A_{j,d})(A_{k,c}+A_{k,d}), \\ h_3(a,b,c,d) &= (A_{j,a}+A_{j,b})(A_{k,c}+A_{k,d}), \quad h_4(a,b,c,d) = (A_{j,c}+A_{j,d})(A_{k,a}+A_{k,b}). \end{aligned}$$

We first compute

$$\sum_{z\in G}\sum_{(a,b)\in S_z}\sum_{(c,d)\in S_z} h_1(a,b,c,d).$$

For a fixed $z \in G$ and $(a,b) \in S_z$, we have

$$\sum_{(c,d)\in S_z} h_1(a,b,c,d) = \sum_{(c,d)\in S_z} (A_{j,a}+A_{j,b})(A_{k,a}+A_{k,b}) = n(A_{j,a}+A_{j,b})(A_{k,a}+A_{k,b}).$$

Thus

$$\begin{aligned}
&\sum_{(a,b)\in S_z}\sum_{(c,d)\in S_z} h_1(a,b,c,d) = n\sum_{(a,b)\in S_z}(A_{j,a}+A_{j,b})(A_{k,a}+A_{k,b})\\
&= n\sum_{(a,b)\in S_z}(A_{j,a}A_{k,a}+A_{j,b}A_{k,b}+A_{j,a}A_{k,b}+A_{j,b}A_{k,a})\\
&= 2n\delta_{j,k} + n\sum_{(a,b)\in S_z}(A_{j,a}A_{k,b}+A_{j,b}A_{k,a}) \quad \text{(by (P1))}.
\end{aligned}$$

Since, by (P2),

$$\sum_{z\in G}\sum_{(a,b)\in S_z} A_{j,a}A_{k,b} = \sum_{a,b=1}^{n} A_{j,a}A_{k,b} = (\sum_{a=1}^{n} A_{j,a})(\sum_{b=1}^{n} A_{k,b}) = 0,$$

it follows that

$$\sum_{z\in G}\sum_{(a,b)\in S_z}\sum_{(c,d)\in S_z} h_1(a,b,c,d) = 2n^2\delta_{j,k}. \tag{17}$$

By symmetry, we also have

$$\sum_{z\in G}\sum_{(a,b)\in S_z}\sum_{(c,d)\in S_z} h_2(a,b,c,d) = 2n^2\delta_{j,k}. \tag{18}$$

Now note that for a fixed $z \in G$ and $(a,b) \in S_z$,

$$\begin{aligned}
&\sum_{(c,d)\in S_z} h_3(a,b,c,d) = \sum_{(c,d)\in S_z}(A_{j,a}+A_{j,b})(A_{k,c}+A_{k,d})\\
&= (A_{j,a}+A_{j,b})\sum_{(c,d)\in S_z}(A_{k,c}+A_{k,d}) = 0 \quad \text{(by (P1))}.
\end{aligned}$$

This shows that

$$\sum_{z\in G}\sum_{(a,b)\in S_z}\sum_{(c,d)\in S_z} h_3(a,b,c,d) = 0.$$

Similarly,

$$\sum_{z\in G}\sum_{(a,b)\in S_z}\sum_{(c,d)\in S_z} h_4(a,b,c,d) = 0.$$

This together with equations (16), (17) and (18) implies the desired result. □

Lemma 3.3. *We have*

$$\sum_{z\in\mathrm{Im}(\tau)}\sum_{(a,b)\in S_z}\sum_{c\in T_z} f(a,b,c,c) = 6n\delta_{j,k} + 2\kappa \sum_{z\in\mathrm{Im}(\tau)}\sum_{(a,b)\in S_z} A_{j,a}A_{k,b}.$$

Proof. Clearly,

$$\begin{aligned} f(a,b,c,c) &= (A_{j,a}+A_{j,b}-2A_{j,c})(A_{k,a}+A_{k,b}-2A_{k,c}) \\ &= (A_{j,a}+A_{j,b})(A_{k,a}+A_{k,b}) + 4A_{j,c}A_{k,c} \\ &\quad -2(A_{j,a}+A_{j,b})A_{k,c} - 2(A_{k,a}+A_{k,b})A_{j,c}, \end{aligned} \tag{19}$$

which is $h_1 + 4h_2 - 2h_3 - 2h_4$ with

$$\begin{aligned} h_1(a,b,c,d) &= (A_{j,a}+A_{j,b})(A_{k,a}+A_{k,b}), \quad h_2(a,b,c,d) = A_{j,c}A_{k,c}, \\ h_3(a,b,c,d) &= (A_{j,a}+A_{j,b})A_{k,c}, \quad h_4(a,b,c,d) = (A_{k,a}+A_{k,b})A_{j,c}. \end{aligned}$$

Fix $z \in \mathrm{Im}(\tau)$ and $(a,b) \in S_z$. Then

$$\sum_{c\in T_z} h_1(a,b,c,d) = \sum_{c\in T_z}(A_{j,a}+A_{j,b})(A_{k,a}+A_{k,b}) = (A_{j,a}+A_{j,b})(A_{k,a}+A_{k,b})\kappa$$

and hence

$$\begin{aligned} &\sum_{(a,b)\in S_z}\sum_{c\in T_z} h_1(a,b,c,d) = \kappa \sum_{(a,b)\in S_z} (A_{j,a}+A_{j,b})(A_{k,a}+A_{k,b}) \\ &= \kappa \sum_{(a,b)\in S_z} (A_{j,a}A_{k,a} + A_{j,b}A_{k,b} + A_{j,a}A_{k,b} + A_{j,b}A_{k,a}) \\ &= \kappa \sum_{a=1}^{n} A_{j,a}A_{k,a} + \kappa \sum_{b=1}^{n} A_{j,b}A_{k,b} + \kappa \sum_{(a,b)\in S_z} (A_{j,a}A_{k,b} + A_{j,b}A_{k,a}) \\ &= 2\kappa\delta_{j,k} + \kappa \sum_{(a,b)\in S_z} (A_{j,a}A_{k,b} + A_{j,b}A_{k,a}), \end{aligned}$$

the last equality resulting from (P1). Consequently,

$$\begin{aligned} &\sum_{z\in\mathrm{Im}(\tau)}\sum_{(a,b)\in S_z}\sum_{c\in T_z} h_1(a,b,c,d) = 2\kappa\delta_{j,k} \cdot \frac{n}{\kappa} + \kappa \sum_{z\in\mathrm{Im}(\tau)}\sum_{(a,b)\in S_z} (A_{j,a}A_{k,b} + A_{j,b}A_{k,a}) \\ &= 2n\delta_{j,k} + 2\kappa \sum_{z\in\mathrm{Im}(\tau)}\sum_{(a,b)\in S_z} A_{j,a}A_{k,b}. \end{aligned} \tag{20}$$

Further, for fixed $z \in \mathrm{Im}(\tau)$ we have that

$$\sum_{(a,b)\in S_z}\sum_{c\in T_z} h_2(a,b,c,d) = \sum_{(a,b)\in S_z}\sum_{c\in T_z} A_{j,c}A_{k,c} = n \sum_{c\in T_z} A_{j,c}A_{k,c}.$$

This gives

$$\sum_{z\in \mathrm{Im}(\tau)} \sum_{(a,b)\in S_z} \sum_{c\in T_z} h_2(a,b,c,d) = n \sum_{z\in \mathrm{Im}(\tau)} \sum_{c\in T_z} A_{j,c}A_{k,c} = n\sum_{c=1}^{n} A_{j,c}A_{k,c} = n\delta_{j,k}, \tag{21}$$

where we again used (P2). For fixed $z \in \mathrm{Im}(\tau)$ and $(a,b) \in S_z$ we also get

$$\sum_{c\in T_z} h_3(a,b,c,d) = \sum_{c\in T_z}(A_{j,a}+A_{j,b})A_{k,c} = (A_{j,a}+A_{j,b})\sum_{c\in T_z} A_{k,c},$$

so

$$\begin{aligned}\sum_{(a,b)\in S_z} \sum_{c\in T_z} h_3(a,b,c,d) &= \left(\sum_{c\in T_z} A_{k,c}\right) \sum_{(a,b)\in S_z}(A_{j,a}+A_{j,b}) \\ &= \left(\sum_{c\in T_z} A_{k,c}\right)\left(\sum_{a=1}^{n} A_{j,a} + \sum_{b=1}^{n} A_{j,b}\right) = 0,\end{aligned}$$

the last equality resulting once more due to (P2). Thus,

$$\sum_{z\in \mathrm{Im}(\tau)} \sum_{(a,b)\in S_z} \sum_{c\in T_z} h_3(a,b,c,d) = 0. \tag{22}$$

Similarly

$$\sum_{z\in \mathrm{Im}(\tau)} \sum_{(a,b)\in S_z} \sum_{c\in T_z} h_4(a,b,c,d) = 0 \tag{23}$$

The result now follows from equalities (16), (20), (21), (22), and (23). □

Lemma 3.4. *We have*

$$\sum_{z\in \mathrm{Im}(\tau)} \sum_{a\in T_z} \sum_{c\in T_z} f(a,a,c,c) = 8\kappa\delta_{j,k} - 8 \sum_{z\in \mathrm{Im}(\tau)} (\sum_{a\in T_z} A_{j,a})(\sum_{c\in T_z} A_{k,c}).$$

Proof. First of all,

$$\begin{aligned} f(a,a,c,c) &= (A_{j,a}+A_{j,a}-A_{j,c}-A_{j,c})(A_{k,a}+A_{k,a}-A_{k,c}-A_{k,c}) \\ &= 4(A_{j,a}-A_{j,c})(A_{k,a}-A_{k,c}).\end{aligned}$$

Thus, for fixed $z \in \mathrm{Im}(\tau)$ and a fixed integer a with $1 \le a \le n$,

$$\begin{aligned}&\sum_{c\in T_z} f(a,a,c,c) = \sum_{c\in T_z} 4(A_{j,a}-A_{j,c})(A_{k,a}-A_{k,c}) \\ &= 4\sum_{c\in T_z} A_{j,c}A_{k,c} - 4A_{j,a}\sum_{c\in T_z} A_{k,c} - 4A_{k,a}\sum_{c\in T_z} A_{j,c} + 4A_{j,a}A_{k,a}\sum_{c\in T_z} 1 \\ &= 4\sum_{c\in T_z} A_{j,c}A_{k,c} - 4A_{j,a}\sum_{c\in T_z} A_{k,c} - 4A_{k,a}\sum_{c\in T_z} A_{j,c} + 4A_{j,a}A_{k,a}\kappa.\end{aligned}$$

Now

$$
\begin{aligned}
&\sum_{a\in T_z}\sum_{c\in T_z} f(a,a,c,c)\\
&= 4\kappa\sum_{c\in T_z} A_{j,c}A_{k,c} - 4\sum_{a\in T_z} A_{j,a}\sum_{c\in T_z} A_{k,c} - 4\sum_{a\in T_z} A_{k,a}\sum_{c\in T_z} A_{j,c} + 4\kappa\sum_{a\in T_z} A_{j,a}A_{k,a}\\
&= 4\kappa\sum_{c\in T_z} A_{j,c}A_{k,c} - 8\sum_{a\in T_z} A_{j,a}\sum_{c\in T_z} A_{k,c} + 4\kappa\sum_{a\in T_z} A_{j,a}A_{k,a}
\end{aligned}
$$

and we finally get

$$\sum_{z\in \mathrm{Im}(\tau)}\sum_{a\in T_z}\sum_{c\in T_z} f(a,a,c,c) = 8\kappa\delta_{j,k} - 8\sum_{z\in \mathrm{Im}(\tau)}\sum_{a\in T_z} A_{j,a}\sum_{c\in T_z} A_{k,c},$$

as desired. □

As already said, the preceding four lemmas yield Theorem 2.3.

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A. Böttcher, Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany
aboettch@mathematik.tu-chemnitz.de

L. Fukshansky, Department of Mathematics, Claremont McKenna College,
850 Columbia Ave, Claremont, CA 91711, USA
lenny@cmc.edu

S. R. Garcia, Department of Mathematics, Pomona College,
610 N. College Ave, Claremont, CA 91711, USA
stephan.garcia@pomona.edu, URL: http://pages.pomona.edu/~sg064747/

H. Maharaj, Department of Mathematics, Pomona College,
610 N. College Ave, Claremont, CA 91711, USA
hirenmaharaj@gmail.com