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Marcy's Dots: A Problem on National Test Revisited

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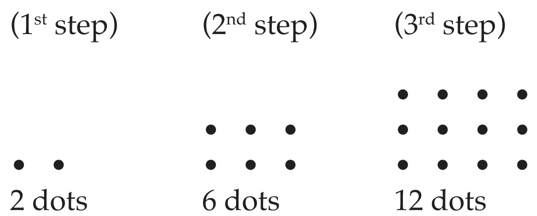
In 1992 the following problem was given to 8th grade students as a part of the National Assessment of Educational Progress (NAEP)'s national assessment:

MARCY'S DOTS
A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues infinitely.



Marcy has to determine the number of dots in the 20th step, but she does not want to draw all 20 pictures and then count the dots.

Explain or show how she could do this and give the answer that Marcy should get for the number of dots.

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The only answer accepted as "correct" was 420. But the quality of each explanation was graded as minimal, partial, or satisfactory or better.

Responses of 8th grade students in the national sample				
		Correct		
No response	Incorrect	Minimal	Partial	Satisfactory or better
16%	63%	10%	6%	6%

This problem is a typical "IQ type" problem common on NEAP tests. In order to solve it, a student needs only 3rd grade mathematics. A student has to know that the number of dots in a rectangular array is the product of the number of rows and the number of columns, and he/she has to be able to make one easy mental multiplication. Finding the intended solution requires no ingenuity if a student has any experience with looking for numerical, and not visual, patterns. But providing a concise and clear explanatory write-up is difficult even for the best students.

The intended solution looks as follows:

1. Marcy should notice that the number of rows in step n = 1, 2 and 3, is n, and that the number of columns is n + 1.
2. Therefore the number of dots is equal to n(n + 1), for n = 1, 2 and 3.
3. If this formula holds for other numbers, then for n = 20 the number of dots is 20 * 21 = 420. This is the number of dots Marcy should get.
4. She should also check that the number of dots that are added increases from one step to the next. That is easy, because the number of dots added in step n is n(n + 1) – (n – 1)n = 2n.

But is there any reason to claim that 420 is the unique correct solution? NO!

There are infinitely many patterns of dots that satisfy the conditions of the problem, and there is no overwhelming reason to claim that the one which seems to one person the most obvious is the "correct" one. Clearly many 8th graders saw patterns that were different from the one seen by the makers of the test.

Below are some possible solutions to the problem. (They are written as if they were students' answers, but they were not.)

EXAMPLE 1
There are many other solutions besides the obvious

420, so I asked myself, what is the smallest possible solution?

The differences between the numbers of dots that are added in each step must increase at least by 1. So we have the following pattern if we make the differences as small as possible.

Step:	1	2	3	4	5	…	n-1	n	…
# of dots:	2	6	12	19	27	…	?	?	…
Differences:	4	6	7	8		n+2	n+3 …		

Thus, the number of dots at the n^{th} step is:

$$12 + 7 + 8 + \dots + (n + 2) + (n + 3)$$
$$= 12 + (n + 10)(n - 3)/2.$$

So the smallest possible solution for the 20^{th} step is 267 dots.

Example 2

The question is about the number of dots, and not about their pattern. So I decided to concentrate just on numbers. The ratios between the numbers of dots are 3 and 2 (6/2 and 12/6), and that suggests an exponential growth. However this cannot be "purely" exponential, because the ratios are not equal, so I fiddled a little with formulas and found this one for the number of dots, d(n), in step n:

$$d(n) = 2^n + 2(n - 1).$$

I used a scientific calculator to compute d(20) = 1,048,614.

Example 3

The pattern can continue by repeating the ratios 3 and 2.

Step:	1	2	3	4	5	6	…
Number of dots:	2	6	12	36	72	216	…

Thus the number of dots in step 2n is 6^n, and the number of dots in step 2n + 1 is $2*6^n$.

Therefore the number of dots in the 20^{th} step is 6^{10} = 60,466,176. (I used a calculator.)

Example 4

We were learning about Fibonacci, so I started looking for Fibonacci numbers. And guess what? I found them!

We have to start with step 0, with no dots, and look at the differences.

Step:	0	1	2	3	?
Number:	0	2	6	12	?
Difference:	2	4	6	?	

onacci pattern! So the next difference is 4 + 6 = 10, the next is 6 + 10 = 16, and so on. Then you have to add all these differences up to step 20 to get the number of dots. I went only up to step 10 and I gave up. Not enough time. I would rather program a calculator to give me the answer.

0	1	2	3	4	5	6	7	8	9	10
2	4	6	10	16	26	42	68	110	178	
0	2	6	12	22	38	64	106	174	284	462

Conclusion

In order to make the solution to the original problem unique, one needs to add a few strong assumptions. For example, the number of dots, d(n), in step n is expressed by a polynomial of second degree. The assumption about the degree of the polynomial is needed because the polynomial

$$d(n) = n(n + 1) + (n - 1)(n - 2)(n - 3)$$

is a third degree polynomial, which is a solution to the original problem.

Questions that are used on national and state tests should be mathematically sound. Questions should be testing mathematical knowledge that is expected at a given grade level. Also, answers should be scored objectively and correctly. The problem of Marcy's Dots fails all three criteria. It is an example of rushing toward a solution, rather than thinking, what is a solution to the stated problem (Buerk 2000).

It is hard to judge whether poor test questions are exceptions or if they are the norm, because test makers protect themselves by keeping the contents of tests secret.

References

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Aphorisms

Lee Goldstein

Editor's note: These are excerpts from a longer paper.

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It is easy to see that even in (ordinary) human life, and first of all in every individual life from childhood up to maturity, the originally intuitive life which creates its originally self-evident structures through activities on the basis of sense-experience very quickly and in increasing measure falls victim to the seduction of language. Greater and greater segments of this life lapse into a kind of talking and reading that is dominated purely by association; and often enough, in respect to the validities arrived at in this way, it is disappointed by subsequent experience.

Edmund Husserl, *The Origin of Geometry*

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- Sense statements may tend to be homeopathic to the mathematical, and mathematical statements tend to be allopathic to the sense world.
- Mathematics (or geometry) opens betwixt infra-realization and super- nominalization, both of which are programs.
- Mathematics is in the thinning of programmaticness, as such a checking of programs and unprogram-maticness.
- Mathematics leans on institutions of objectivity.
- Hollow mathematicians are at least correct.
- Upon a people's limited language, nonverbal mathematics was the first mathematics.
- I believe in nonverbal universals.
- We may read silence in dreams.
- Statements containing "there exists" could be that penultimate resort of the nonverbal.
- The nonverbal could hypothetically be as nominal, not nominalist.
- Mathematics is nominalism's self.
- A referential statement about mathematics might be as an unending hypothesis
- Science and the beginning of the world are not referentism.
- The reference is the residue.

The basic idea of the above aphorisms is that a quickening and underlying programmatization of the understructure of things quickens the tendency to words and language and not the intuitive realism which precedes programs. This answers Husserl's question, then, and opens up a new field coincidental or before grammar. As our grammar is hard-wired into our brains, so is the programmetrical structure of the world.