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A TWO POINT BOUNDARY VALUE PROBLEM WITH JUMPING NONLINEARITIES

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ABSTRACT. We prove that a certain two point BVP with jumping nonlinearities has a solution. Our result generalizes that of [2]. We use variational methods which permit giving a minimax characterization of the solution. Our proof exposes the similarities between the variational behavior of this problem and that of other semilinear problems with noninvertible linear part (see [5]).

1. Introduction and notations. Here we study the two point BVP

$$\text{(I)} \quad \begin{cases} u''(t) + g(u(t)) = p(t), & t \in [0, \pi], \\ u(0) = u(\pi) = 0. \end{cases}$$

We assume that $g\colon \mathbf{R} \to \mathbf{R}$ and $p\colon [0, \pi] \to \mathbf{R}$ are continuous functions such that

(1.1) $g(u) = u$ for $u \geqslant 0$.

(1.2) There exists $\alpha > 0$ such that $g(u)/u \to 1 + \alpha$ as $u \to -\infty$.

(1.3) $\int_0^\pi p(t)\sin(t)\,dt \leqslant 0$.

The purpose of this paper is to give a variational proof of

THEOREM A. *If* (1.1), (1.2) *and* (1.3) *are satisfied then* (I) *has a solution.*

Theorem A is a generalization of a result due to L. Aguinaldo and K. Schmitt (see [2]). The main difference between our approach and that of [2] is that we use variational methods while the proof of [2] is based on degree theoretical arguments. As a byproduct of our technique for proving Theorem A we observe the functional J, to be defined below, has a variational behavior similar to that of the functional corresponding to other semilinear problems with noninvertible linear part. We use a variant of a minimax principle proved first by P. Rabinowitz to obtain a variational proof of the theorem due to Ahmad, Lazer and Paul (see [3]).

Let $H = H_0^{1,2}[0, \pi]$ (see [1, p. 44]) be the Sobolev space of square integrable functions defined on $[0, \pi]$ vanishing on $\{0, \pi\}$ with generalized first derivative in $L_2[0, \pi]$. The inner product and norm in H are given by

$$\langle u, v \rangle = \int_0^\pi u'(t)v'(t)\,dt \quad \text{and} \quad \|u\|^2 = \langle u, u \rangle.$$

According to Sobolev's lemma (see [1, p. 95]) H can be imbedded in the space of continuous functions defined on $[0, \pi]$. Thus, there exists a real number $c > 0$ such

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that

$$\max_{t\in[0,\pi]} |u(t)| \leq c\|u\| \quad \text{for all } u \in H. \tag{1.4}$$

We let $J\colon H \to R$ be defined by

$$J(u) = \int_0^\pi \left(\frac{(u'(t))^2}{2} - G(u(t)) + p(t)u(t) \right) dt, \tag{1.5}$$

where $G(u) = \int_0^u g(s)\, ds$. It is easy to check that

$$\begin{aligned} \langle \nabla J(u), v \rangle &\equiv \lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} \\ &= \int_0^\pi u'v' - g(u)v + pv \quad \text{for all } u, v \in H. \end{aligned} \tag{1.6}$$

By standard regularity theory it follows that if $\nabla J(u) = 0$ then u is a solution of (I). Therefore, from now on we aim our work towards proving that J has a critical point. In the rest of this paper the symbol $\int$ means integral from 0 to π unless the integration limits are specified. For future reference, we remark that because of (1.1) and (1.2) there exists a real number M_1 such that

$$|G(u)| \leq ((1 + 2\alpha)u^2)/2 + M_1 \quad \text{for all } u \in \mathbf{R}. \tag{1.7}$$

2. Preliminary lemmas. If $\int p(t)\sin(t)\, dt = 0$ then it is easily verified that $\{u'' + u = p(t),\ t \in [0, \pi],\ u(0) = u(\pi) = 0\}$ has a positive solution u_0. Therefore u_0 is a solution of (I). Thus it is sufficient to restrict ourselves to the case

$$\int p(t)\sin(t)\, dt < 0. \tag{2.1}$$

LEMMA 1. *The functional J satisfies $J(\lambda\sin(t)) \to -\infty$ as $|\lambda| \to \infty$.*

PROOF. That $J(\lambda\sin(t)) \to -\infty$ as $\lambda \to \infty$ follows immediately from (1.1) and (2.1). Because of (1.2) there exists a real number M such that

$$G(u) \geq ((1 + \alpha/2)u^2)/2 + M \quad \text{for all } u \leq 0. \tag{2.2}$$

Therefore, for $\lambda < 0$ we have

$$\begin{aligned} J(\lambda\sin(t)) \leq (\lambda^2/2)\int \cos^2(t)\, dt - (1 + \alpha/2)\left(\int \sin^2(t)\right)(\lambda^2)/2 \\ - M\pi + \lambda \int p(t)\sin(t)\, dt. \end{aligned} \tag{2.3}$$

Since $\alpha > 0$, it is clear that (2.3) implies that $J(\lambda\sin(t)) \to -\infty$, as $\lambda \to -\infty$, and the lemma is proved.

We let Y be the closed subspace of H generated by $\{\sin(2t), \sin(3t), \ldots\}$. It is readily verified that Y is the orthogonal complement of the subspace generated by $\sin(t)$. Observe that for $y \in Y$

$$4\int y^2(t)\, dt \leq \int (y'(t))^2\, dt. \tag{2.4}$$

LEMMA 2. *There exist real numbers* $r_0 > 0$ *and* $\rho_0 > 0$ *such that if* $y \in Y$ *and* $\|y\| = r_0$ *then*

$$J(\rho_0 \sin(t) + y(t)) \geq \sup_{\lambda \in \mathbf{R}} J(\lambda \sin(t)) + 1.$$

PROOF. Because of (1.4), given $\varepsilon > 0$ there exists $\rho > 0$ such that if $y \in Y$ and $\|y\| = 1$ then

$$y(t) + \rho\sin(t) > 0 \quad \text{for all } t \in [\varepsilon, \pi - \varepsilon].$$

Furthermore, for all $t \in [0, \pi]$, $y(t) + \rho\sin(t) \geq -c$. Consequently, for $b > 0$ and $y \in Y$ with $\|y\| = 1$,

$$\begin{aligned} J(b(\rho\sin t + y(t))) \geq{} & (b^2\rho^2)/2\int \cos^2(t)\,dt - (b^2\rho^2)/2\int \sin^2(t)\,dt \\ & + b^2/2 - b^2\int y^2(t)\,dt + b\rho\int p(t)\sin(t)\,dt \\ & + b\int p(t)y(t)\,dt - \int_0^{\varepsilon} G(b(\rho\sin(t) + y(t)))\,dt \\ & - \int_{\pi-\varepsilon}^{\varepsilon} G(b(\rho\sin(t) + y(t)))\,dt. \end{aligned}$$

Combining this with (2.2) and (2.4) we have

$$\begin{aligned} J(b(\rho\sin(t) + y(t))) \geq{} & b^2(3/8) - b\rho\int p(t)\sin(t)\,dt \\ & - 2\varepsilon(1 + (\alpha/2))b^2c^2 - 2\varepsilon M. \end{aligned}$$

Thus, choosing ε small enough and b sufficiently large we see that $r_0 = b$ and $\rho_0 = \rho b$ satisfy the conditions of the lemma. Hence, Lemma 2 is proved.

From now on ρ_0 and r_0 denote two fixed real numbers satisfying Lemma 2. Because of (1.1) and (1.2), J is bounded on bounded sets. Therefore, there exists a real number c_1 such that if $y \in Y$ and $\|y\| = r_0$ then

$$J(\rho_0\sin(t) + y(t)) \geq c_1. \tag{2.5}$$

We let $\lambda_0 > \rho_0$ be such that

$$\max\{J(\lambda_0\sin(t)), J(-\lambda_0\sin(t))\} \leq c_1 - 1. \tag{2.6}$$

We denote by Σ the family of all continuous functions σ: $[0, 1] \to S \equiv H - \{\rho_0\sin(t) + y(t); y \in Y \text{ and } \|y\| = r_0\}$ such that

(a) $\sigma(0) = -\lambda_0\sin(t)$, $\sigma(1) = \lambda_0\sin(t)$, and

(b) σ is homotopic on S to a map σ_0 through a homotopy which keeps end points fixed, where σ_0 is defined by $\sigma_0(s) = 2s\lambda_0\sin(t) - \lambda_0\sin(t)$.

An elementary topological argument shows that if $\sigma \in \Sigma$ then there exists $s \in [0, 1]$ and $y \in Y$ with $\sigma(s) = \rho_0\sin(t) + y(t)$ and $\|y\| < r_0$.

THEOREM 3. *Let* J, r_0, ρ_0 *and* Σ *be as before. If every sequence* $\{x_n\} \subset H$ *such that* $\nabla J(x_n) \to 0$ *and* $\{J(x_n)\}$ *is bounded has a convergent subsequence, then* J *has a critical point* u_0. *Moreover*

$$J(u_0) = \inf_{\sigma \in \Sigma}\Big(\max_{s \in [0,1]} J(\sigma(s))\Big).$$

Since Theorem 3 is a slight variant of Theorem 1.2 of [5] we do not give a proof of it here.

3. Proof of Theorem A. Let $\{x_n\} \subset H$ be a sequence such that $\nabla J(x_n) \to 0$ and $\{J(x_n)\}$ is bounded. According to Theorem 3 and the remark following (1.6) we only need show that $\{x_n\}$ has a convergent subsequence.

By (1.6), $\nabla J(x) = x + g_1(x)$, where $g_1: H \to H$ is continuous. Moreover, since the inclusion of H into $L_2[0, \pi]$ is compact (see [1, Theorem 6.2]), g_1 is compact. In addition, g_1 maps weakly convergent sequences into convergent sequences.

Suppose $\{x_n\}$ does not have a convergent subsequence. First we observe that $\{x_n\}$ does not have a weakly convergent subsequence $\{x_{n_k}\}$. For if it does, then $\{g_1(x_{n_k})\}$ is a convergent sequence; since $x_{n_k} + g_1(x_{n_k}) \to 0$ we have that $\{x_{n_k}\}$ converges, a contradiction. Hence we can assume that $\{\|x_n\|\}$ tends to $+\infty$.

Let $\{x_{n_j}/\|x_{n_j}\|\}$ be a subsequence of $\{x_n/\|x_n\|\}$ converging weakly to a point x_0 in H. For each $v \in H$ we have

$$\langle \nabla J(x_{n_j}), v\rangle/\|x_{n_j}\| = \left(\int x'_{n_j}v' - g(x_{n_j})v + pv\right)/\|x_{n_j}\| \to 0.$$

Therefore

$$\int\left(x'_0v' - \left(g(x_{n_j})/\|x_{n_j}\|\right)\right)v \to 0 \quad \text{as } j \to \infty. \tag{2.7}$$

Because of (1.1) and (1.2) we have $g(x) = f_1(x) + f_2(x)$ where f_1 is defined by $\{f_1(x) = x \text{ for } x \geqslant 0 \text{ and } f_1(x) = (1 + \alpha)x \text{ for } x < 0\}$, and f_2 satisfies

$$(f_2(x))/x \to 0 \quad \text{as } |x| \to +\infty. \tag{2.8}$$

Thus, we have

$$\int\left(g(x_{n_j})/\|x_{n_j}\|\right)v = \int f_1\left(x_{n_j}/\|x_{n_j}\|\right)v + \left(f_2(x_{n_j})v\right)/\|x_{n_j}\|.$$

Because of (2.8) and (2.9) we obtain

$$\int x'_0v' - f_1(x_0)v = 0 \quad \text{for all } v \in H.$$

Therefore x_0 satisfies $\{x''_0 + f_1(x_0) = 0, x(0) = x(\pi) = 0\}$, which implies that $x_0 = \zeta\sin(t)$ for some $\zeta \geqslant 0$. In case $\zeta = 0$, by (1.7) we have that given $\varepsilon > 0$ there exists j_0 such that

$$\int G\left(x_{n_j}(t)\right) dt < \varepsilon(1 + 2\alpha)\|x_{n_j}\|^2/2 + M_1\pi \quad \text{for all } j > j_0. \tag{2.9}$$

From this inequality we have

$$J(x_{n_j}) > \|x_{n_j}\|^2/2 - \varepsilon(1 + 2\alpha)\|x_{n_j}\|^2/2 - M_1\|x_{n_j}\| - \|x_{n_j}\|\int(px_{n_j})/\|x_{n_j}\|. \tag{2.10}$$

When $\varepsilon(1 + 2\alpha) < 1$ our assumption that $\|x_n\| \to \infty$ contradicts the boundedness of $\{\|J(x_n)\|\}$.

It remains to consider the case $\zeta > 0$. Note that $\{g(x_{n_j})/\|x_{n_j}\|\}$ converges in $L_2[0, \pi]$ to $f_1(x_0)$ and $g_1(x_n/\|x_n\|)$ converges to $g_1(x_0)$. We are assuming $\nabla J(x_n) = x_n + g_1(x_n) \to 0$, so $\{x_{n_j}/\|x_{n_j}\|\}$ converges to x_0. Since x_0 is positive on $(0, \pi)$ with $x'(0) > 0$ and $x'(\pi) < 0$, we have that there exists j_1 and a real number c such that $x_{n_j}(t) \geqslant c$ for all $t \in [0, \pi]$ and all $j \geqslant j_1$. Thus, as $j \to \infty$,

$$(2J(x_{n_j}))/\|x_{n_j}\| = \int\left((x'_{n_j})^2 + 2px_{n_j}\right)/\|x_{n_j}\| - \left(\int_{x_{n_j}>0} x_{n_j}^2 + \int_{x_{n_j}<0} G(x_{n_j})\right)/\|x_{n_j}\| \to 0,$$

and

$$\langle \nabla J(x_{n_j}), x_{n_j}/\|x_{n_j}\|\rangle = \int\left((x'_{n_j})^2 + px_{n_j}\right)/\|x_{n_j}\| - \left(\int_{x_{n_j}>0} x_{n_j}^2 + \int_{x_{n_j}<0} \left(g(x_{n_j})x_{n_j}\right)\right)/\|x_{n_j}\| \to 0$$

By the hypotheses on $\{x_n\}$ we find $(\int p(x_{n_j}/\|x_{n_j}\|)) \to 0$. Then by (2.1), ζ cannot be positive. This final contradiction implies that $\{x_n\}$ has a convergent subsequence and Theorem A is proved.

REMARK. With obvious modifications of the method above one can prove results analogous to Theorem A for other type of boundary conditions. For example, it can be shown that

$$\text{(II)} \quad \begin{cases} u'' + g(u(t)) = p(t), & t \in [0, \pi], \\ u'(0) = u'(\pi) = 0 \end{cases}$$

has a solution if

(1′) $g(u) = 0$ for $u \geqslant 0$.

(11′) for some $\alpha > 0$, $g(u)/u \to \alpha$ as $u \to -\infty$.

(111′) $\int p(t)\,dt < 0$.

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