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Cover Page Footnote

My sincere thanks to Andrew Fish, David Freund, Jennifer Hom, Josh Howie and Jessica S. Purcell for very kindly agreeing to be a part of this research.

Untangling Knots: Embodied Diagramming Practices in Knot Theory

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Abstract

The low visibility and specialised languages of mathematical work pose challenges for the ethnographic study of communication in mathematics, but observation-based study can offer a real-world grounding to questions about the nature of its methods. This paper uses theoretical ideas from linguistic pragmatics to examine how mutual understandings of diagrams are achieved in the course of conference presentations. Presenters use shared knowledge to train others to interpret diagrams in the ways favoured by the community of experts, directing an audience's attention so as to develop a shared understanding of a diagram's features and possible manipulations. In this way, expectations about the intentions of others and appeals to knowledge about the manipulation of objects play a part in the development and communication of concepts in mathematical discourse.

1. Introduction

Observational study of mathematics as it is being shared and created between individuals today can serve to demystify the discipline and put forward an account of how mathematical work proceeds that brings real-world interactions between persons and materials into the heart of that work. The more vividly we can demonstrate the ways that real-world mathematics inhabits artefacts, people, and discourses, the more we can build up an understanding of mathematics as study that is conducted by hard interactive work.

This paper presents some first steps in an ongoing project that aims to do just that. What follows is a study of knot theory diagramming practices as observed in presentations of mathematical work, considered using relevance theory, a contemporary linguistic pragmatic theory put forward by Dan Sperber and Deirdre Wilson [21]. In Section 2, the basic approach of the paper is laid out, and relevance theory is briefly introduced. In Section 3, the materials and settings of communication at conferences are discussed, and diagramming is considered as a communicative medium. In Section 4, a number of examples are put forward that demonstrate the kind of resources that are leveraged in order for an audience and speaker to arrive at a shared understanding of the affordances of a diagram.

The intention is to present these examples as a proposal for a way of looking at interactions between mathematicians that takes into account materials, settings and the subtleties of communication that go beyond simple interpretation of a code. By considering mathematical communication in these terms it may be possible to build up a picture of mathematical work as part of a spectrum of situated, interactive human practices, not encapsulated by its codified elements. This also constitutes an exploration of the ways in which ideas developed in linguistic pragmatics to describe the mechanics of everyday conversation might extend to more specialised and unusual domains.

2. Observation in the wild

Ethnographer of science Bruno Latour expressed concerns about the difficulty of making "inspectable and accountable topics" [17, page 444] of the processes of mathematics, but in spite of this, observational studies of mathematics and exploratory perspectives that place more emphasis on practice are growing in number [1, 11, 14, 16, 19]. Efforts have increased to carry out observations in the mathematical world that take up the social and material aspects of the discipline as being inherently involved in the progress of mathematics, notably from sociologist Christian Greiffenhagen, and in the theses of Lorenzo Lane and Michael Barany [1, 14]. In one paper, Greiffenhagen focuses on the blackboard as a site for the laying-out of ideas and analyses features of the inscription practices used (after Latour), in a methodology designed to bring the materiality of mathematics into sociological investigation [12]. Barany and Mackenzie's study of chalk goes on to posit "an

essential relationship between the supposedly abstract concepts and methods of advanced mathematics and the material substituents and practices that constitute them" [2, page 2]. If, indeed, progress in the mathematical world would be substantially hindered by the loss of the materials, interactions and practices that are involved in mathematical work, these should be taken seriously in a holistic view of what mathematics is.

The question of whether an observer of science should be an insider, with the knowledge to follow technical conversations, or an outsider, less susceptible to the problem of 'going native', is much debated. This paper gives some initial results from a research project aimed at subjecting mathematics to consideration as part of a spectrum of other activities, and in this instance the perspective of a person not immersed in and committed to the norms of mathematical culture, aided by discussion with people familiar with the field, can be considered a distinct advantage. This paper presents close analysis of a selection of details observed in mathematical talks to consider how diagrams are used in mathematical communication.

2.1. A brief introduction to relevance theory

This analysis has its basis in relevance theory, a set of ideas from linguistic pragmatics that is considered to be a theory of communication, rather than of language, applying broadly to actions that are intended to be recognised as intentional and communicative. The theory was developed by Dan Sperber and Deirdre Wilson, and is considered a cognitivist theory of real-world communication [21]. The case has been made (for example in [6]) that it is a very appropriate tool to use in analysing multimodal communication, due to its treatment of different kinds of stimuli functioning in a spectrum of ways (see Section 2.2 on showing and meaning).

Relevance theory rests on a communicative principle of relevance that was greatly influenced by Paul Grice's maxims of cooperative communication. The basic principle is that when a speaker pronounces an utterance or, more broadly, behaves ostensively, the hearer becomes aware of an intention to communicate. This produces an expectation in that hearer that whatever the speaker is trying to communicate will have sufficient effects to justify the effort that the hearer must invest — and indeed, will keep that effort to an absolute minimum. The hearer therefore looks for the most relevant interpre-

tation to satisfy the expectation of relevance that this recognition of intention promises. Relevance theory describes a process by which hearers weigh the cognitive effects of an utterance — what an utterance does to strengthen, develop or contradict an existing assumption — against processing effort, and seek an interpretation that satisfies the expected balance.

Both effort and effects will depend on a hearer's cognitive environment, the set of assumptions manifest to that person at that time, so the speaker brings extensive beliefs about her audience into play as she selects an optimally relevant stimulus. For this cooperative work to be successful, relevance theory holds that both speaker and hearer take their knowledge of one another's beliefs and intentions into account in order to communicate as efficiently and effectively as possible. This means that contextual knowledge about the speaker and shared environment, beliefs about one another's knowledge and assumptions, and attribution of intention are all deeply involved in the comprehension process that Sperber and Wilson outline [21].

This might be illustrated by considering how an identical stimulus might be given entirely different interpretations in different contexts, depending on the intentions ascribed to the communicator. If a knot theorist were to be shown a symbol like that in Figure 1 on the board by a colleague, she will assume that he intended the image to be a diagram of a knot in the mathematical sense — a trefoil knot — and that some operation on or example involving that knot was to follow. She would have a great many assumptions present in her mind pertaining to crossings and equivalence, and the problems and objectives of knot theory, and many experiences of explanations involving knots such as this. As a result, she would be particularly aware of the features of the diagram most pertinent to the study of knot theory, such as its sequence of crossings, and expecting the discussion to proceed down a certain kind of route that will have cognitive effects in that domain.

If, however, she encountered the same diagram drawn on the wall above the leftmost one of a pair of bins, she might assume that perhaps the usual signage had broken and that the symbol had been drawn as a temporary replacement by somebody intending to communicate to someone such as herself that the bin on the left was for material to be recycled. This person would, she would suppose, have selected a conventionalised sign that simply and effectively distinguishes the two bins to provide immediate effects (letting her know where to put her paper napkin).

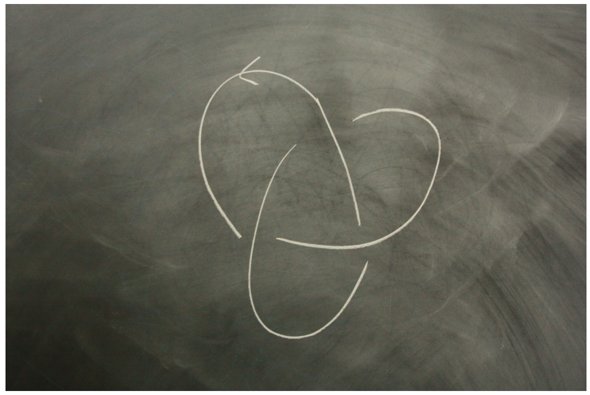

Figure 1: Trefoil knot diagram from "Heegaard Floer Homology and the Knot Concordance Group" by Jennifer Hom [23]. All boardwork in this article is reproduced by the author from board notes in the talks cited.

When we are communicating, we can expect that a speaker will select the most efficient route possible, and this expectation helps us as we go about interpreting his utterances. The principle is that in interpreting an utterance, an audience looks for an interpretation that is optimally relevant. Relevance here is given a technical definition as follows:

> "*(1) Relevance of an input to an individual*
>
> *a. Other things being equal, the greater the positive cognitive effects achieved by processing an input, the greater the relevance of the input to the individual at that time.*
>
> *b. Other things being equal, the greater the processing effort expended, the lower the relevance of the input to the individual at that time.*" [25, page 252]

It is important to note that relevance theory puts forward a view of communication that is not based in retrieving a message or any simple sense of acquiring knowledge. The cognitive effects mentioned may include "implications, strengthenings and contradictions resulting in the erasure of premises," [21, page 115], and are described in terms of establishing *mutual manifestness*. This is an interesting notion, weaker than that of mutual knowledge, which might best be understood as that which is made available to the hearer, rather than that which is consciously brought to mind. A speaker can aim to make certain assumptions mutually manifest in the audience's cognitive environment, and so develop shared assumptions.

2.2. A relevance-theoretic approach — how do we understand a diagram?

Grice famously drew a distinction between what he called non-natural meaning (meaning$_{\mathrm{NN}}$) and showing, distinguishing the kind of stimulus interpretation that requires consideration of intentions, and that which can occur without reference to a speaker's intentions. Grice claimed that while an utterance of "St. John is dead" *means*$_{\mathrm{NN}}$ that that is the case, Herod's presentation of St. John's head on a spike is rather an example of *showing*; after all, "Salome can infer that St. John the Baptist is dead solely on the strength of the evidence presented, and independent of any intentions Herod has in presenting her with his head" [13, page 218]. Though Grice's account draws a clear divide between the two, relevance theorists have reconsidered this question, arguing that it is something more like an interplay of both direct and indirect evidence that is considered in utterance interpretation.

Tim Wharton makes the case that relevance theory "recognises both showing and meaning$_{\mathrm{NN}}$ as instances of overt intentional or ostensive-inferential communication", and argues that showing and meaning ought be considered as existing on a spectrum. Wharton emphasises "the precise nature of the evidence provided" [24, page 16], noting that "ostensive stimuli are often highly complex composites of different, inter-related behaviours which fall at various points between 'showing' and 'meaning$_{\mathrm{NN}}$'" [24, page 19]. Wharton refers to Grice's case of a private detective attempting to let Mr X know that his spouse has been unfaithful. If the detective is able to acquire a photograph showing Mrs X embracing another person then Grice argues that this need only be left somewhere where Mr X might find it for him to reach that belief, whereas if the detective were to draw a picture of the same event (a more indirect form of 'evidence'), Mr X would surely need to wonder why he was being shown this to reach the desired conclusion. Wharton makes the point that directness is not the only criterion; if the photograph is a little blurry so that the faces are not clearly seen, Mr X may well need to know or to guess that he is being intentionally shown this photograph by the private detective to realise that his spouse is in the picture and reach that conclusion. What's more, in practice, whenever we encounter a picture in the world we have some theories in our minds as to why it is we're seeing it and who (roughly) has put it there. This picture of showing and meaning suggests that intentions are considered to varying extents in all kinds of situations, and that this is not simply dependent on the type of evidence provided, but a complex set of contextual attributes.

Danielle Macbeth considers diagrams in Euclid's elements to be "icons with Gricean non-natural meaning", but supplements this with an interesting discussion of the diagram as something that is itself used to reason with, stating that "one reasons in the diagram in Euclidean geometry, actualizing at each stage some potential of the diagram" [18, page 265]. This seems to hint at more complexity, making a space for something more like direct evidential reasoning to have a part in diagrammatic functioning. Diagrams clearly occupy an interesting position between these types of meaning, as tools with language-like properties but more direct, evidential aspects as well, and one that is perhaps better described using relevance theory's spectrum-like picture than standardly Gricean ideas.

A problem that has been noted with moves to discuss images using theory with its origin in discussion of language is that even relatively straightforward pictures can admit an excess of possible interpretations, as demonstrated by Charles Forceville and Billy Clarke in "Can Pictures Have Explicatures?" [7]. Because even the simplest picture could be transformed into a wide variety of descriptive sentences, they argue that pictures cannot have explicatures in the classical, language-oriented, Gricean sense, though they do make the case for a kind of very explicature-like functioning in some cases where elements are combined in sufficiently rule-like ways as to more closely constrain translation [7]. Sperber and Wilson define an explicature as follows:

> "*An assumption communicated by an utterance U is explicit if and only if it is a development of a logical form encoded by U.*" [21, page 182]

There is an assumption (brought into play in Forceville and Clark's discussion) that this logical form is language-like, in the sense of involving conventionally-defined, combinable elements with some kind of grammar, that thus constrain the possible interpretations.

In the case of diagrams as used in mathematical work, interpretation can indeed be constrained to a great extent. With training in the use of a particular diagram, just such a conventional, constraining grammar can be established. Valeria Giardino in *A Practice-Based Approach to Diagrams* notes that as with language, we must learn the correct ways to interpret a diagram, as "... the diagram user — its interpreter — is not interested in many of the visual properties of the diagram; she only attends to a selection of them ...

diagrams are not simply 'seen' but must be 'read', i.e., interpreted" [9, page 142]. This is a part of her account that hovers between the linguistic and the visual.

With training, attention can be focused on certain features of a diagram to the exclusion of others, as Giardino notes:

> "*Diagram-users share something like the experience of seeing in a diagram what they have to, focusing their vision on a selection of relevant features, which will bring them, as rational agents, to understand and reproduce the relevant features of diagrams in a non-mechanical way and without 'damages'.*" [9, page 146]

What this means is that a diagram-user develops an understanding of a diagram as having certain features that are important and relevant and others that are not. This understanding means that the re-drawing of the diagram relies not upon imitation of the particular form, but upon a reconstruction built up from certain important features.

Let's return to our mathematician encountering the symbol in Figure 1. If she were to encounter a sketch like this in a simple but expensive frame in a serious gallery, she might take quite a different approach to it than in the case of the recycling symbol. Assuming it to be a work of art, she would be far more likely to pay attention to the particulars of the actual form before her, considering the exact trajectory that each line takes and wondering what kind of cycle or vortex the artist was hoping to evoke. Her beliefs about the communicator and about the intended audience inform not only her interpretation of the form, but also the extent to which it is encountered as an image, rather than a symbol with conventional meaning in the case of the recycling sign interpretation, or a codified, structured form in the case of the knot diagram interpretation. In making a distinction between text and image, Victor Fei Lim uses the example of a smiley face icon, which, if its mouth were rendered as a red brushstroke rather than a black printed line, would be read very differently; the meaning of the image would alter, whereas for a text the meaning would still stay essentially the same [5, page 227]. If it is interpreted as such, factors such as the quality and colour of the lines of a knot diagram are, to a person who is familiar with them, unlikely to alter the way in which the diagram is 'read'. In the example given in Figure 1,

the mathematician is very much less likely to pay attention to whether the image is drawn in white or in vivid red if she encounters it in the setting of the classroom, rather than that of the gallery.

To a trained audience member, then, it may be that a diagram acquires a more language-like status. Only certain elements are considered to be useful, and their combination is constrained by sets of rules. Diagrams are treated with suspicion by some mathematicians as a result of what some consider to be their perilous underdeterminedness [8].

The attribution of intention may even account for the effectiveness of even the messiest of diagrams; often the knots drawn in talks are so hastily sketched that the lines either side of a crossing are wildly misaligned, yet this causes no problem for the audience's comprehension. As Giardino states: "... the diagram does not need to be properly drawn, as long as the user is aware of the prescriptions contained in the instructions for its construction and is aware of its intended meaning," [9, page 149]. Only the discrete, relevant features are important and need to be reproduced, and many of the qualitative features may be discarded along the way.

3. The Materials and Settings of a Conference Talk

The audience attending a talk at a conference can be very diverse in terms of the level of understanding of the work being presented. Some attendees will be working on something very closely related, and understand everything that is being discussed, some will be working in the same field and understand some of the content, and some will be students, there simply to familiarise themselves with the field.

Such presentations play a very different role when compared with the publication of papers. A piece of work is established and accepted by publishing a paper in a peer-reviewed journal, where the paper will be carefully read by reviewers and each step checked; once the paper passes, it is considered reliable enough to be built upon. A conference presentation instead outlines the main ideas in the argument (not necessarily covering all of the detail), and 'sells' both the work and the topic as interesting areas to work on. The detailed, exhaustive presentation in a paper is necessary if the reader is to work with a topic, but in a presentation a presenter aims to pique interest,

outline basic principles, and give the audience a window into a field of enquiry. As such, the setting demands a particular kind of careful, effective communication.

3.1. Blackboard use

Greiffenhagen notes the ubiquity and centrality of blackboards in mathematics departments [12, page 506], very unusual in an age when most disciplines are happily migrating to slide presentations. In a set of interviews with attendees at the Knots in Hellas conference in Greece, 2016,[1] I encountered several arguments in favour of blackboard use. These properties are certainly not exclusive to blackboards; some or all might be found in other methods, such as slides or handouts. If we are to explain the particular popularity of blackboards, though, it seems worthwhile to consider the combination of features that make them effective and exactly how these are manifested.

1. Pacing. Blackboards slow a presenter down, by virtue of the laborious writing process. This can create long pauses, and these can also encourage a presenter to elaborate or add more detail while drawing or writing.
2. Incrementality. A presenter must write or draw everything, from the ground up. This may re-enact something of the process of enquiry, or allow an audience member to gradually build up an understanding of the material on the board that matches the presenter's.
3. Spatiality. Purpose-built pictures can be drawn according to need, which can be difficult to do on a computer. A presenter can also give the audience clues about the importance or role of a particular statement by placement or style; for example, writing a statement in large letters in the centre of the board would indicate that this is a key statement in the main flow of the talk, whereas writing it in cramped letters in a corner might suggest that it was an interesting and related side note or relevant background assumption.
4. Acting as an 'external memory'. Blackboards can often be moved up and down, allowing notes to remain visible for a long time before they are rubbed off again, unlike a slide projection in which the contents of

[1] Interviews conducted with Dr. Andrew Fish, Dr. Josh Howie, and David Freund, mathematicians and students of mathematics.

> a slide are invisible as soon as the presenter moves on. For the audience this allows artificial ease of recall if they want to refer back to points mentioned earlier in the talk, which compensates for their relative lack of familiarity with the topic.

These arguments might be understood in terms of James Gibson's notion of *affordances*, which can be understood as that which an object or environment makes available to a person [10], for example, the affordance for holding provided by the handle on a teapot. The affordances of a blackboard might facilitate practices that help an audience member to achieve comprehension. This is particularly useful because it is very difficult as a presenter to remain adequately conscious of the fact that an audience member does not have the same background knowledge and familiarity with the material.

3.2. The diagram as communicative device

Knot theory is a field with a particularly interesting, formalised diagramming practice, described by Marcus Giaquinto in his extended study of visual thinking in mathematics [8], and philosopher James Robert Brown in a chapter that also notes the landscape of knot diagrams and the absence of research on the topic [3, pages 84–98].

A remarkable amount of time and energy is spent on diagramming in presentations. The explanation of a particular concept in a talk tends to involve many small, useful sketches and illustrations in comparison with the more austere, textual presentation seen in the paper. To understand why there is this difference we can consider the aims and goals of presentations: to give ready access to the main ideas of a piece of mathematics, to "sell" a mathematical field as a good and interesting place to study, and to show the audience something of the way that those working in the field are understanding the content.

Diagrams have been said to offer various helpful affordances, such as the possibility of using devices such as proximity and visual rhythm to establish associations between elements [15, page 71]. Silvia De Toffoli and Valeria Giardino describe diagrams as tapping into a manipulative imagination that allows us to use them like tools, which they outline as follows:

> "*Experts perform actions on diagrams by re-drawing them in appropriate ways, according to the way they interpret them. For*

this reason, novices need to train their imagination in order to recognize the various possible moves on diagrams, and then be able to effectively use them. Moreover, these manipulations are similar to the manipulations we can perform on concrete objects, but instead of having a pragmatic aim, they have an epistemic one (Kirsh and Maglio 1994). The use of diagrams triggers a form of manipulative imagination that gets enhanced by the practice." [4, page 836]

The gradual, enactive work of blackboard use may help to train us in these manipulations. Seeing a diagram being built up by a live presenter gives us a different kind of access to its possible uses and constraints.

4. Guided interpretation in conference presentations

Although many forms of diagram are commonly used, innovation is always occurring, and when a new usage or form crops up it can be necessary to guide an audience member through its interpretation and manipulation, in order to establish an understanding of its affordances that is shared between speaker and audience. Line drawings can be given quite a range of interpretations; in particular it is important to establish the permissible manipulations of an element. This could be understood as training a viewer to know how the diagram can be manipulated and what the 'grammar' of the permitted moves might be. A solid diagonal line met on either side by two shorter perpendicular lines might potentially be seen as a crossing, or as a cut, or as the joining of two unlike elements, or of three, depending on the perspective and experience of the viewer. Although such basic elements can be very well established, new applications or developments are happening all the time (such as in the establishment of new conventions for virtual knots) and training may be taking place more broadly than is immediately apparent. In one talk, in which the presenter demonstrates certain geometrical operations on knots, the presenter often speaks as though training, using phrasings like:

"*Step two … is you cut the whole thing in half along the projection plane …*" [22]

In relevance theoretic terms, this might be described as a process of adding information about the speaker's intentions for the diagram, in order to strengthen the manifestness of the interpretation used by experts and make it the

easiest one to reach. This is achieved through a process of guiding attention, creating expectations, and bringing in contextual knowledge that can be assumed to be mutual. These are particularly visible to an outsider — more so than communication that occurs simply by code interpretation — because often this training makes use of explanatory forms that require little specialist knowledge, such as information that is made available and referable on the blackboard, and metaphorical explanations that use far more commonly-known contextual references.

Observing as a non-mathematician, communicative strategies that used resources outside of the code of mathematics were particularly evident to me. The following is a non-exhaustive list:

1. Combining images and notation to make use of the properties of more than one mode
2. The strategic adding of ostension to increase the manifestness of certain aspects of a diagram
3. Bringing in commonly-shared contextual knowledge by means of metaphor
4. Providing an appropriate cognitive environment by means of surrounding statements or images that ape the cognitive environment of the expert

Below I detail some real-world examples that exemplify these strategies being used in presentation scenarios.

4.1. Labelling shifts perceptions

Labelling can transfer qualities from one mode to the other; from text to image, for example. Bringing a second mode into a diagram creates an expectation that some feature of, say, letter-labels is going to prove useful, causing an audience member to look for an interpretation of the diagram that will make use of the features of letter-labels that they know. This might strengthen the manifestness of features that have some aspect in common with some already-known property of the imported mode. In Figure 2, the board notes that accompanied a section using such a strategy are shown.

At this point during the talk, the speaker is identifying the ways in which components of the drawing will correspond to components in the polyhedron she is beginning to describe.

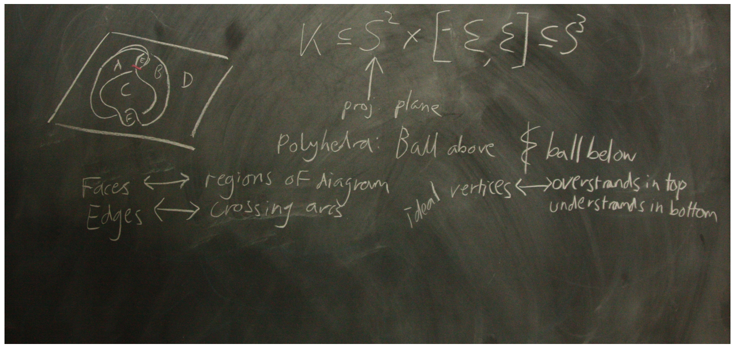

Figure 2: Author's recreation of the board work from Jessica S. Purcell's talk "Geometric Techniques in Knot Theory" [22]. Time stamp on IAS / YouTube video: 08:44

She labels the regions in between the loops of the knot with the letters A-F: "*They're gonna meet in faces. So lemme label these faces ABCDEF [. . .] So the faces are going to correspond to regions of the diagram.*" One property of letter-labels is that each label is equally weighted; they look different and sound different when pronounced, but are always interchangeable.[2] This labelling move makes use of the audience's contextual knowledge of letters and uses this to strengthen an interpretation of the diagram: that of understanding the areas between the loops as belonging to a class of things, 'regions in a diagram'.

Next, when she wants to draw attention to another component she refers back to the letters, making use of another property, their facility for identification: "*. . . you'll get an edge when um– two of the faces bump into each other– and they're going to bump into each other at crossings. So right here for example there's gonna be an edge between face E and face B on this polyhedron that's coming up above. [. . .] That's gonna give you an edge of the polyhedron.*" Having drawn attention very efficiently to two particular regions she adds a line that draws attention to the crossing at which they meet, speaking about a class of 'areas where the regions meet'; the line is very small, so she uses red chalk to make it more salient. This time she simply adds one annotation to one section of the diagram; the idea that such a move will apply to every such

[2] Goodman's theory of symbols contain a set of reasonable assumptions about diagrams, and this includes the assumption that things given a similar graphical representation should be similar [20].

instance in the diagram has been favoured and made available to the audience by the labelling move. In the previous labelling process, the audience was trained to understand that the move is to be repeated for each example of a particular thing in the diagram, so repetition isn't necessary.

Finally, the speaker states "*then the vertices are the remnants of the knot, in fact they're going to correspond to overstrands*", and traces over one of the crossing arcs she had drawn for a second time. This time she doesn't add anything to the diagram but ostensively reinforces existing features of the diagram, the effect being to draw attention to these strands as similarly a class of things with a new place in the polyhedron.

4.2. Emphasis as signal

By adding emphasis with vocal cues or by spending extra time on something, a speaker creates an expectation in a hearer that there must be additional effects to justify the extra interpretive effort that such moves create. This can make it clear that a certain aspect requires particular attention, or create expectation that there is extra work to be done. A discussion of Figure 3 will illustrate this.

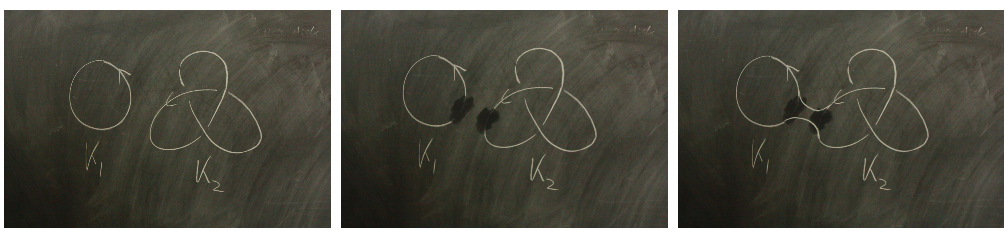

Figure 3: Author's recreation of the board work from Jennifer Hom's talk "Heegaard Floer Homology and the Knot Concordance Group" [23] explaining the connected sum operation. Time stamp on IAS / YouTube video: 01.21–01.34

The speaker begins by stating:

> "*So there's an operation on knots called the connected sum [...] you just have two knots and um you connect them.*"

To exemplify this operation, the speaker keeps the images on the board and acts directly upon them. Having connected the two images, she makes a gesture that emphasises the orientedness already suggested by the arrows.

This added layer of ostension makes the shared orientation more salient to the audience. She highlights it using a hand gesture, introducing animation into the diagram arrangement. The implication of the arrows suggesting a particular route becomes considerably more manifest, using the familiar progress of movement around a track to favour the interpretation of the knot diagrams as specifying a particular route that must be respected.

4.3. Building up a shared understanding of a diagram

Figure 4 shows the board notes from a particular section of a talk.

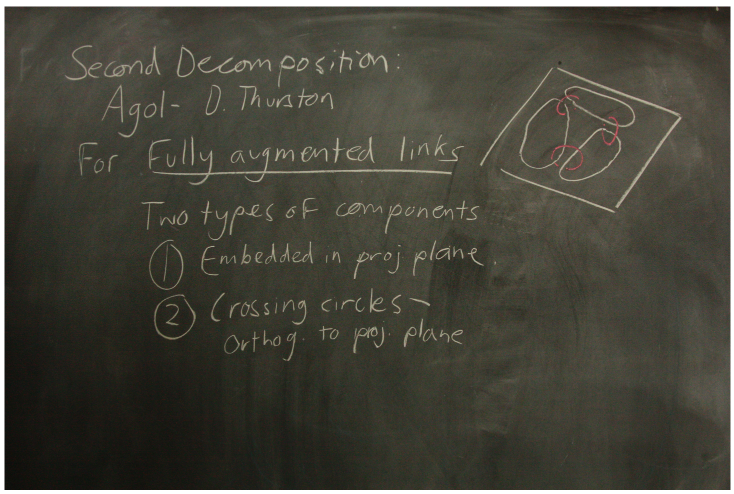

Figure 4: Author's recreation of the board work from Jessica S. Purcell's talk "Geometric Techniques in Knot Theory" [22]. Time stamp on IAS / YouTube video: 11.50

The speaker:

1. Draws the projection plane,
2. Draws the embedded links (labels),
3. Draws small example orthogonal links,
4. Highlights where they'd 'cross',
5. Draws them in on the diagram.

The speaker states, "*there are components which lie entirely in the projection plane*", and draws a parallelogram, a common diagramming device that represents a two-dimensional plane. By sketching a roughly perspectival tabletop-like rectangular area, the diagrammer sets up a limited two-dimensional field

in which the rest of the diagramming can proceed. The ostensive stimulus, the sketch, is extremely rough, but the shared knowledge of diagramming conventions picks up a lot of the slack, and the speaker is aware of this, meaning that any additional time spent on this would likely be wasted. Once the plane has been drawn, it sets up an expectation that its two-dimensionality will be used, like Chekov's gun. She states "*there are some components which just sit in here*"; the three "embedded links" sit within the area of the board that she has delineated and also within the remit of the diagramming conceit, to be read as lying within it.

In the next stage, she names the second type of component 'crossing circles' and states that they are orthogonal to the projection plane. At this stage, rather than adding to the main diagram she draws a new, miniature one, beginning with a vertical oval, then a small parallelogram, and adding two additional lines on either side: "*so here's the projection plane. It's orthogonal to that and exactly two strands of – of the link run through it.*" She emphasises two dots, one on each side of the oval (see Figure 5).

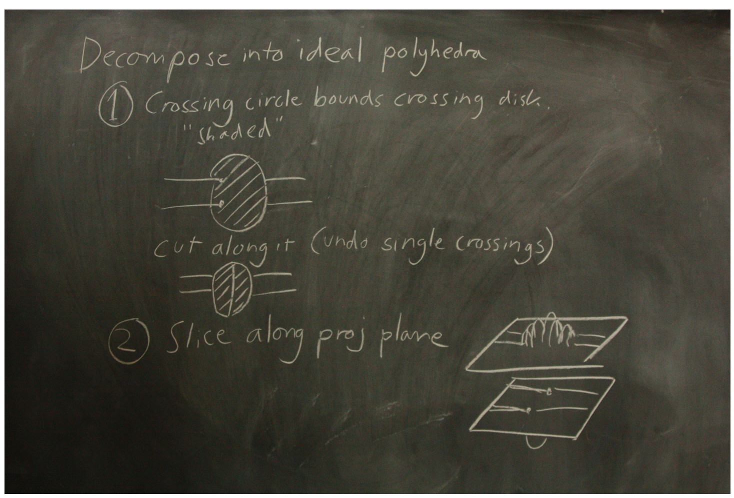

Figure 5: Author's recreation of the board work from Jessica S. Purcell's talk "Geometric Techniques in Knot Theory" [22]. Time stamp on IAS / YouTube video: 17.02

This extra effort is to propose the idea that the circle is not lying within the established plane but at 90 degrees, intersecting the plane at two points. This move is introduced in a much more careful way, and the ways in which the plane and circle are to intersect are carefully outlined using the dots;

adding this evidence of intention encourages the audience to go slightly further in interpreting the relationship of the circle to the plane. The small extra diagram uses the audience's familiarity with spatial interactions and increases the manifestness of the two points where the circle and plane might intersect so that the slightly more effortful interpretation that the circle is not within the plane, but at ninety degrees to it, will be reached by the audience.

The audience having been persuaded to 'see' the circle and two lines as a three-dimensional arrangement, Purcell goes on to expand on these in another set of side diagrams. Purcell states, "*the first thing you're gonna do is view each of these crossing circles as bounding a disk*", and shades in the interior of the circle. These moves serve to morph the audience's understanding of the sketched circles on the board, shifting them from circle-like things (round, empty, constructed from one line) to disk-like things (flat, solid). In this side diagram, the lines also truncate earlier than usual, as they would when passing through an opaque surface.

The next move is outlined verbally as follows:

> "*and then I'm gonna cut along it ok <writes 'cut along it'>*
>
> *so just you can think of this this is a full slice of pitta bread and its <draws over drawing>*
>
> *I I sliced it down the middle so it's open inside*
>
> *so you can kind of push it apart from itself ok <pushing apart gesture>*"

The speaker is working to increase the mutual manifestness of the two-sidedness of a disk form. By invoking the pitta bread comparison, the speaker strengthens the idea of two sides that could potentially be separated, bringing in a tactile example whose manipulation would be familiar from everyday life for much of the audience — an embodied, action-based analogy that helps the audience to see how the diagram should be treated. This strengthens the idea of two-sidedness even to the point that the idea of flatness can be slightly violated, as the notion of separating the two sides comes in. Having established this idea, she transfers it to the main diagram, indicating this by physically shading the crossing circles that occur within it.

Purcell makes use of understandings of objects to gradually alter and develop our understanding of the diagram. Each of these diagrams can be understood as nested within the original one; she is not altering the thing being represented, only the audience's understanding of the diagram itself. She makes additional features of our understanding of space and forms manifest at different points in such a way as to alter an audience's view of a knot or link of this form. It isn't a case of simple substitution, though; the added description increases the manifestness of properties that are already weakly manifest in our very large and assumable shared experience of space and form.

The idea is to get the understanding of the diagram across as quickly and efficiently (i.e., as relevantly) as possible. With a mixed audience, the speaker benefits from making use of this kind of shared experience (rather than a more coded, notational presentation requiring a lot more shared knowledge and work) which provides essential pieces of ready-made context. The pitta is not only something that everyone has used, it's something that everyone has manipulated in just the way she describes, so that manipulation easily grows into her presentation of the way to read the disk.

5. Conclusion

I have endeavoured to sketch a process by which materials and broad communicative resources are included in mathematical practice, in a way that establishes shared understandings of even relatively codified forms. The speaker encourages the audience to see a chalked form as she does, and so to appreciate the moves that are permitted in its manipulation, and understand its affordances. By careful direction, and the addition of ostension, a speaker encourages the audience members to build up a subtle understanding of the object grounded in their own cognitive environments, using not only the specialist knowledge that some proportion of the audience will share but also broader contextual knowledge deployed in careful and selective ways.

There is a question here about whether diagrams operate in chiefly language-like or picture-like ways, and the answer appears to be that aspects of each are at play, in complex and interlocking ways. On the one hand, this process of training shows both that their interpretation according to the norms of the discipline is dependent on sharing and recognising the intentions behind

their use; on the other, the use of metaphorical descriptions to build up an understanding of a diagram as a manipulable object-like thing points to the ways in which diagrams selectively make use of non-conventionalised, direct experience.

Study of mathematics that takes seriously the interpersonal and material resources used in the development of mathematics will help put forward an image of the discipline as bound up with the same messy, material, interactive negotiations as the rest of human endeavour.

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