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An Exponential Formula for Random Variables Generated by Multiple Brownian Motions

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Approval of the Dissertation Defense

This dissertation has been duly read, reviewed, and critiqued by the Committee listed below, which hereby approves the manuscript of Maximilian Lawrence Baroi as fulfilling the scope and quality requirements for meriting the degree of Doctor of Philosophy in Education.

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To my parents Deb and Ion

My sister Liz

And my teammate Shannon

Abstract

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The frozen operator has been used to develop Dyson-series like representations for random variables generated by classical Brownian motion, Lévy processes and fractional Brownian with Hurst index greater than $\frac{1}{2}$. The relationship between the conditional expectation of a random variable (or fractional conditional expectation in the case of fractional Brownian motion) and that variable's Dyson-series like representation is the exponential formula. These results had not yet been extended to either fractional Brownian motion with Hurst index less than $\frac{1}{2}$, or d-dimensional Brownian motion. The former is still out of reach, but we hope our review of stochastic integration for fractional Brownian motion and our results for the later will provide a framework.

Examining the case of d-dimensional Brownian motion in general, and two-dimensional Brownian motions in specific, have led to a number of new insights. The first of which, is realizing the component operators in the Dyson-series expansion can be written concisely as iterated applications of the Gross Laplacian. The original choice of "Dyson-series" as nomenclature was to suggest some connection between the original results and expressions which occur in quantum field theory. There have always been connections between financial and mathematical physics, and expressing the exponential formula in terms of a foundational operator originally used to study the theory of potentials on Hilbert space suggests another.

The second major insight: is to realize the natural domain of the freezing operator on stochastic integrals is as an operator over Stratonovich integrals. Freezing a Skorokhod integral has always been a challenge. The naïve assumption of how they interact is wrong, and there was little intuition in the ensuing calculations. However, the naïve assumption does work for Stratonovich integrals. In retrospect, this is understandable. The mental model for the frozen operator is in terms of realizations of paths. One would expect the integral which emphasizes the geometric nature of

stochastic processes to be a more natural fit.

These two developments have allowed us to prove an exponential formula for random variables generated by two Brownian motions, and apply those results to the SABR model. There has been work in this field, but the extension from one Brownian motion to two Brownian motions was ad hoc. Our development is more systematic and more readily extended to d-dimensional Brownian motion.

Contents

Chapter 1

Introduction

A centered Gaussian process $\{B_t^H, t \geq 0\}$ is called a fractional Brownian motion (also written as fBm) with Hurst parameter $H \in (0,1)$ when the autocovariance function is of the form

$$R_H(s,t) := \mathbb{E}\left[B_s^H B_t^H\right] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right). \tag{1.1}$$

When H is fixed or clear from the surrounding context, we may simply write B_t or R. We may also write B_t to denote classical Brownian motion, but in those contexts we will explicitly state so, and any fractional Brownian motion referred to within the same context will be written as B_t^H.

Though it was examined before, the phrase "fractional Brownian motion" and "Hurst" index first appears in [MN68]. Within that paper, Mandelbrot and Van Ness state and prove the fundamental analytical and probabilistic properties of fBm; including the moving average representation

$$B^H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}} (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} dB(s), \tag{1.2}$$

of an fBm B_t^H (where $(x)_+$ denotes the maximum of x and zero) as an integral against a classical Brownian motion B. Hurst noticed a self-similar pattern in the historical data concerning frequency and volume of excess discharge from the Nile river [Hur51]. Hurst, along with coauthors Black and Simaikah, cataloged further empirical examples of self-similar phenomenon [HBS65]. Fractional

Brownian motion is not only of empirical interest, but is a natural generalization of classical Brownian motion. We borrow a diagram which we first found in a much better dissertation [Ros09]. It is a

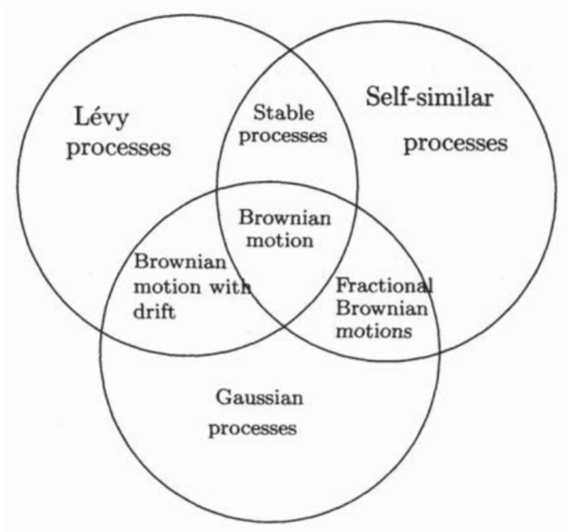

Figure 1.1: Interrelation between families of stochastic processes. Taken from [CT03]

wonderful diagram even if it is only mostly true. There are self-similar Gaussian processes which are not fractional Brownian motions, such as sub-fractional Brownian motions [BGT04], but we will not concern ourselves with that class. And the correct characterization of an fBm B_t^H is as the unique centered Gaussian processes which satisfies the self-similarity condition

$$B_{ta}^H \sim |a|^H B_t^H, \tag{1.3}$$

for every positive a, and whose increments are stationary

$$B_{t+h}^H - B_h^H \sim B_t^H. \tag{1.4}$$

We fix $\mathrm{Var}[B_1^H] = 1$ to preclude admission of arbitrary scalings. To prove this characterization is a simple exercise. A process which satisfies the above conditions has some autocovariance function $R_H(s,t)$. We know $R_H(t,t) = t^{2H} R_H(1,1) = t^{2H}$ from the self-similarity condition. Then consider

$\mathrm{Var}(B^H_{s+h} - B^H_s)$. Since increments are stationary, this quantity must equal the variance of B^H_h, which is h^{2H}. And a direct calculation shows

$$\begin{aligned} \mathrm{Var}\, B^H_{s+h} - B^H_s =& \mathbb{E}\left[(B^H_{s+h} - B^H_s)^2\right] \\ =& \mathbb{E}\left[B^H_{s+h} B^H_{s+h}\right] - 2\mathbb{E}\left[B^H_{s+h} B^H_s\right] + \mathbb{E}\left[B^H_s B^H_s\right] \\ =& (s+h)^{2H} - 2R_H(s+h, s) + s^{2H}. \end{aligned} \tag{1.5}$$

Combining these two results, we see

$$\begin{aligned} h^{2H} =& (s+h)^{2H} - 2R_H(s+h, s) + s^{2H} \\ \Rightarrow R_H(s+h, s) =& \frac{1}{2}\left((s+h)^{2H} + s^{2H} - h^{2H}\right). \end{aligned} \tag{1.6}$$

If we substitute $h = t - s$, remember how we tacitly assumed $t \geq s$, and note R_H must be symmetric, then we find

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{1.7}$$

which is precisely what we wish to prove. A nice and practical consequence of the stationarity of increments of fBm is we can appeal to more advanced and efficient methods of simulation like [WC94]. However, our focus is to utilize the frozen path operator introduced in [JPS15a] to *avoid* simulations. We want to develop methods to estimate the expectations of random variables using only deterministic expressions and integrals.

The majority of our attention will be focused on fBm with Hurst index $H < \frac{1}{2}$. Our goal, whose time horizon lays beyond this dissertation, is to eventually develop an exponential formula for random variables generated by a set of fBm with Hurst index $H < \frac{1}{2}$. To accomplish this will require us to extend the exponential formulae in [JPS15a] to processes generated by multiple classical Brownian motion (Chapter 4), and then try to generalize that result using the knowledge of stochastic integration against fBm with Hurst index $H < \frac{1}{2}$ (Chapter 3). Underpinning these two developments will be our general knowledge of isonormal Gaussian processes and re-interpreting old results under that framework (Chapter 2). Hopefully, this dissertation will provide enough of a framework to successfully tackle this original problem in future papers.

As we survey stochastic integration of fBm with Hurst index $H < \frac{1}{2}$, we will learn more about the qualitative differences between the various classes of fractional Brownian motions. An immediate difference is seen when examining the covariance between disjoint increments.

- When $H > \frac{1}{2}$, they are historical; the correlation is positive
- When $H = \frac{1}{2}$, they are ahistorical; they are independent
- When $H < \frac{1}{2}$, they are anti-historical; the correlation is negative

If we were able to rewrite the literature, we would use the terms "(historical/anti-historical) fBm" instead of "fBm with Hurst index (greater/less) than $\frac{1}{2}$". But it is for the best we do not have that power.

Chapter 2

The Malliavin Calculus on Isonormal Gaussian Processes

We develop the Malliavin calculus for isonormal Gaussian processes. Our program will mainly follow Chapter 2 of [NP12], but we will also use Chapter 1 of [Nua06] as a guide. Any result which we do not ourselves prove will be appropriately cited.

2.1 Isonormal Gaussian Processes

One is first taught to consider stochastic processes as a collection of time-indexed random variables. But there is a more expansive interpretation which allows us to more readily leverage functional analytic tools. Instead of indexing over a parameter or set of parameters, we instead consider a process indexed over a Hilbert space.

We fix a real separable Hilbert space $\mathfrak{H}$ with an inner-product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$, and consider a process $X = \{X(h); h \in \mathfrak{H}\}$ indexed by $\mathfrak{H}$ (one can also say a process over $\mathfrak{H}$). If X is a collection of centered jointly Gaussian random variables, over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}[X(h)X(h')] = \langle h, h' \rangle_{\mathfrak{H}}, \tag{2.1}$$

then we call X an isonormal Gaussian process over $\mathfrak{H}$. The map $h \mapsto X(h)$ is indeed linear since

$$\begin{aligned}
&\mathbb{E}\left[(X(ah) - aX(h))^2\right] \\
=&\mathbb{E}\left[X(ah)^2\right] - 2a\mathbb{E}[X(ah)X(h)] + a^2\mathbb{E}\left[X(h)^2\right] \\
=&\|ah\|_{\mathfrak{H}}^2 - 2a\langle ah, h\rangle_{\mathfrak{H}} + a^2\|h\|_{\mathfrak{H}}^2 \\
=&0,
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&\mathbb{E}\left[(X(h+h') - (X(h) + X(h')))^2\right] \\
=&\mathbb{E}\left[X(h+h')^2\right] - 2\mathbb{E}\left[X(h+h')X(h)\right] - 2\mathbb{E}\left[X(h+h')X(h')\right] \\
&+ \mathbb{E}\left[(X(h) + X(h'))^2\right] \\
=&\|h+h'\|_{\mathfrak{H}}^2 - 2\langle h+h', h\rangle_{\mathfrak{H}} - 2\langle h+h', h'\rangle_{\mathfrak{H}} \\
&+ \left(\mathbb{E}[X(h)^2] + 2\mathbb{E}[X(h)X(h')] + \mathbb{E}[X(h')^2]\right) \\
=&-\|h\|_{\mathfrak{H}}^2 - 2\langle h, h'\rangle_{\mathfrak{H}} - \|h'\|_{\mathfrak{H}}^2 \\
&+ \|h\|_{\mathfrak{H}}^2 + 2\langle h, h'\rangle_{\mathfrak{H}} + \|h'\|_{\mathfrak{H}}^2 \\
=&0.
\end{aligned} \tag{2.3}$$

By construction this linear map is an isometry.

If we had a "traditional" centered Gaussian process $X = \{X_t\}$ with a covariance function $R(s,t) = \mathbb{E}[X_s X_t]$ we can construct a representation of X as an isonormal Gaussian process. This representation is entirely induced by the canonical map $1_{[0,t]} \mapsto X_t$, which we extend by linearity. Any simple function of the form $h = \sum_i a_i 1_{[0,t_i]}$ is mapped to the random variable $\sum_i a_i X_{t_i}$. The space of simple functions has an inner-product defined by $\langle 1_{[0,s]}, 1_{[0,t]}\rangle = R(s,t)$ and extended by bi-linearity. We set $\mathfrak{H}$ to be closure of simple functions with respect to the norm induced by this inner-product, and X can be considered an isonormal Gaussian process over $\mathfrak{H}$. So we can "embed" our old notion of stochastic processes within the field of isonormal Gaussian processes.

2.2 Wiener-Itô Chaos Decomposition

The path before us is laid out in Section 1.1 of [Nua06]. To prove the Wiener-Itô chaos decomposition we first need some ancillary results and calculations. We start with an isonormal Gaussian process X over the Hilbert space $\mathfrak{H}$. Unless explicitly noted otherwise, we always assume the σ-algebra of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is generated by the collection X and is complete. Furthermore, we assume the probability space has no atoms i.e, there is no set A such that $\mathbb{P}[A]$ is positive but $\mathbb{P}[B]$ is zero for every proper subset $B \subset A$.

In general, if we let Y and Z be standard normal random variables then $sY + tZ$ is a normal random variable with mean zero and variance equal to $s^2 + t^2 + 2\mathbb{E}[YZ]$. e^{sY+tZ} is of course log-normal and therefore

$$\mathbb{E}\left[e^{sY-\frac{s^2}{2}}e^{tZ-\frac{t^2}{2}}\right] = e^{\mathbb{E}[YZ]}. \tag{2.4}$$

We know $e^{tx-\frac{t^2}{2}}$ is the generating function of the Hermite polynomials, see Proposition A.7, and therefore $e^{tx-\frac{t^2}{2}} = \sum_k t^k H_k(x)$. Plugging this relation into Equation (2.4) produces the identity

$$\sum_{k,\ell=0}^{\infty} s^k t^\ell \mathbb{E}[H_k(Y)H_\ell(Z)] = \sum_{m=0}^{\infty} s^m t^m \frac{1}{m!}\mathbb{E}[YZ]^m. \tag{2.5}$$

These two expressions must have equal coefficients when considered as power series in s and t, and we therefore have the following result

Proposition 2.1. *For any two standard normal random variables Y and Z*

$$\mathbb{E}[H_k(Y)H_\ell(Z)] = \begin{cases} 0 & \text{if } k \neq \ell \\ \frac{1}{k!}\mathbb{E}\left[YZ\right]^k & \text{if } k = \ell. \end{cases} \tag{2.6}$$

Our process X might be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, but our results will generally refer to the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ where $\mathcal{G}$ is the complete σ-algebra generated by the random variables $\{X(h); h \in \mathfrak{H}\}$. Understandably, our results can generally only reference information which can actually be gleaned from observing X.

Examining the family of log-normal random variables $\{e^{X(h)}; h \in \mathfrak{H}\}$ is often useful.

Lemma 2.2. *The family $\{e^{X(h)}; h \in \mathfrak{H}\}$ is a total subset of $L^2(\Omega, \mathcal{G}, \mathbb{P})$ i.e., the span of this collection is dense in $L^2(\Omega, \mathcal{G}, \mathbb{P})$.*

Consider an L^2 random variable Z that is orthogonal to any member of the above collection. We follow Nualart's proof of Lemma 1.1.2 in [Nua06]. The crux of the proof is recognizing that an expression $\mathbb{E}[Ze^{t_1 X(h_1)+\ldots+t_m X(h_m)}]$ is always zero by assumption. We then recognize this expression is the Laplace transform of the signed measure $\mu(A) = \mathbb{E}[Z1_A(X(h_i), \ldots, X(h_m))]$ on $\mathbb{R}^m$ evaluated at $(t_1, \ldots, t_m)$. Our choice of parameters $t_1, \ldots, t_m$ was arbitrary and therefore the Laplacian of μ is identically zero and thus μ itself is identically zero. Again, our choices of A and the h_i are completely arbitrary, and we can thus conclude $\mathbb{E}[Z1_G] = 0$ for any $G \in \mathcal{G}$. From this we can conclude Z is zero. Hence, the span of the collection $\{e^{X(h)}; h \in \mathfrak{H}\}$ is dense in $L^2(\Omega, \mathcal{G}, \mathbb{P})$.

Now we define the family $\mathcal{H}_n = \{H_n(X(h)); h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$ for any natural n. $\mathcal{H}_n$ is called the n-th Wiener chaos and our L^2 space has the following orthogonal decomposition

Theorem 2.3.

$$L^2(\Omega, \mathcal{G}, \mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n. \tag{2.7}$$

Orthogonality between the Wiener chaoses follows from Proposition 2.1. We proceed as in the proof of Lemma 2.2 and consider a square-integrable random variable Z that is orthogonal to every n-th Wiener chaos. If we then fix a particular h, then $\mathbb{E}[ZH_n(X(h))]$ is zero for every natural n. Then $\mathbb{E}[ZX(h)^n]$ is zero for every n, and thus $\mathbb{E}[Ze^{X(h)}]$ is zero. Since this holds for any $h \in \mathfrak{H}$, then as in the proof of Lemma 2.2, we must have Z equal to zero. Therefore, the only element orthogonal to the above direct sum is zero, and hence the direct sum is equal to $L^2(\Omega, \mathcal{G}, \mathbb{P})$. Define J_n to be the projection from $L^2(\Omega, \mathcal{G}, \mathbb{P})$ to the n-th Wiener chaos, then we can alternatively express our theorem as

Theorem 2.4.

$$F = \sum_{n=0}^{\infty} J_n F, \tag{2.8}$$

holds for any F in $L^2(\Omega, \mathcal{G}, \mathbb{P})$.

Our eventual goal is to express the chaos expansion as a sum of iterated stochastic integrals with symmetric integrands. We will finally exploit our assumption that $\mathfrak{H}$ is separable. Let $\{e_n\}$

be an orthonormal basis of $\mathfrak{H}$. Then the countable collection of random variables $\{X(e_n)\}$ is an orthonormal system in $L^2(\Omega, \mathcal{G}, \mathbb{P})$. Now we generalize the Hermite polynomials. Let $\mathcal{J}$ be the collection of sequences of natural numbers with finite support i.e., $\boldsymbol{a} = (a_1, a_2, a_3, \ldots) \in \mathcal{J}$ where each a_i is a natural number and all but finitely many terms are zero. Then the norm $|\boldsymbol{a}| = \sum_i a_i$ and $\boldsymbol{a}! = \prod_i a_i!$ are of course well-defined. We can then define a collection of random variables indexed by $\mathcal{J}$ by the equation

$$\boldsymbol{H_a} = \sqrt{\boldsymbol{a}!} \prod_{i=1}^{\infty} H_{a_i}(X(e_i)). \tag{2.9}$$

If i and j differ, then $X(e_i)$ and $X(e_j)$ are independent and so are $H_m(X(e_i))$ and $H_n(X(e_j))$ for any natural m and n. Once again, we appeal to Proposition 2.1 to determine the inner product of elements in this system.

$$\begin{aligned} \mathbb{E}\left[\boldsymbol{H_a H_b}\right] &= \prod_{i=1}^{\infty} \sqrt{a_i! b_i!} \mathbb{E}\left[H_{a_i}(X(e_i)) H_{b_i}(X(e_i))\right] \\ &= \begin{cases} 1 & \text{if } \boldsymbol{a} = \boldsymbol{b} \\ 0 & \text{if } \boldsymbol{a} \neq \boldsymbol{b}. \end{cases} \end{aligned} \tag{2.10}$$

Thus, the collection of all $\boldsymbol{H_a}$ as $\boldsymbol{a}$ ranges over $\mathcal{J}$ is an orthonormal system.

The next step is to show $\{\boldsymbol{H_a}; |\boldsymbol{a}| = n\}$ is a complete orthonormal system for the n-th Wiener chaos $\mathcal{H}_n$. The actual argument is a fair amount of low-level calculation. Essentially, given an element $H_n(X(h))$ in $\mathcal{H}_n$ we construct a suitable approximation from elements of $\{\boldsymbol{H_a}; |\boldsymbol{a}|\}$. The crux is to calculate $\mathbb{E}[H_n(X(h))\boldsymbol{H_a}]$ and, while not difficult, it is not particularly illuminating. So we will relegate the calculation to the Appendix B. Since each collection $\{\boldsymbol{H_a}; |\boldsymbol{a}| = n\}$ is an orthonormal system for the n-th Wiener chaos, the overall collection $\{\boldsymbol{H_a}; \boldsymbol{a} \in \mathcal{J}\}$ is an orthonormal system for $L^2(\Omega, \mathcal{G}, \mathbb{P})$.

We now construct a space of symmetric kernels and define an abstract mapping I_n from the basis of this space of kernels to elements of $\{\boldsymbol{H_a}; |\boldsymbol{a}| = n\}$. When $\mathfrak{H}$ is the space of square-integrable functions on a measurable space without atoms, then our abstract operator I_n precisely coincides with iterated stochastic integrals. But for now we remain as abstract as possible since, asides from classical Brownian motion, the underlying Hilbert space which generates fractional Brownian motion is not of this form.

Let $\mathfrak{H}^{\odot n}$ be the n-fold symmetric tensor product of $\mathfrak{H}$. Given an orthonormal basis $\{e_i\}$ of $\mathfrak{H}$ we can construct a basis of $\mathfrak{H}^{\odot n}$. But first, we have to introduce a bit of notation. Now that we have fixed a basis for $\mathcal{H}$, then for any finitely supported $\boldsymbol{a} \in \mathcal{J}$ we can define an element

$$e_{\boldsymbol{a}} = \otimes_{i=1}^{\infty} e_i^{\otimes a_i}. \tag{2.11}$$

Then $e_{\boldsymbol{a}}$ is an element of $\mathfrak{H}^{\otimes n}$ (not $\mathfrak{H}^{\odot n}$!) whenever $|\boldsymbol{a}| = n$. For our second piece of notation, we introduce the symmetrization operator which takes elements from $\mathfrak{H}^{\otimes n}$ to $\mathfrak{H}^{\odot n}$ characterized by its action on basis elements

$$\operatorname{Sym}\left(e_{i_1} \otimes \cdots \otimes e_{i_n}\right) = \frac{1}{n!} \sum_{\sigma \in S_n} e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(n)}}. \tag{2.12}$$

Here, σ ranges over all permutations on the set $\{1, \dots, n\}$. If an expression e is of small enough width, we will write $\tilde{e}$ for $\operatorname{Sym} e$. For those not familiar with this tensor algebra notation, we can give some examples:

$$\operatorname{Sym}\left(e_1 \otimes e_2\right) = \frac{1}{2}\left(e_1 \otimes e_2 + e_2 \otimes e_1\right), \tag{2.13}$$

and the slightly more complicated

$$\begin{aligned}\operatorname{Sym}\left(e_1^{\otimes 2} \otimes e_2\right) =& \frac{1}{6}\left(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_2\right.\\ &\left. + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_1\right)\\ =& \frac{1}{3}\left(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1\right).\end{aligned} \tag{2.14}$$

In general, the tensor product is the only reasonable way to define a notion on multiplication for an arbitrary Hilbert space $\mathfrak{H}$. However, if $\mathfrak{H}$ is a collection of actual functions we can sometimes have a more concrete definition of $\mathfrak{H}^{\otimes n}$ and symmetrization. For example, if $\mathfrak{H}$ is $L^2([0,T])$, which would mean our isonormal Gaussian process is classical Brownian motion, then $\mathfrak{H}^{\otimes n}$ is isomorphic to the space of square-integrable multi-variable functions $L^2([0,T]^n)$ and symmetrization would simply be

$$\tilde{f}(x_1,\dots,x_n) = \sum_{\sigma\in S_n} f(x_{\sigma(1)},\dots,x_{\sigma(n)}), \tag{2.15}$$

as we would hope. Quick aside: when $\mathfrak{H} = L^2([0,T]^m)$, then our isonormal Gaussian process is a Brownian sheet [Kho14].

Returning from our digression; we imbue the space $\mathfrak{H}^{\odot n}$ with the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{H}^{\odot n}}$, which is defined as $n!\langle\cdot,\cdot\rangle_{\mathfrak{H}^{\otimes n}}$. Where the inner product on $\mathfrak{H}^{\otimes n}$ has the natural definition

$$\langle f_1\otimes\cdots\otimes f_n, g_1\otimes\cdots\otimes g_n\rangle_{\mathfrak{H}^{\otimes n}} = \langle f_1,g_1\rangle_{\mathfrak{H}}\cdots\langle f_n,g_n\rangle_{\mathfrak{H}}. \tag{2.16}$$

Now, consider the collection $\{\tilde{e}_{\boldsymbol{a}}; |\boldsymbol{a}| = n\}$; whose span is obviously dense within $\mathfrak{H}^{\odot n}$. Let us do the immediate ask of determining the inner-product between two elements $\tilde{e}_{\boldsymbol{a}}$ and $\tilde{e}_{\boldsymbol{b}}$. If we write $e_{\boldsymbol{a}} = e_{i_1}\otimes\cdots\otimes e_{i_n}$ and $e_{\boldsymbol{b}} = e_{j_1}\otimes\cdots\otimes e_{j_n}$ then we calculate

$$\begin{aligned}\langle\tilde{e}_{\boldsymbol{a}},\tilde{e}_{\boldsymbol{b}}\rangle_{\mathfrak{H}^{\odot n}} =& n!\langle\tilde{e}_{\boldsymbol{a}},\tilde{e}_{\boldsymbol{b}}\rangle_{\mathfrak{H}^{\otimes n}}\\ =&\frac{1}{n!}\sum_{\sigma,\sigma'\in S_n}\langle e_{i_{\sigma(1)}}\otimes\cdots\otimes e_{i_{\sigma(n)}}, e_{j_{\sigma'(1)}}\otimes\cdots\otimes e_{j_{\sigma'(n)}}\rangle_{\mathfrak{H}^{\otimes n}}\\ =&\sum_{\sigma\in S_n}\langle e_{i_1}\otimes\cdots\otimes e_{i_n}, e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(n)}}\rangle_{\mathfrak{H}^{\otimes n}}\\ =&\sum_{\sigma\in S_n}\langle e_{i_1}, e_{j_{\sigma(1)}}\rangle_{\mathfrak{H}}\cdots\langle e_{i_n}, e_{j_{\sigma(n)}}\rangle_{\mathfrak{H}}.\end{aligned} \tag{2.17}$$

If $\boldsymbol{a}$ is not equal to $\boldsymbol{b}$, then there exists some index k for which $a_k \neq b_k$. Here, e_k appears a_k times in the tensor product $e_{\boldsymbol{a}}$ and b_k times in the tensor product $e_{\boldsymbol{b}}$. So regardless of how we re-order the factors in $e_{\boldsymbol{b}}$, we cannot match every instance of e_k in $e_{\boldsymbol{b}}$ with a corresponding e_k in $e_{\boldsymbol{a}}$. Hence, each summand is zero. If instead $\boldsymbol{b}$ is equal to $\boldsymbol{a}$, and we are calculating $\|\tilde{e}_{\boldsymbol{a}}\|^2$, then a summand is one is if a permutation of the factors of $e_{\boldsymbol{a}}$ is again $e_{\boldsymbol{a}}$, and zero otherwise. A permutation would need to fix the a_1 copies of e_1, the a_2 copies of e_2, and etc. There are precisely $\boldsymbol{a}!$ such permutations, and thus $\|\tilde{e}_{\boldsymbol{a}}\|^2_{\mathfrak{H}^{\odot n}}$ is $\boldsymbol{a}!$. Therefore, $\{\frac{1}{\sqrt{\boldsymbol{a}!}}\tilde{e}_{\boldsymbol{a}}; |\boldsymbol{a}| = n\}$ is an orthonormal basis for $\mathfrak{H}^{\odot n}$.

Finally, we can introduce the operator $I_n : \mathfrak{H}^{\odot n} \to \mathcal{H}_n$, defined on the basis by the rule $I_n(\tilde{e}_{\boldsymbol{a}}) =$

$\sqrt{\boldsymbol{a}!}\boldsymbol{H_a}$. Our lengthy calculation was to convince ourselves that I_n is indeed an isometry between the domain of symmetric kernels and the n-th Wiener chaos $\mathcal{H}_n$. We can summarize our calculations into the pithy algebraic statement Hence

Theorem 2.5.

$$L^2(\Omega, \mathcal{G}, \mathbb{P}) = \bigotimes_{n=1}^{\infty} \sqrt{n!}\mathcal{H}_n. \tag{2.18}$$

So for any F in $L^2(\Omega, \mathcal{G}, \mathbb{P})$ there exists a sequence of $f_n \in \mathfrak{H}^{\odot n}$ such that

$$F = \sum_{n=1}^{\infty} I_n(f_n). \tag{2.19}$$

This is the Fock space construction which appears in Quantum Mechanics. Our use of symmetric tensors corresponds to the Fock space derived for a boson. The Fock space for a fermion would involve antisymmetric tensors instead [Mey95]. We can make a stronger statement if $\mathfrak{H}$ is the space of square-integrable functions over an atom-free measure space $(T, \mathcal{B}, \mu)$. In this case, an element of $\mathfrak{H}^{\odot n}$ is a square-integrable function on T^n with the measure $\mu^{\otimes n}$ which is symmetric in its n variables. We denote the space of all square-integrable symmetric function $\hat{L}^2(T^n)$. In this case, our isometry I_n perfectly coincides with iterated stochastic integration against our isonormal Gaussian process X. One only needs to examine how the image of some indicator function $1_{A_1 \times \cdots \times A_n}$, once appropriately symmetrized, under I_m is precisely $X(A_1) \cdots X(A_n)$. Let us rephrase our previous result with more earthiness.

Theorem 2.6 (Wiener-Itô Chaos Decomposition)**.** *Given an isonormal Gaussian process X over the space $L^2(T, \mathcal{B}, \mu)$. If $(T, \mathcal{B}, \mu)$ is atom-free, and F is in $L^2(\Omega, \mathcal{G}, \mathbb{P})$, then there exists a sequence of symmetric functions $f_n \in \hat{L}^2(T^n)$, where $f_0 = \mathbb{E}[F]$, such that*

$$F = \sum_{n=0}^{\infty} I_n(f_n). \tag{2.20}$$

2.3 The Malliavin Derivative

We are now in a place to introduce the fundamental operators we will use; beginning with the Malliavin derivative. To begin we consider a subspace $\mathcal{S}$ of $L^2(\Omega)$ of all random variables which can be expressed as $f(X(h_1), \ldots, X(h_m))$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function such that f and all its derivatives have at most polynomial growth. Our usage of "$\mathcal{S}$" to denote the family of smooth cylindrical random variables is a mnemonic as every random variable within can be expressed as the application of a function which resides in some Schwartz space [SS03] to elements of our isonormal Gaussian process.

Definition 2.7 (The Malliavin Derivative)**.** *Consider a smooth cylindrical random variable $F \in \mathcal{S}$ of the form $f(X(h_1, \ldots, X(h_m))$, then the Malliavin derivative of F is defined as the $\mathfrak{H}$-valued process*

$$DF = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(X(h_1)), \ldots, X(h_m))h_i \tag{2.21}$$

.

Consequently, the p-th Malliavin derivative of F is an element of $L^2(\Omega \times \mathfrak{H}^{\odot n})$ and has the closed-form expression

$$D^k F = \sum_{i_1,\ldots,i_k=1}^{m} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(X(h_1), \ldots, X(h_m))h_{i_1} \otimes \cdots \otimes h_{i_k}. \tag{2.22}$$

Symmetry stems from the realization that f is smooth, so the order of partial differentiation is irrelevant. The above expression is sufficient for almost all direct calculations. However, sometimes an alternative expression for D^p will ease our burden. If we return to our vector notation, we can write $h_{\boldsymbol{a}}$ for $h_1^{a_1} \otimes \cdots \otimes h_m^{a_m}$, and $\tilde{h}_{\boldsymbol{a}}$ for the symmetrization of $h_{\boldsymbol{a}}$. Then

$$D^p F = \sum_{|\boldsymbol{a}|=p} \frac{p!}{\boldsymbol{a}!} \frac{\partial^p f}{\partial x^{\boldsymbol{a}}}(X(h_1), \ldots, X(h_m))\tilde{h}_{\boldsymbol{a}}. \tag{2.23}$$

When F and f is clear from context, we may denote $F_{\boldsymbol{a}}$ for $\frac{\partial^p f}{\partial x^{\boldsymbol{a}}}(X(h_1), \ldots, X(h_m))$, and more succinctly write

$$D^p F = \sum_{|\boldsymbol{a}|=p} \frac{p!}{\boldsymbol{a}!} F_{\boldsymbol{a}} \tilde{h}_{\boldsymbol{a}}. \tag{2.24}$$

When $\mathfrak{H}$ is a space functions over an interval, then DF is a stochastic process indexed by some parameter. In that case, we can write $DF = \{D_s F\}$ where $D_s F$ is the random variable the process

DF assumes at "time" s.

If we fix an $h \in \mathfrak{H}$ and consider the above F, we can calculate the inner product between h and DF

$$\begin{aligned} \langle DF, h\rangle_{\mathfrak{H}} &= \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(X(h_1), \ldots, X(h_m))\, \langle h_i, h\rangle \\ &= \lim_{\epsilon \to 0} \frac{f(X(h_1) + \epsilon\langle h_1, h\rangle_{\mathfrak{H}}, \ldots, X(h_m) + \epsilon\langle h_m, h\rangle_{\mathfrak{H}}) - F}{\epsilon}, \end{aligned} \tag{2.25}$$

which justifies the interpretation of the Malliavin derivative as a Fréchet derivative of operators on the space of paths. The density of $\mathcal{S}$ in $L^2(\Omega)$ is fairly obvious; $\mathcal{S}$ contains our orthonormal system $\{\boldsymbol{H_a}\}_{\mathcal{J}}$.

We need to give a suitable domain and topology for the Malliavin derivative. The following spaces are ubiquitous in the literature though it seems they are rarely given a proper name. Given the definition and setting, "Malliavin Sobolev spaces" [Imk+16] seems appropriate. For a smooth cylindrical random variable $F \in \mathcal{S}$ we define the norm

$$\|F\|_{k,p} = \left(\mathbb{E}[|F|^p] + \sum_{i=0}^{k} \mathbb{E}\left[\|D^i F\|_{\mathfrak{H}^{\otimes i}}\right]\right)^{\frac{1}{p}}. \tag{2.26}$$

In a deterministic setting, these are the norms which define the Sobolev spaces. We define $\mathbb{D}^{k,p}$ as the closure of $\mathcal{S}$ with respect to the $\|\cdot\|_{k,p}$ norm. Then we can consider D^k as a map from $\mathbb{D}^{k,p}$ to $L^p(\Omega \times \mathfrak{H}^{\odot k})$. Finally, we define the space $\mathbb{D}^{\infty,p}$ as the intersection of all $\mathbb{D}^{k,p}$. Of special interest are the spaces of the form $\mathbb{D}^{k,2}$, since they are Hilbert spaces with respect to the inner product

$$\langle F, G\rangle = \mathbb{E}[FG] + \sum_{i=1}^{k} \mathbb{E}\left[\langle D^i F, D^i G\rangle_{\mathfrak{H}^{\otimes n}}\right]. \tag{2.27}$$

Chapter 1 of [Nua06] and Chapter 2 [NP12] prove relevant properties like inclusion and compatibility among the Malliavin-Sobolev spaces.

We conclude with this simple result.

Proposition 2.8 (The Chain Rule)**.** *If we have a series of random variables $F_1, \ldots, F_m \in \mathbb{D}^{1,p}$, and a continuously differentiable function $\phi : \mathbb{R}^m \to \mathbb{R}$ with bounded partial derivatives, then*

$\phi(F_1, \ldots, F_m)$ is also in $\mathbb{D}^{1,p}$ and

$$D\phi(F_1, \ldots, F_m) = \sum_{i=1}^{m} \frac{\partial \phi}{\partial x_i}(F_1, \ldots, F_m) DF_i. \tag{2.28}$$

The proof of which just follows the definitions. So the Malliavin calculus is a first-order calculus, which already distinguishes it from the Itô calculus.

2.4 The Divergence Operator

Naturally, the next operator we introduce after the Malliavin derivative corresponds to integration. We define the divergence operator δ^p to be the adjoint of the k-th Malliavin derivative $D^k : \mathbb{D}^{k,2} \to L^2(\Omega \times \mathfrak{H}^{\odot k})$. The domain of this operator is denoted $\operatorname{Dom} \delta^k$ and contains the processes u in $L^2(\Omega \times \mathfrak{H}^{\otimes n})$ such that there exists a constant C such that

$$\left|\mathbb{E}\left[\langle D^k F, u \rangle\right]\right| \leq C\sqrt{\mathbb{E}[F^2]}, \tag{2.29}$$

for all F in $\mathbb{D}^{k,2}$. Once we fix a particular u in $\operatorname{Dom} \delta^k$, the linear functional which maps an $L^2(\Omega)$ random variable F to $\mathbb{E}\left[\langle D^k F, u \rangle\right]_{\mathfrak{H}^{\otimes k}}$ is continuous. A routine appeal to the Reisz Representation Theorem implies there is a unique random variable, which will be our $\delta^k(u)$, such that $\mathbb{E}\left[\langle D^k F, u \rangle\right]_{\mathfrak{H}^{\otimes k}}$ is equal to $\mathbb{E}[F\delta^k(u)]$ for any square-integrable random variable F. The k-th order divergence operator $\delta^k : \operatorname{Dom} \delta^k \to L^2(\Omega)$ is defined by this integration by parts formula.

To actually begin integrating we will need one result.

Proposition 2.9. *Let F be a smooth cylindrical random variable in $\mathcal{S}$, and h be an element of $\mathfrak{H}$. Then*

$$\mathbb{E}\left[\langle DF, h \rangle_{\mathfrak{H}}\right] = \mathbb{E}\left[FX(h)\right]. \tag{2.30}$$

This one of the few times we will actually evaluate an expectation as an integral against a probability density function. Under the assumptions, F is of the form $\phi(X(h_1), \ldots, X(h_m))$. Of course, we can assume $h_1, \ldots, h_m$ are orthonormal by replacing ϕ with $\phi \circ O$ where O is the change of coordinates transformation. Then

$$\begin{aligned}&\mathbb{E}\left[\langle DF, h\rangle_{\mathfrak{H}}\right]\\&=\sum_{i=1}^{m} a_i \mathbb{E}\left[\frac{\partial \phi}{\partial x_i}(X(h_1),\ldots,X(h_m))\right],\end{aligned} \tag{2.31}$$

where each a_i is simply $\langle h, h_i\rangle_{\mathfrak{H}}$. $(X(h_1),\ldots,X(h_m))$ is a standard normal random vector and therefore

$$\begin{aligned}&\mathbb{E}\left[\frac{\partial \phi}{\partial x_i}(X(h_1),\ldots,X(h_m))\right]\\&=\frac{1}{(2\pi)^{\frac{m}{2}}}\int_{\mathbb{R}^m}\frac{\partial \phi}{\partial x_i}(x_1,\ldots,x_m)e^{-\frac{\sum_{i=1}^m x_i^2}{2}}dx\\&=\frac{1}{(2\pi)^{\frac{m}{2}}}\int_{\mathbb{R}^m}\phi(x_1,\ldots,x_m)x_i e^{-\frac{\sum_{i=1}^m x_i^2}{2}}dx\\&=\mathbb{E}\left[\phi(X(h_1),\ldots,X(h_m))X(h_1)\right].\end{aligned} \tag{2.32}$$

Substituting these terms back into our previous equation we see

$$\mathbb{E}\left[\langle DF, h\rangle_{\mathfrak{H}}\right]=\mathbb{E}\left[FX(h)\right]. \tag{2.33}$$

We extend to all F by density, and remembering the definition of the divergence operator, we conclude $\delta(h) = X(h)$.

Of special importance are the family of smooth elementary processes $\mathcal{S}_{\mathfrak{H}}$. A process u resides within $\mathcal{S}_{\mathfrak{H}}$ if u is of the form $\sum_{i=1}^n F_i h_i$ where each F_i is a smooth cylindrical random variable. Note, the image of smooth cylindrical random variables $\mathcal{S}$ under the Malliavin derivative lies within $\mathcal{S}_{\mathfrak{H}}$. $\mathcal{S}_{\mathfrak{H}}$ is usually the first input into the "standard machine", and theorems are bootstrapped from there.

Proposition 2.10. *For any elementary process $u = \sum_{i=1}^n F_i h_i$ in $\mathcal{S}_{\mathfrak{H}}$, u is in* $\operatorname{Dom}\delta$ *and its divergence is*

$$\delta(u)=\sum_{i=1}^{n} F_i X(h_i)-\sum_{i=1}^{n}\langle DF_i, h_i\rangle_{\mathfrak{H}}. \tag{2.34}$$

Let us avoid using more indices than we have to and consider $u = Fh$, and only then extend to

$\mathcal{S}_{\mathfrak{H}}$ by linearity. We proceed by duality. For any $G \in \mathbb{D}^{1,2}$ we have

$$\begin{aligned} &\mathbb{E}\left[G\delta(Fh)\right] = \mathbb{E}\left[\langle DG, Fh\rangle_{\mathfrak{H}}\right] \\ =&\mathbb{E}\left[\langle FDG, h\rangle_{\mathfrak{H}}\right] \\ =&\mathbb{E}\left[\langle D(FG), h\rangle_{fH}\right] - \mathbb{E}\left[\langle GDF, h\rangle_{fH}\right] \\ =&\mathbb{E}\left[FG\delta(h)\right] - \mathbb{E}\left[G\langle DF, h\rangle_{fH}\right] \\ =&\mathbb{E}\left[G\left(FX(h) - \langle DF, h\rangle_{fH}\right)\right]. \end{aligned} \tag{2.35}$$

By the uniqueness of the divergence we conclude $\delta(u) = FX(h) - \langle DF, h\rangle_{\mathfrak{H}}$.

Depending on the day, we are either amused or annoyed at how many integration-by-parts formulae there are within the Malliavin calculus. We currently have one for integrating a random variable against a deterministic process. We now expand it to one which covers integrating a random variable against a stochastic process.

Proposition 2.11 (Integration By Parts)**.** *Given $F \in \mathbb{D}^{1,2}$ and $u \in \operatorname{Dom}\delta$ such that $F\|u\|_{\mathfrak{H}}$, $F\delta(u)$ and $\langle DF, u\rangle_{\mathfrak{H}}$ are square integrable, the integral of their product is*

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathfrak{H}}. \tag{2.36}$$

As before, we begin by assuming is an extremely elementary process $u = F'h$. From proposition 2.10 we know

$$\begin{aligned} &\delta(Fu) = \delta(FF'h) = FF'X(h) - \langle D(FF'), h\rangle_{\mathfrak{H}} \\ =&F\left(F'X(h) - \langle DF', h\rangle_{\mathfrak{H}}\right) - \langle DF, F'h\rangle_{\mathfrak{H}} \\ =&F\delta(u) - \langle DF, u\rangle_{\mathfrak{H}}. \end{aligned} \tag{2.37}$$

As we promised, the proposition holds for $\mathcal{S}_{\mathfrak{H}}$ by linearity and $\operatorname{Dom}\delta$ by density; the standard machine in operation. The restriction to random variables and processes which satisfy the stated square-integrability conditions is what makes the algebraic rearrangements true and not merely formal.

We would be remiss if we did not prove the Heisenberg commutativity principle. This is the rule of the symbol-pushing game which allows us to interchange divergence and the Malliavin derivative.

Lemma 2.12 (Heisenberg commutivity property). *Consider a $u \in \operatorname{Dom}\delta$ such that $\delta(u) \in \mathbb{D}^{2,2}$. Then*

$$D\delta(u) - \delta(Du) = u. \tag{2.38}$$

Let $u = Fh$ where $F \in \mathcal{S}$, so $F = \phi(X(h_1,),\dots,X(h_m))$ for some smooth ϕ of at most polynomial growth. Nothing to it but to do it. To save on ink, write F_j for $\frac{\partial\phi}{\partial x_j}(X(h_1),\dots,X(h_m))$ and F_{jk} for $\frac{\partial^2\phi}{\partial x_j \partial x_k}(X(h_1,),\dots,X(h_m))$. We can use the same trick as before and assume h is in the span of $h_1,\dots,h_m$. Furthermore, for now we let $u = Fh_i$. Then

$$\begin{aligned} D\delta(u) &= D\left(FX(h_i) - \langle DF, h_i\rangle_{\mathfrak{H}}\right) \\ &= Fh_i + X(h_i)DF - D\sum_j DF\langle h_j, h_i\rangle \\ &= u + X(h_i)DF - DF_i. \end{aligned} \tag{2.39}$$

Let us make explicit the rule "first things first", so $\delta(h\otimes h') = X(h)h'$ and $\langle h, h'\otimes h''\rangle_{\mathfrak{H}} = \langle h, h'\rangle_{\mathfrak{H}} h''$. Making the former choice fixes the other by the duality relation which defines the divergence. Usually, everything is already symmetrized so all our arbitrary choices lead to the same result. But for right now, we should specify. We also use the convention that taking a derivative "adds a variable at the end" while integrating "consumes the first variable", which is how we would normally calculate if $\mathfrak{H}$ was a family of univariate functions.

$$\begin{aligned} \delta(Du) &= \sum_j \delta(F_j h_i \otimes h_j) \\ &= \sum_j F_j \delta(h_i \otimes h_j) - \sum_j \langle DF_j, h_i \otimes h_j\rangle_{\mathfrak{H}} \\ &= X(h_i)\sum_j F_j h_j - \sum_{j,k} F_{jk}\langle h_k, h_i \otimes h_j\rangle_{\mathfrak{H}} \\ &= X(h_i)DF - \sum_j F_{ji} h_j. \end{aligned} \tag{2.40}$$

Our desired result immediately follows.

Iterating on this idea we have the following result.

Lemma 2.13 (The better Heisenberg commutivity property)**.**

$$D\delta^k u - \delta^k Du = k\delta^{k-1}u, \tag{2.41}$$

which follows from a simple induction proof with Proposition 2.12 as the base case.

$$\begin{aligned} &D\delta^{k+1}u - \delta^{k+1}Du = (D\delta^k)\delta u - \delta^{k+1}Du \\ =&(k\delta^{k-1} + \delta^k D)\delta u - \delta^{k+1}Du = k\delta^k u + \delta^k(D\delta u - \delta Du) = k\delta^k u + \delta^k u \\ =&(k+1)\delta^k u. \end{aligned} \tag{2.42}$$

At the moment, we have two different operators I_k and δ^k. The former is defined by mapping basis vectors of the Hilbert space $\mathfrak{H}^{\odot k}$ to n-th Wiener chaos of $L^2(\Omega, \mathcal{G}, \mathbb{P})$. The latter is defined as the adjoint of the n-th Malliavin derivative. We know prove they coincide. After fixing an orthonormal basis $\{e_i\}$ of $\mathfrak{H}$, we consider the orthonormal basis $\{\frac{1}{\sqrt{\boldsymbol{a}!}}\tilde{e}_{\boldsymbol{a}}; |\boldsymbol{a}| = k\}$. In Section 2.2, $I_k(\frac{1}{\sqrt{\boldsymbol{a}!}}\tilde{e}_{\boldsymbol{a}})$ and $\boldsymbol{H}_{\boldsymbol{a}}$ are equal definitionally. What remains is to show $\delta^k(\frac{1}{\sqrt{\boldsymbol{a}!}}\tilde{e}_{\boldsymbol{a}})$ is also equal to $\boldsymbol{H}_{\boldsymbol{a}}$. Consider two elements $\frac{1}{\sqrt{\boldsymbol{a}!}}\tilde{e}_{\boldsymbol{a}}$ and $\frac{1}{\sqrt{\boldsymbol{b}!}}\tilde{e}_{\boldsymbol{b}}$ of $\mathfrak{H}^{\odot k}$. For the moment, consider a multi-index $\boldsymbol{a} \in \mathcal{J}$ such that $|\boldsymbol{a}| = |\boldsymbol{b}| = k$. By the definition of δ^k we have

$$\mathbb{E}\left[\boldsymbol{H}_{\boldsymbol{a}}\delta^k\left(\frac{1}{\boldsymbol{b}!}\tilde{e}_{\boldsymbol{b}}\right)\right] = \frac{1}{\sqrt{\boldsymbol{b}!}}\mathbb{E}\left[\langle D^k\boldsymbol{H}_{\boldsymbol{a}}, \tilde{e}_{\boldsymbol{b}}\rangle_{\mathfrak{H}^{\otimes k}}\right]. \tag{2.43}$$

Recall $\boldsymbol{H}_{\boldsymbol{a}} = \sqrt{\boldsymbol{a}!}\prod_{i=1}^{\infty} H_{a_i}(X(e_i))$. We want to calculate the k-th Malliavin derivative of this expression. After some relabelling of the e_i we can express $\boldsymbol{H}_{\boldsymbol{a}}$ as a smooth cylindrical random variable of the form $f(X(e_1'), \dots, X(e_m'))$ where $f(x_1, \dots, x_m)$ is equal to $\sqrt{\boldsymbol{a}!}H_{a_1'}(x_1)\cdots H_{a_m'}(x_m)$ where all the a_i' are positive and $a_1' + \cdots + a_m' = k$. Then we calculate the k-th Malliavin derivative directly. From Equation (A.5) we can conclude $\frac{d^i}{dx^i}H_j(x) = H_{j-i}(x)$ when $i \leq j$ and is identically zero otherwise. Using Equation (2.23) we see

$$D^k\boldsymbol{H}_{\boldsymbol{a}} = \sum_{|\boldsymbol{\mu}|=k} \frac{k!}{\boldsymbol{\mu}!}\frac{\partial^k f}{\partial x^{\boldsymbol{\mu}}}(X(e_1'), \dots, X(e_m'))\tilde{e_{\boldsymbol{\mu}}'}. \tag{2.44}$$

By a pigeon-hole argument, $\frac{\partial^k f}{\partial x^{\boldsymbol{\mu}}}$ is zero unless $\boldsymbol{\mu}$ is equal to $\boldsymbol{a}' = (a_1', \dots, a_m')$, in which case it is

$\sqrt{\boldsymbol{a}!}$. So the only summand which is non-zero is when $\boldsymbol{\mu}$ is equal to $\boldsymbol{a}'$. Then we return to our original labelling to see

$$D^k \boldsymbol{H}_{\boldsymbol{a}} = \frac{k!}{\sqrt{\boldsymbol{a}!}} \tilde{e}_{\boldsymbol{a}}. \tag{2.45}$$

Plugging this result back into our original equation we find

$$\begin{aligned} &\mathbb{E}\left[I_k\left(\frac{1}{\sqrt{\boldsymbol{a}!}}\tilde{e}_{\boldsymbol{a}}\right)\delta^k\left(\frac{1}{\boldsymbol{b}!}\tilde{e}_{\boldsymbol{b}}\right)\right] = \frac{k!}{\sqrt{\boldsymbol{a}!\boldsymbol{b}!}}\mathbb{E}\left[\langle \tilde{e}_{\boldsymbol{a}}, \tilde{e}_{\boldsymbol{b}}\rangle_{\boldsymbol{H}^{\otimes k}}\right] \\ &= \left\langle \frac{1}{\boldsymbol{a}!}\tilde{e}_{\boldsymbol{a}}, \frac{1}{\boldsymbol{b}!}\tilde{e}_{\boldsymbol{b}!}\right\rangle_{\mathfrak{H}^{\odot k}}. \end{aligned} \tag{2.46}$$

Therefore,

$$\mathbb{E}\left[\boldsymbol{H}_{\boldsymbol{a}}\delta^k\left(\frac{1}{\boldsymbol{b}!}\tilde{e}_{\boldsymbol{b}}\right)\right] = \mathbb{E}\left[\boldsymbol{H}_{\boldsymbol{a}}\boldsymbol{H}_{\boldsymbol{b}}\right], \tag{2.47}$$

for all $|\boldsymbol{a}| = |\boldsymbol{b}| = k$. We can at least conclude the projection of $\delta^k(\frac{1}{\boldsymbol{b}!}\tilde{e}_{\boldsymbol{b}})$ onto the k-th Wiener chaos is $\boldsymbol{H}_{\boldsymbol{b}}$ when $|\boldsymbol{b}| = k$. Now consider $\mathbb{E}[\boldsymbol{H}_{\boldsymbol{a}}\delta^k(\frac{1}{\boldsymbol{b}!}\tilde{e}_{\boldsymbol{b}})]$ where $|\ell| < k$. As we retrace the steps of the above proof we see $D^k\boldsymbol{H}_{\boldsymbol{a}}$ is identically zero. And therefore $\mathbb{E}[\boldsymbol{H}_{\boldsymbol{a}}\delta^k(\frac{1}{\boldsymbol{b}!}\tilde{e}_{\boldsymbol{b}})]$ is zero whenever $|\boldsymbol{a}| < |\boldsymbol{b}| = k$. Finally, we must consider when $|\boldsymbol{a}| > |\boldsymbol{b}| = k$. In this third pass-through of the proof, $\langle D^k\boldsymbol{H}_{\boldsymbol{a}}, \tilde{e}_{\boldsymbol{b}}\rangle_{\boldsymbol{H}^{\otimes n}}$ is an element the $(|\boldsymbol{b}| - |\boldsymbol{a}|)$-th Wiener chaos and therefore has mean zero. We can conclude for any two multi-indices $\boldsymbol{a}, \boldsymbol{b}$ where $|\boldsymbol{a}| = k$ (we place no restriction on $|\boldsymbol{b}|$), $\mathbb{E}[\boldsymbol{H}_{\boldsymbol{a}}\delta^k(\frac{1}{\boldsymbol{b}!}\tilde{e}_{\boldsymbol{b}})]$ is identically zero unless $\boldsymbol{a} = \boldsymbol{b}$, in which case, the expression is equal to one. Since the $\{\boldsymbol{H}_{\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathcal{J}}$ form a complete orthonormal system of $L^2(\Omega, \mathcal{G}, \mathbb{P})$ we are forced to conclude $\delta^k(\frac{1}{\boldsymbol{b}!}\tilde{e}_{\boldsymbol{b}})$ is $\boldsymbol{H}_{\boldsymbol{b}}$, and therefore I_k and δ^k agree on $\mathfrak{H}^{\odot k}$.

Our next operators are not as common in the literature.

2.5 The Gross Laplacian

The Malliavin calculus admits many different Laplace-type operators. There is the Lévy Laplacian [Fel05] and its more exotic variants which have no direct finite-dimensional analog. There is the Volterra Laplacian [Hid+93] which is a proper infinite-dimensional analog of the classical Laplacian but a bit abstract for our purposes. We will work with the Gross Laplacian [Gro67] which is more concrete a specialization of the Volterra Laplacian. For a cylindrical random variable $F =$

$f(X(h_1), \ldots, X(h_m))$ where all the h_i are orthonormal we define the Gross Laplacian to be

$$\Delta F = \sum_{i=1}^{m} \frac{\partial^2 f}{\partial x_i^2}(X(h_1), \ldots, X(h_m)), \tag{2.48}$$

which is intuitive enough of a definition to justify the nomenclature.

We now prove that we can loosen the orthonormality requirement. Let $F = f(X(h_1), \ldots, X(h_m))$ be a smooth cylindrical random variable, but let us drop the requirement that the h_i are orthonormal. Let $\{e_i\}_1^\ell$ be an orthonormal basis for the span of the $\{h_i\}_1^m$. Then F can be expressed as

$$F = f(X(h_1), \ldots, X(h_m)) = f\left(A \begin{bmatrix} X(e_1) \\ \vdots \\ X(e_\ell) \end{bmatrix} \right), \tag{2.49}$$

where the (possibly non-square) matrix $A = (a_{ij})$ is the change of coordinates between the $\{h_i\}$ and the $\{e_i\}$. If we let $g = f \circ A$, then

$$\Delta F = \sum_{i=1}^{\ell} \frac{\partial^2 g}{\partial x_i^2}(X(e_1), \ldots, X(e_\ell)). \tag{2.50}$$

We calculate the first derivatives of g

$$\begin{aligned} \frac{\partial g}{\partial x_i}(\boldsymbol{x}) =& \frac{\partial}{\partial x_i}\left(f(A\boldsymbol{x})\right) = \sum_{j=1}^{m} \left(\frac{\partial f}{\partial x_j}(A\boldsymbol{x}) \right) \frac{\partial}{\partial x_i} \left(\sum_{k=1}^{\ell} a_{jk} x_k \right) \\ =& \sum_{j=1}^{m} \left(\frac{\partial f}{\partial x_j}(A\boldsymbol{x}) \right) a_{ji}. \end{aligned} \tag{2.51}$$

The second derivatives are

$$\frac{\partial^2 g}{\partial x_i^2} = \sum_{j,k=1}^{m} \left(\frac{\partial f}{\partial x_j \partial x_k}(A\boldsymbol{x}) \right) a_{ji} a_{ki}, \tag{2.52}$$

and therefore

$$\frac{\partial^2 g}{\partial x_i^2}(X(e_i), \cdot, W(e_\ell)) = \sum_{j,k=1}^{m} \left(\frac{\partial f}{\partial x_j \partial x_k} \left(X(h_1), \ldots, X(h_m) \right) \right) a_{ji} a_{ki}. \tag{2.53}$$

Once we recognize that $\langle h_j, h_k \rangle_{\mathfrak{H}} = \sum_{i=1}^{\ell} a_{ji} a_{ki}$, we can see

$$\begin{aligned} \Delta F =& \sum_{i=1}^{\ell} \sum_{j,k=1}^{m} \left(\frac{\partial f}{\partial x_j \partial x_k} \left(X(h_1), \ldots, X(h_m) \right) \right) a_{ji} a_{ki} \\ =& \sum_{j,k=1}^{m} \left(\frac{\partial f}{\partial x_j \partial x_k} \left(X(h_1), \ldots, X(h_m) \right) \right) \sum_{i=1}^{\ell} a_{ji} a_{ki} \\ =& \sum_{j,k=1}^{m} \left(\frac{\partial f}{\partial x_j \partial x_k} \left(X(h_1), \ldots, X(h_m) \right) \right) \langle h_j, h_k \rangle_{\mathfrak{H}}. \end{aligned} \tag{2.54}$$

Note

$$D^2 F = \sum_{j,k=1}^{m} \partial^{jk} f (h_j \otimes h_k). \tag{2.55}$$

So the Gross Laplacian is some sort of trace of $D^2 F$, which corresponds with the classical idea of the Laplacian as the trace of the Hessian matrix.

If X was a classical BM, then $\langle f, g \rangle = \int_0^T f(s) g(s) ds$ and the Gross Laplacian would be equal to the expression $\int_0^T D_s D_s F ds$. The Gross Laplacian is actually one of the fundamental operators within [JPS15a], and we can pithily rephrase a particular result as

$$\mathbb{E}[F] = \omega^0 (e^{\frac{1}{2}\Delta} F) = \sum_k \frac{1}{2^k k!} \omega^0 (\Delta^k F). \tag{2.56}$$

The meaning of ω^0 will be revealed shortly in Section 2.6. For now, we do not know if rewriting exponential formulae using the Gross Laplacian is only notational concision, or whether there is some more interesting connection between the exponential formulae and potential theory on Hilbert spaces.

2.6 The Frozen Path Operator

Introduced in [JPS15a], the frozen path operator was used for a new representation theorem for smooth martingales generated by classical Brownian motion. This Dyson series [Zei06] was further used to characterize solutions for path-dependent parabolic PDEs [JS16] and later extended [JPS15b] to random variables generated by fBm with Hurst index greater than $\frac{1}{2}$.

If we have a smooth cylindrical random variable $F = f(B_{t_1}, \ldots, B_{t_m})$ where B is classical Brownian Motion then we can "freeze" the random variable at time $0 \leq t \leq T$ by defining the operator ω^t such that

$$\omega^t(F) = f(B_{t_1 \wedge t}, \ldots, B_{t_m \wedge t}). \tag{2.57}$$

Alternatively, we may write $F(\omega^t)$ for $\omega^t(F)$. Section 3.2 of [JPS15b] gives a proof that this operator is indeed well-defined. Classical Brownian Motion is an isonormal Gaussian process over the Hilbert space $L^2([0,T])$. So for a smooth cylindrical random variable $F = f(B(h_1), \ldots, B(h_m))$ we have $B(h_i) = \int_0^T h_i(x)dB(x)$, where each h_i is square-integrable. As we would expect, freezing this random variable gives us

$$\omega^t(F) = f\left(\int_0^t h_1(x)dB(x), \ldots, \int_0^t h_m(x)dB(x)\right). \tag{2.58}$$

All claims in this section continue to hold when B is replaced by a fBm B^H regardless of whether the Hurst index H is greater or less than $\frac{1}{2}$.

For an adapted process u, we do have the formula $\omega^t \int_0^T u_s ds = \int_0^t u_s ds + u_t(T-t)$. However, for stochastic integrals of nondeterministic processes, the naïve guess that $\omega^t \int_0^T u_s dB_s$ is simply $\int_0^t u_s dB_s$ immediately fails. If we let $u_s = B_s$, then $\int_0^T B_s dB_s = \frac{1}{2}B_T^2 - \frac{T}{2}$, and freezing this result at time t produces $\frac{1}{2}B_t^2 - \frac{T}{2}$, while $\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{t}{2}$. Neither can we approximate the result of freezing an integral by freezing an approximation. We have the same witness of $\int_0^T B_s dB_s$ for a counter-example. Given a partition of $[0,T]$, freeze the approximation $\sum_i B_{t_i}(B_{t_{i+1}} - B_{t_i})$, then let the mesh become arbitrarily fine. Once again, we obtain the result $\frac{1}{2}B_t^2 - \frac{t}{2}$ instead of our desired conclusion of $\frac{1}{2}B_T^2 - \frac{T}{2}$. Later, will claim a simple formula for freezing the Stratonovich integral of an adapted process.

2.7 d-dimensional BM as an Isonormal Gaussian Process

We wish to make explicit what we have only seen hinted at within the usual references; describing d-dimensional Brownian Motion within the framework of isonormal Gaussian processes. After this construction we will examine the rules of "multi-variable" stochastic calculus.

The underlying Hilbert space we will use is the space $L^2([0,T];\mathbb{R}^d)$ of $\mathbb{R}^d$-valued functions $\boldsymbol{h}$ such that $\int_0^T |\boldsymbol{h}(x)|^2 dx$ is finite. It may feel backwards to specify an indexing Hilbert space to generate a Gaussian process instead of determining the underlying Hilbert space given a Gaussian process, but Proposition 2.1.1 of [NP12] provides a simple construction of a Gaussian process given a real separable Hilbert space and a sequence of i.i.d. standard normal random variables. If we let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ be the standard basis $\mathbb{R}^d$, then a function $\boldsymbol{h} \in L^2([0,T];\mathbb{R}^d)$ has a component-wise decomposition $\boldsymbol{h} = \sum_{i=1}^d h_i \boldsymbol{e}_i$ where each h_i is a square-integrable real function on $[0,T]$. Well, $L^2([0,T])$ is the underlying Hilbert space when considering classical Brownian Motion as an isonormal Gaussian process. Intuitively, an isonormal Gaussian process over $L^2([0,T];\mathbb{R}^d)$ should contain d copies of Brownian motions.

To begin, let B be an isonormal Gaussian process which has $L^2([0,T];\mathbb{R}^d)$ as its underlying Hilbert space. Now define the linear map $B_i : L^2([0,T]) \to L^2(\Omega)$ by $B_i(h) = B(h\boldsymbol{e}_i)$. When we want to emphasize B_i as a process over time we adhere to the usual notation and write $B_i(t)$ for $B_i(1_{[0,t]}) = B(1_{[0,t]}\boldsymbol{e}_i)$. Then $B(\boldsymbol{h}) = \sum_i B_i(h_i)$ and in particular $B(t) = \sum_i B_i(t)$. $\{B_i(h)\}_{h\in L^2(0,T)}$ is a family of centered Gaussian random variables, and if we look at the auto-covariance of B_i we see

$$\begin{aligned}\mathbb{E}\left[B_i(s)B_i(t)\right] =& \mathbb{E}\left[B\left(1_{[0,s]}\boldsymbol{e}_i\right)B\left(1_{[0,t]}\boldsymbol{e}_i\right)\right] = \left\langle 1_{[0,s]}\boldsymbol{e}_i, 1_{[0,t]}\boldsymbol{e}_i\right\rangle_{L^2} \\ =& \int_0^T 1_{[0,s]}(u)1_{[0,t]}(u)du = s \wedge t.\end{aligned} \tag{2.59}$$

The only Gaussian process which satisfies these properties is classical Brownian motion. Furthermore, if we consider B_i and B_j with distinct indices i and j then

$$\mathbb{E}\left[B_i(s)B_j(t)\right] = \left\langle 1_{[0,s]}\boldsymbol{e}_i, 1_{[0,t]}\boldsymbol{e}_j\right\rangle_{L^2} = 0, \tag{2.60}$$

which is all we need to conclude that $B_1, \dots, B_d$ are independent Brownian motions.

There is a slightly different representation which [Nua06] uses when it briefly comments on d-dimensional Brownian motion as an isonormal Gaussian process. Which is understandable; there are many Hilbert spaces isomorphic to $L^2([0,T];\mathbb{R}^d)$. If we let $\mathbb{N}_d$ be the set $\{1,\dots,d\}$ we can alternatively consider $L^2([0,T]\times\mathbb{N}_d)$ as the underlying Hilbert space. Then an element h has a unique decomposition defined by

$$h(t,j) = \sum_{i=1}^{d} h_i(t) 1_{\{i\}}(j). \tag{2.61}$$

Where, once again, each h_i is an element of $L^2([0,T])$ and $h_i(t) = h(t,i)$. We want to avoid using Kronecker deltas when discussing an indicator function on a single point because with the Skorokhod integral we have too many deltas as is. Obviously the isomorphism between these two Hilbert spaces is induced from mapping the indicator function $1_{\{i\}}$ to the basis vector $\boldsymbol{e}_i$. Some analysis is easier when considering $L^2([0,T]\times\mathbb{N}_d)$ but instead of having two notations we write $\boldsymbol{h} = \sum_i h_i \boldsymbol{e}_i$ and trust in the maturity of the reader to understand when we consider $\boldsymbol{e}_i$ as a basis vector of $\mathbb{R}^d$ or when we consider it as the indicator function on $\{i\}$ in $\mathbb{N}_d$. Let us quickly run through the Malliavin operators.

Given a smooth cylindrical random variable $F = f(B(\boldsymbol{h}_1),\dots,B(\boldsymbol{h}_m))$ the Malliavin derivative is the following $\mathbb{R}^d$-valued process.

$$DF = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}\left(B(\boldsymbol{h}_1),\dots,B(\boldsymbol{h}_m)\right)\boldsymbol{h}_i. \tag{2.62}$$

Then we can define partial Malliavin derivatives by the relation

$$DF = \begin{bmatrix} D_1 F \\ \vdots \\ D_d F \end{bmatrix}. \tag{2.63}$$

Each $D_i F$ is simply the i-th component of DF. One slight nuance is to calculate $D_i F$, F needs to be measurable with respect to B and not just B_i, and hence F must be measurable with respect to the

σ-algebra generated jointly by d Brownian motions. Algorithmically, calculating D_iF is equivalent to calculating the Malliavin derivative as normal but treating all the Brownian motions aside from B_i as deterministic parameters. It is slightly unwieldy, but we also write $D_{i,s}F$ to denote the value of the stochastic process D_iF at "time" s.

Now let δ be the divergence operator induced by B. Then for a scalar-valued process u we can define $\delta_i(u) = \delta(u\boldsymbol{e}_i)$. By construction, for a $\mathbb{R}^d$-valued process $\boldsymbol{u} = \sum_{i=1}^d u_i\boldsymbol{e}_i$ we have $\delta(\boldsymbol{u}) = \sum_i \delta_i(u_i)$. We would like to show δ_i coincides with the Skorokhod integral with respect to B_i. To do so, we consider a square-integrable random variable F and a process u such that $u\boldsymbol{e}_i$ is in $\operatorname{Dom}\delta$

$$\begin{aligned}\mathbb{E}\left[F\delta_i(u)\right] =&\mathbb{E}\left[F\delta(u\boldsymbol{e}_i)\right] = \mathbb{E}\left[\langle DF, u\boldsymbol{e}_i\rangle_{L^2([0,T];\mathbb{R}^d)}\right] \\ =&\mathbb{E}\left[\langle D_iF, u\rangle_{L^2([0,T])}\right].\end{aligned} \tag{2.64}$$

In the special case when F and u are generated only by B_i this collapses into the definition of the Skorokhod integral for a single BM. δ_i really is the Skorokhod integral with respect to B_i. We can prove it follows the expected integration-by-parts rule

$$\begin{aligned}\delta_i(Fu) =&\delta\left(Fu\boldsymbol{e}_i\right) = F\delta\left(u\boldsymbol{e}_i\right) - \langle DF, u\boldsymbol{e}_i\rangle_{L^2([0,T];\mathbb{R}^d)} \\ =&F\delta_i(u) - \langle D_iF, u\rangle_{L^2([0,T])}.\end{aligned} \tag{2.65}$$

We follow [Nua06] to argue that when u is adapted then $\delta_i(u)$ is equal to the Itô integral $\int_0^T u_s dB_i(s)$. It follows from Lemma 1.3.2 after consider the random variable u_y as $\mathcal{F}_{[0,t]\times\{i\}}$.

The last component which is essential to computation is determining how D_i interacts with δ_j as operators. Here we follow the proof of Proposition 2.5.4 in [NP12] which we had already seen in Section 2.4. Take the elementary process Fh where h is in $L^2([0,T])$.

$$\begin{aligned}D_i\delta_j(Fh) =&D_i\left(FB_j(h) - \langle D_jF, h\rangle_{L^2([0,T])}\right) \\ =&B_j(h)D_i + FD_iB_j(h) - \langle D_iD_jF, \rangle_{L^2([0,T])},\end{aligned} \tag{2.66}$$

while

$$\delta_j D_i(Fh) = \delta_j \left(h \otimes D_i F\right) = B_j(h) D_i f - \langle D_j D_i F, h\rangle_{L^2([0,T])}. \tag{2.67}$$

D_i and D_j obviously commute, so when we take the difference of these two expressions, we see

$$D_i\delta_j(Fh) - \delta_j D_i(Fh) = F D_i B_j(h). \tag{2.68}$$

When i and j are equal this is the Heisenberg commutativity relation. When they differ, the right-hand side is 0 and thus D_i and δ_j commute.

For a cylindrical random variable $F = f(B(\boldsymbol{h}_1), \ldots, B(\boldsymbol{h}_m))$, we want to arrange the terms of the Gross Laplacian in a particular way. We express each $\boldsymbol{h}_i$ as $\sum_i h_{ki}\boldsymbol{e}_k$, and then see

$$\begin{aligned} \Delta F &= \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j} \left(B(\boldsymbol{h}_1), \ldots, B(\boldsymbol{h}_m)\right) \langle \boldsymbol{h}_i, \boldsymbol{h}_j\rangle_{L^2([0,T];\mathbb{R}^d)} \\ &= \sum_{k=1}^{d} \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j} \left(B(\boldsymbol{h}_1), \ldots, B(\boldsymbol{h}_m)\right) \langle h_{ki}, h_{kj}\rangle_{L^2([0,T])}. \end{aligned} \tag{2.69}$$

Each summand will be denoted $\Delta_k F$, which is equivalent to the Gross Laplacian of F if the Brownian motions other than B_k were constant. Then we could express the Gross Laplacian as $\Delta = \Delta_1 + \cdots + \Delta_d$. The freezing operator is even simpler and behaves like freezing all the Brownian motions at the same point in time.

To express a chaos expansion of a random variable generated by d independent Brownian motions, we should have a better idea of what symmetric kernels look like. Switching back to our scalar notation, the second symmetric kernel would be of the form

$$f((s,i),(t,j)) = \sum_{k,\ell} f_{k,\ell}(s,t)\boldsymbol{e}_{kl}(i,j). \tag{2.70}$$

In this case, the constraint is $f((s,i),(t,j)) = f((t,j),(s,i))$ and

$$\begin{aligned} f((t,j),(s,j)) &= \sum_{k,\ell} f_{k\ell}(t,s)\boldsymbol{e}_{kl}(j,i) = \sum_{k,\ell} f_{k\ell}(t,s)\boldsymbol{e}_{lk}(i,j) \\ &= \sum_{k,\ell} f_{\ell k}(t,s)\boldsymbol{e}_{kl}(i,j). \end{aligned} \tag{2.71}$$

As this expression is equal to $f((s,i),(t,j))$ we must have $f_{\ell k}(t,s) = f_{k\ell}(s,t)$. To provide a concrete example to disabuse us of faulty intuition, consider the following matrix-valued function

$$f(s,t) = \begin{bmatrix} 0 & st^2 \\ s^2t & 0 \end{bmatrix}. \tag{2.72}$$

Not all the components of f are symmetric functions nor is f generally a symmetric matrix. But if we write f as

$$f(s,t) = \begin{bmatrix} s \\ s^2 \end{bmatrix} \otimes \begin{bmatrix} t \\ t^2 \end{bmatrix} - \begin{bmatrix} s \\ 0 \end{bmatrix} \otimes \begin{bmatrix} t \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ s^2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ t^2 \end{bmatrix}, \tag{2.73}$$

then we can readily see f is a symmetric element of $L^2([0,T];\mathbb{R}^d) \otimes L^2([0,T];\mathbb{R}^d)$. Our key point is f is a symmetric kernel, but that neither implies that all the components of f are symmetric functions nor f must always be a symmetric matrix for any fixed value of s and t.

We wish to rewrite the n-chaos $I_n(f_n)$ in terms iterates of our operators $\delta_1, \ldots, \delta_d$ applied to the components of f_n. f_n is of the form

$$f_n(s_1, \ldots, s_n) = \sum_{1 \le j_1, \ldots, j_n \le d} f_{j_1 \cdots j_n}(s_1, \ldots, s_n)\boldsymbol{e}_{j_1, \cdots j_n}. \tag{2.74}$$

We repeat our previous symmetry argument in a more long-winded fashion. Consider f_n as a symmetric scalar-valued function on $([0,T] \times \mathbb{N}_d)^n$, and take a permutation σ on $\mathbb{N}_d$. Then

$$\begin{aligned} & f_n((s_{\sigma(1)}, i_{\sigma(1)}), \ldots, (s_{\sigma(n)}, i_{\sigma(n)})) \\ =& \sum_{j_1,\cdot,j_n} f_{j_1 \ldots j_n}(s_{\sigma(1)}, \ldots, s_{\sigma(n)}) \boldsymbol{e}_{j_1 \cdots j_n}(i_{\sigma(1)}, \ldots, i_{\sigma(n)}) \\ =& \sum_{j_1,\cdot,j_n} f_{j_1 \ldots j_n}(s_{\sigma(1)}, \ldots, s_{\sigma(n)}) \boldsymbol{e}_{\sigma^{-1}(j_1) \cdots \sigma^{-1}(j_n)}(i_1, \ldots, i_n) \\ =& \sum_{j_1,\cdot,j_n} f_{\sigma(j_1) \ldots \sigma(j_n)}(s_{\sigma(1)}, \ldots, s_{\sigma(n)}) \boldsymbol{e}_{j_1 \cdots j_n}(i_1, \ldots, i_n). \end{aligned} \tag{2.75}$$

So the symmetry relation among components is as follows: for any multi-index $(j_1, \cdots, j_n)$ and permutation σ, we must have

$$f_{\sigma(j_1) \ldots \sigma(j_n)}(s_{\sigma(1)}, \ldots, s_{\sigma(n)}) = f_{j_1 \cdots j_n}(s_1, \ldots, s_n). \tag{2.76}$$

For the second symmetric kernel $f_2 = \sum_{k\ell} f_{k\ell} \boldsymbol{e}_{k\ell}$ we have

$$\begin{aligned} I_2(f_2) =& \delta\left(\sum_{\ell} \delta\left(\sum_k f_{k\ell} \boldsymbol{e}_k\right) \otimes \boldsymbol{e}_\ell\right) = \delta\left(\sum_{\ell}\left(\sum_k \delta_k f_{k\ell}\right) \boldsymbol{e}_\ell\right) \\ =& \sum_{k,\ell} \delta_\ell \delta_k f_{k\ell}. \end{aligned} \tag{2.77}$$

Generalizing, we see if $f_N = \sum_{j_1,\ldots,j_n} f_{j_1 \cdots j_n} \boldsymbol{e}_{j_1 \cdots j_n}$ then

$$I_n(f_n) = \sum_{1 \le j_1,\ldots,j_n \le d} \delta_{j_n} \cdots \delta_{j_1} f_{j_1 \cdots j_n}. \tag{2.78}$$

Now to exploit the symmetry condition we have established. For any index $j_1, \ldots, j_n$ there is a permutation σ such that $j_{\sigma(1)} \le \cdots \le j_{\sigma(n)}$. Let us call this reordering $j'_1, \cdots, j'$. If we examine the contribution of $f_{j_1\cdots j_n}$ to $I_n(f_n)$ we see

$$\begin{aligned}
&\delta_{j_n}\cdots\delta_{j_1}f_{j_1\cdots j_n}\\
=&\int_0^T\cdots\int_0^T f_{j_1\cdots j_n}(s_1,\dots,s_n)\delta B_{j_1}(s_1)\cdots\delta B_{j_n}(s_n)\\
=&\int_0^T\cdots\int_0^T f_{j_{\sigma(1)}\cdots j_{\sigma(n)}}(s_{\sigma(1)},\dots,s_{\sigma(n)})\delta B_{j_1}(s_1)\cdots\delta B_{j_n}(s_n)\\
=&\int_0^T\cdots\int_0^T f_{j'_1\cdots j'_n}(s_{\sigma(1)},\dots,s_{\sigma(n)})\delta B_{j_{\sigma(1)}}(s_{\sigma(1)})\cdots\delta B_{j_{\sigma(n)}}(s_{\sigma(n)})\\
=&\int_0^T\cdots\int_0^T f_{j'_1\cdots j'_n}(s_1,\dots,s_n)\delta B_{j'_1}(s_1)\cdots\delta B_{j'_n}(s_n)\\
=&\delta_{j'_n}\cdots\delta_{j'_1}f_{j'_1\cdots j'_n}.
\end{aligned}\tag{2.79}$$

To recap; we exploited the symmetry condition, re-ordered integration (one only needs to examine elementary process to see re-ordering is valid), and re-labeled dummy variables. Thus, two components contribute equally to $I_n(f_n)$. We call a component $f_{j_1\cdots j_n}$ whose index satisfies the constraint $j_1\le\cdots\le j_n$ a canonical component. A canonical component can be indexed by a vector $\boldsymbol{\mu}=(\mu_1,\dots,\mu_d)\in\mathbb{N}^d$ such that $j_1=\cdots=j_{\mu_1}=1$, the next μ_2 indexes are all equal to 2, etc., and where the last μ_d indexes are all equal d. Then we can label the canonical component $f_{\boldsymbol{\mu}}$, and we have

$$I_n(f_n)=\sum_{\boldsymbol{\mu}\in\mathbb{N}^d;|\boldsymbol{\mu}|=n}\frac{n!}{\boldsymbol{\mu}!}\delta^{\boldsymbol{\mu}}f_{\boldsymbol{\mu}},\tag{2.80}$$

where $\delta^{\boldsymbol{\mu}}=\delta_d^{\mu_d}\cdots\delta_1^{\mu_1}$. The symmetry relation manifests quite nicely for a canonical component. For $f_{\boldsymbol{\mu}}$, the function is symmetric under any permutation of the first μ_1 variables. $f_{\boldsymbol{\mu}}$ is also symmetric under permutation among the next μ_2 variables, and so on. The $\frac{n!}{\boldsymbol{\mu}!}$ is the number of components whose contribution is equal to that of $f_{\boldsymbol{\mu}}$. The advantage of this representation is there are no redundancies between components. It is the same combinatorial exercise as when expressing the higher-order Malliavin derivatives in terms of the symmetric basis. We can conclude this excursion with the chaos expansion in terms of these canonical components.

Theorem 2.14. *For a square-integrable random variable F generated by d independent copies of*

Brownian Motion. There is a unique decomposition

$$F = \sum_{\boldsymbol{\mu} \in \mathbb{N}^d} \frac{|\boldsymbol{\mu}|!}{\boldsymbol{\mu}!} \delta^{\boldsymbol{\mu}} f_{\boldsymbol{\mu}}, \tag{2.81}$$

where the kernel

$$f_{\boldsymbol{\mu}}(s_{1,1}, \ldots, s_{1,\mu_1}, s_{2,1}, \ldots, s_{2,\mu_2}, \ldots, s_{d,1}, \ldots, s_{d,\mu_d}), \tag{2.82}$$

is square-integrable on $[0,T]^{|\boldsymbol{\mu}|}$ *and is symmetric among the first variables* $s_{1,1}, \ldots, s_{1,\mu_1}$*, symmetric among the next variables,* $s_{2,1}, \ldots, s_{2,\mu_2}$*, and so forth.*

There is really nothing in this section unique to classical Brownian Motion. We can, repeat this construction to utilize a Malliavin calculus for random variables and processes generated by d independent copies of fractional Brownian Motion with Hurst index less than $\frac{1}{2}$. Dealing with classical Brownian Motion is a better introduction, and it feels fairer to proceed with d-dimensional fBm after we are more intimately acquainted with stochastic calculi specific to fBm.

Chapter 3

A Survey of Stochastic Integration for Fractional Brownian Motion

Until now, we have tried to work as abstractly as possible; partly to emphasize the elastic nature of the Malliavin calculus, and partly to inspire ourselves about possible generalizations of later results. However, now we turn away from the analysis of isonormal Gaussian processes in general to the study of fractional Brownian motion with $H < \frac{1}{2}$ in particular. For the proceeding sections we now fix the Hurst index $H < \frac{1}{2}$, and we often employ the variables $\alpha = \frac{1}{2} - H$. α is then an extremely crude measure of how close our fBm is to classical Brownian motion. Many of the following results will collapse into the standard ones after setting α equal to zero.

Every survey of stochastic calculus on fractional Brownian motion should reference Coutin's [Cou07]. One of our many failings in writing this dissertation is not discovering this reference sooner. We are able to extend on Coutin's survey in some aspects, but we have the slight advantage of writing over a decade after her.

Our trouble begins when realizing fBm is not a semimartingale when H differs from $\frac{1}{2}$. Analysis of fBm then falls outside the scope of the usual machinery of stochastic analysis via Itô integrals.

3.1 Representation of fBm with $H < \frac{1}{2}$ and the Transfer Principle

We have worked somewhat backwards. We have our desired autocovariance function, and there is an isonormal Gaussian process which has this autocovariance function. Ontology is not a problem, but to better analyze fBm with $H < \frac{1}{2}$ we need a better understanding. If we return to isonormal Gaussian processes as our lens then our fBm B^H is an isometry between some Hilbert space $\mathfrak{H}$ into the space of square-integrable random variables generated by our fBm. We need to characterize $\mathfrak{H}$.

3.1.1 Primer on Fractional Calculus

The usual guide on fractional calculus for the uninitiated is [SKM93]. Consider a function f in $L^1([0,T])$. Definition 2.1 in [SKM93] of the fractional integral is

Definition 3.1 (The Riemann-Liouville fractional integral)**.** *The left-sided fractional integral of order α is the function*

$$(\mathcal{I}^{\alpha}_{0+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(u)}{(x-u)^{1-\alpha}} du, \tag{3.1}$$

while the right-sided fractional integral of order α is

$$(\mathcal{I}^{\alpha}_{T-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^T \frac{f(u)}{(u-x)^{1-\alpha}} du. \tag{3.2}$$

For our purposes, α will lie in the interval $(0, \frac{1}{2})$, but this definition is for any positive α.

A simple calculation proves the Riemann-Liouville fractional integrals enjoy the semigroup property for any positive β and γ:

$$\mathcal{I}^{\beta}_{0^+}\mathcal{I}^{\gamma}_{0^+} f = \mathcal{I}^{\beta+\gamma}_{0^+} f \quad \text{and} \quad \mathcal{I}^{\beta}_{T^-}\mathcal{I}^{\gamma}_{T^-} f = \mathcal{I}^{\beta+\gamma}_{T^-} f. \tag{3.3}$$

The proof follows:

$$
\begin{aligned}
(\mathcal{I}^{\beta}_{0+}\mathcal{I}^{\gamma}_{0+}f)(x) =&\frac{1}{\Gamma(\beta)}\int_0^x \frac{(\mathcal{I}^{\beta}_{0+}f)(u)}{(x-u)^{1-\beta}}du \\
=&\frac{1}{\Gamma(\beta)}\int_0^x\int_0^u \frac{f(v)}{(x-u)^{1-\beta}(u-v)^{1-\gamma}}dvdu \\
=&\frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_0^x f(v)\int_v^x \frac{1}{(x-u)^{1-\beta}(u-v)^{1-\gamma}}dudv.
\end{aligned} \tag{3.4}
$$

The last line is non-trivial and is an application of Dirichlet's formula and noting $f(v)$ is independent of u. We make the substitution of $s=\frac{u-v}{x-v}$ into the inner integral. The differential is $ds=\frac{du}{x-v}$, then end points become $s=0$ and $s=1$. $x-u$ is equal to $(x-v)(1-s)$ while $u-v$ is $s(x-v)$. Therefore, the inner integral is

$$
\begin{aligned}
&\int_v^x \frac{1}{(x-v)^{1-\beta}(u-v)^{1-\gamma}}du \\
=&\int_0^1 \frac{(x-v)}{\left((x-v)(1-s)\right)^{1-\beta}\left(s(x-v)\right)^{1-\gamma}}ds \\
=&\frac{1}{(x-v)^{1-\beta-\gamma}}\int_0^1 (1-s)^{\beta-1}s^{\gamma-1}ds \\
=&\frac{B(\beta,\gamma)}{(x-v)^{1-\beta-\gamma}},
\end{aligned} \tag{3.5}
$$

where B is the Beta function (Euler integral of the kind). Then

$$
\begin{aligned}
(\mathcal{I}^{\beta}_{0+}\mathcal{I}^{\gamma}_{0+}f)(x) =&\frac{B(\beta,\gamma)}{\Gamma(\beta)\Gamma(\gamma)}\int_0^x \frac{f(v)}{(x-v)^{1-\beta-\gamma}}dv \\
=&(\mathcal{I}^{\beta+\gamma}_{0+}f)(x),
\end{aligned} \tag{3.6}
$$

where we use the relation $B(\beta,\gamma)=\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$. The proof of the right-sided fractional integral is similar, and would collapse to that of the left-sided fractional integral once one exploits what is referred to as the reflection operator.

We also need an appropriate fractional analogue of the classical derivative. The domain for the operator is more restricted since we need to assume a modicum of regularity. But the definitions are intuitive, if we integrate a function to order α, we expect the result to be differentiable up to order α.

Definition 3.2 (The Riemann-Liouville fractional derivative). *Fix $p > 1$, then define $\mathcal{I}^{\alpha}_{0+}(L^p)$ (or $\mathcal{I}^{\alpha}_{T-}(L^p)$) to be the image of $L^p([0,T])$ under the operator $\mathcal{I}^{\alpha}_{0+}$ (or $\mathcal{I}^{\alpha}_{T-}$). If f is in $\mathcal{I}^{\alpha}_{0+}(L^p)$ (or $\mathcal{I}^{\alpha}_{T-}(L^p)$) then we define the left-sided (right-sided) Riemann-Liouville fractional derivatives as*

$$(\mathcal{D}^{\alpha}_{0+}f)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{x^{\alpha}} + \alpha\int_0^x \frac{f(x)-f(u)}{(x-u)^{\alpha+1}}du\right) \tag{3.7}$$

and

$$(\mathcal{D}^{\alpha}_{T-}f)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(T-x)^{\alpha}} + \alpha\int_x^T \frac{f(x)-f(u)}{(u-x)^{\alpha+1}}du\right). \tag{3.8}$$

Unlike integration, this definition only holds for α in $(0,1)$.

We write the definitions differently than in [SKM93]. Our presentation is the same as the one found in [LN06]. To extend the fractional derivatives to higher orders we break the order γ into $\gamma = \lfloor\gamma\rfloor + \{\gamma\}$. Here, $\lfloor\gamma\rfloor$ is the largest integer no greater than γ, $\{\gamma\} = \gamma - \lfloor\gamma\rfloor$ is the fractional part of γ. Then we define $\mathcal{D}^{\gamma}_{0+}f = \frac{d^{\lfloor\gamma\rfloor}}{dx^{\lfloor\gamma\rfloor}}\mathcal{D}^{\{\gamma\}}_{0+}f$, and $\mathcal{D}^{\gamma}_{T-}f = (-1)^{\lfloor\gamma\rfloor}\frac{d^{\lfloor\gamma\rfloor}}{dx^{\lfloor\gamma\rfloor}}\mathcal{D}^{\{\gamma\}}_{T-}f$. The slight difference in definition is due to our desired for left-sided derivative to generalize $\frac{d}{dx}$ while the right-handed should extend $-\frac{d}{dx}$. With all these definitions in place, we can state the most basic of the fractional versions of the fundamental theorems of calculus

$$\begin{aligned}
\mathcal{D}^{\alpha}_{0+}\mathcal{I}^{\alpha}_{0+}f &= f, \quad \forall f \in L^1([0,T]) \\
\mathcal{I}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}f &= f, \quad \forall f \in \mathcal{I}^{\alpha}_{0+}(L^1) \\
\mathcal{D}^{\alpha}_{T-}\mathcal{I}^{\alpha}_{T-}f &= f, \quad \forall f \in L^1([0,T]) \\
\mathcal{I}^{\alpha}_{T-}\mathcal{D}^{\alpha}_{T-}f &= f, \quad \forall f \in \mathcal{I}^{\alpha}_{T-}(L^1).
\end{aligned} \tag{3.9}$$

The Clark-Ocone-Haussmann Theorem for fBm will come later, but its more technical arguments rely on a series of function space embeddings which we present now since we are currently on the subject of fractional calculus. If $\alpha < \frac{1}{p}$, then q satisfies $\frac{1}{p} + \frac{1}{q} = \alpha$ (so q is similar to a Hölder conjugate of p) then the following inclusion holds

$$\mathcal{I}^{\alpha}_{0+}(L^p) = \mathcal{I}^{\alpha}_{T-}(L^p) \subset L^q([0,T]). \tag{3.10}$$

Conversely, if $\alpha > \frac{1}{p}$ then

$$\mathcal{I}^{\alpha}_{0+}(L^p) \cup \mathcal{I}^{\alpha}_{T-}(L^p) \subset C^{\alpha-\frac{1}{p}}([0,T]), \tag{3.11}$$

and we have the following inversion formulae

$$\begin{aligned} \mathcal{D}^{\alpha}_{0+}\mathcal{I}^{\alpha}_{0+}f &= f, \quad \forall f \in L^p([0,T]) \\ \mathcal{I}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}f &= f, \quad \forall f \in \mathcal{I}^{\alpha}_{0+}(L^p) \\ \mathcal{D}^{\alpha}_{T-}\mathcal{I}^{\alpha}_{T-}f &= f, \quad \forall f \in L^p([0,T]) \\ \mathcal{I}^{\alpha}_{T-}\mathcal{D}^{\alpha}_{T-}f &= f, \quad \forall f \in \mathcal{I}^{\alpha}_{T-}(L^p). \end{aligned} \tag{3.12}$$

Appendix B of [Bia+10] collects a nice summary of relations in the fractional calculus. The left-sided and right-sided derivatives are further related by the integration by parts

$$\int_0^T f(x)(\mathcal{D}^{\alpha}_{0+}g)(x)dx = \int_0^T (\mathcal{D}^{\alpha}_{T-}f)(x)g(x)dx, \tag{3.13}$$

for $f \in \mathcal{I}^{\alpha}_{T-}(L^p)$ and $g \in \mathcal{I}^{\alpha}_{0+}(L^q)$ where $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, $\alpha \in (0,1)$, and p, q are both greater than 1. The corresponding integration by parts formula for fractional integration is

$$\int_0^T f(x)(\mathcal{I}^{\alpha}_{0+}g)(x)dx = \int_0^T (\mathcal{I}^{\alpha}_{T-}f)(x)g(x)dx, \tag{3.14}$$

for $f \in L^p([0,T])$, $g \in L^2([0,T])$ where $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ where α must be positive and p, q greater than 1.

We will use results from papers like [CN05] which decline to use Riemann-Liouville fractional derivatives and integrals for functions on bounded intervals. Instead, they consider fractional calculus on the whole real line. Ultimately the difference occurs because while we consider fBm on a fixed time interval $[0,T]$ they opt to consider fBm as a stochastic process whose time domain is the entire real line. Consequently, their fractional calculus is based on the Marchaud fractional derivatives

$$\mathcal{D}^{\alpha}_{\pm}f(X) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x \mp u)}{u^{1+\alpha}} du, \tag{3.15}$$

and the fractional integrals on the entire real axis

$$
\begin{aligned}
(\mathcal{I}_{+}^{\alpha} f)(x) =& \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(u)du}{(x-u)^{1-\alpha}} \\
(\mathcal{I}_{-}^{\alpha} f)(x) =& \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(u)du}{(u-x)^{1-\alpha}}.
\end{aligned}
\tag{3.16}
$$

Translating results will not be difficult, but for completion we should mention this nuance.

3.1.2 Representation of fBm with $H < \frac{1}{2}$

Let $\mathcal{E}$ denote the family of step functions on the interval $[0,T]$. We can induce an inner product on $\mathcal{E}$ be defining

$$
\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{E}} = R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{3.17}
$$

and then extending by bilinearity to all of $\mathcal{E} \times \mathcal{E}$. R_H is of course the auto-covariance function for a fBm with Hurst index equal to H. We let $\mathfrak{H}$ denote the completion of $\mathcal{E}$ with respect to the norm derived from the above inner product, and $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ denote the inner-product of this resulting Hilbert space. Given a fBm B^H we can consider the map $1_{[0,t]} \mapsto B_t^H$. The above map can be extended to a linear transformation from $\mathfrak{H}$ into square-integrable random variables generated by the process $\{B_t^H\}_{0 \le t \le T}$. Let us abuse notation and denote this map by $B^H : \mathfrak{H} \to L^2(\Omega)$. Again, let us abuse notation and consider the image of this map $B^H = \{B^H(h); h \in \mathfrak{H}\}$. By construction, $\mathbb{E}\left[B^H(h_1)B^H(h_2)\right] = \langle h_1, h_2 \rangle_{\mathfrak{H}}$. B^H is therefore an isonormal Gaussian process over the underlying Hilbert space $\mathfrak{H}$. Our current goal is to characterize $\mathfrak{H}$.

$\mathfrak{H}$ is called the reproducing kernel Hilbert space (RKHS). The nomenclature refers to our search for a square-integrable kernel K on $[0,T]^2$ such that our fractional Brownian motion B^H can be expressed as the Volterra process

$$
B_t^H = \int_0^t K_H(t,s) dW_s, \tag{3.18}
$$

where W is a classical Brownian motion. Furthermore, the kernel will satisfy the identity

$$
R_H(s,t) = \int_0^{s \wedge t} K_H(t,r) K_H(s,r) dr. \tag{3.19}
$$

Examining K_H as H varies can often explain qualitative differences between different fBms than the auto-covariance function. Later on, we will see a sharp distinction between fBm with $H \leq \frac{1}{4}$ as opposed to $\frac{1}{4} < H < \frac{1}{2}$. The differing regularity of K_H in those two cases will explain the differing behavior of fBms from between those two classes.

Our detour into fractional calculus now pays dividends. Proposition 6 of [AMN01] states $\mathfrak{H}$ is the space $\mathcal{I}^{\alpha}_{T-}(L^2)$ where $\alpha = \frac{1}{2} - H$. So $\mathfrak{H}$ is the image of $L^2([0,T])$ under the right-sided Riemann-Liouville fractional integral of order α. To emphasize the dependence of the RKHS on both the interval $[0,T]$ and Hurst parameter (and to save some more ink), we will write $\Lambda^H_T = \mathcal{I}^{\frac{1}{2}-H}_{T-}(L^2)$. We are not often blessed when considering fBm with $H < \frac{1}{2}$ instead of $H > \frac{1}{2}$, but here we have a small grace: the RKHS is an actual space of classical functions. When $H > \frac{1}{2}$, the RKHS is a space of proper distributions. We can enjoy the simple pleasure of evaluating an element of Λ^T_H at a point. When $H > \frac{1}{2}$ we often need to modify an argument to consider only the subspace of the RKHS which can be represented by actual functions.

Then we wish to define an operator $K^*_H : \Lambda^H_T \to L^2([0,T])$ such that 3.18 generalizes to

$$B^H(h) = \int_0^T (K^*_H h)(t) dW_t. \tag{3.20}$$

At that point we will have a way to calculate elements of the first chaos by calculating an associated element of the first chaos generated by classical Brownian motion.

[Nor+99] and Section 5.1 of [Nua06] trudge through the necessary calculus. [DÜ99] proves the Relation (3.19), while we present the kernel ex nihilo:

$$K_H(t,s) = C_H \left[\left(\frac{t}{s}\right)^{-\alpha} (t-s)^{-\alpha} + \alpha s^{\alpha} \int_s^t \frac{du}{u^{1+\alpha}(u-s)^{\alpha}} \right], \tag{3.21}$$

where

$$C_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}}, \tag{3.22}$$

and β is the usual beta function. With kernel in hand we now turn our attention to the map K^*_H. The transformation is centered around the mapping $(K^*_H 1_{[0,t]}(r)) = K_H(t,r)1_{[0,t]}(r)$. To verify this initial choice, consider times t and u.

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T (K_H^* 1_{[0,s]})(r)dW_r \int_0^T (K_H^* 1_{[0,t]})(r)dW_r\right] \\
=&\mathbb{E}\left[\int_0^T K_H(s,r)1_{[0,s]}(r)dW_r \int_0^T K_H(t,r)1_{[0,t]}(r)dW_r\right] \\
=&\int_0^{t\wedge u} K_H(t,r)K_H(s,r)dr = R_H(t,s).
\end{aligned} \tag{3.23}$$

So the process $t \mapsto \int_0^T (K_H^* 1_{[0,t]})(s)dW_s$ does have the same distribution as fBm. When we extend this definition from step functions to all of Λ_T^H we are fortunate in that we actually do have a closed-form formula for this map K_H^*. The inner product on Λ_T^H is given by

$$\langle g, h\rangle_{\Lambda_T^H} = \langle K_H^* g, K_H^* h\rangle_{L^2([0,T])}, \tag{3.24}$$

where

$$(K_H^* h)(s) = C_H \Gamma(1-\alpha) s^{\alpha} \left(\mathcal{D}_{T-}^{\alpha} u^{-\alpha} h(u)\right)(s). \tag{3.25}$$

The foundation with which to understand fBm with $H < \frac{1}{2}$ as an isonormal Gaussian process is now complete, and we can now begin to consider the particulars of the Malliavin calculus specific to fBm with $H < \frac{1}{2}$.

3.2 The Russo-Valois Symmetric Integral

We turn our attention to the integration of stochastic processes against a fBm B^H with Hurst index less than $\frac{1}{2}$. A naïve attempt to define integration against fBm path-wise fails immediately. We need only consider $\int_0^1 B_t^H dB_t^H$. Consider a partition $0 \le t_0 \le \cdots \le t_N = 1$, and suppose we approximate the $\int_{t_i}^{t_{i+1}} B_t^H dB_t^H$ by $(\alpha B_{t_i}^H + \beta B_{t_{i+1}}^H)(B_{t+1}^H - B_{t_i}^H)$ where $\alpha + \beta = 1$ and both parameters are non-negative. So the "point of evaluation" is some convex combination of the values at the end-points of the interval. Since this is only a sketch, we can make life more pleasant and assume the partition

$t_i = \frac{i}{N}$. The expectation of the approximation would be

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=0}^{N-1}(\alpha B_{t_i}^H + \beta B_{t_{i+1}}^H)(B_{t_{i+1}}^H - B_{t_i}^H)\right] \\
=&\sum_{i=0}^{N-1}\left(\alpha\left(R_H\left(\frac{i+1}{N},\frac{i}{N}\right) - R_H\left(\frac{i}{N},\frac{i}{N}\right)\right)\right. \\
&\left.+\beta\left(R_H\left(\frac{i+1}{N},\frac{i+1}{N}\right) - R_H\left(\frac{i+1}{N},\frac{i}{N}\right)\right)\right) \\
=&\frac{\alpha}{2}\sum_{i=0}^{N-1}\left(\frac{(i+1)^{2H}}{N^{2H}} - \frac{i^{2H}}{N^{2H}} - \frac{1}{N^{2H}}\right) \\
&+\frac{\beta}{2}\sum_{i=0}^{N-1}\left(\frac{(i+1)^{2H}}{N^{2H}} - \frac{i^{2H}}{N^{2H}} + \frac{1}{N^{2H}}\right) \\
=&\frac{\alpha}{2}\left(1 - \frac{N}{N^{2H}}\right) + \frac{\beta}{2}\left(1 + \frac{N}{N^{2H}}\right) \\
=&\frac{1}{2} + \frac{\beta-\alpha}{2}N^{1-2H}.
\end{aligned}
\tag{3.26}
$$

Since $H < \frac{1}{2}$, the term N^{1-2H} grows without bound. The only way for our approximation to actually converge to anything would be if $\alpha = \beta = \frac{1}{2}$, which would correspond to Stratonovich integration. Our exercise hints that the way forward with respect to path-wise integration of fBm will be some notion of a symmetric integral.

We will remark upon Stratonovich integration later, and rough-path theory in general, but our chosen method is the Russo-Valois symmetric integral [RV93] [RV95]. With Gradinaru and Nourdin, Russo and Vallois were able to extend their results to arrive at an Itô formula for $H > \frac{1}{6}$ [GRV03] [Gra+05]. This is one of the few results we are aware of which manages to pierce the boundary at $H = \frac{1}{4}$. Given continuous processes X and Y we can define the forward integral by the limit in probability of the quantity

$$\frac{1}{\epsilon}\int_0^T Y_u(X_{(u+\epsilon)\wedge T} - X_u)du, \tag{3.27}$$

as ϵ approaches 0 from above and is denoted $\int_0^T Y_u d^- X_u$. Similarly, we define the symmetric integral as the limit in probability of

$$\frac{1}{2\epsilon}\int_0^T Y_u(X_{(u+\epsilon)\wedge T} - X_{(u-\epsilon)\vee 0})du, \tag{3.28}$$

as ϵ approaches 0 from above and is denoted $\int_0^T Y_u d^\circ X_u$. We are restricting ourselves to fBm considered as a process on the interval $[0, T]$. The definitions in [CN05] are slightly different, but as that paper explains in Remark 5.2, the differences are superficial when we consider continuous processes. The correction term between the forward and symmetric integrals is Russo and Vallois's covariation $[X, Y]_t$, which is the limit in probability of

$$\frac{1}{\epsilon}\int_0^T (X_{(u+\epsilon)\wedge T} - X_u)(Y_{(u+\epsilon)\wedge T} - Y_u)du, \tag{3.29}$$

as ϵ shrinks to zero. The relation between the three quantities is what you would expect as generalization of the Itô and Stratonovich integrals.

$$\int_0^T Y_u d^\circ X_u = \int_0^T Y_u d^- X_u + \frac{1}{2}[X, Y]_T. \tag{3.30}$$

When X and Y are continuous Martingales, the above relation is precisely the translation between Itô and Stratonovich integration. But the above expression can also hold for processes X, Y which do not have a finite quadratic variation. Theorem 5.3 of [CN05] is a fundamental theorem of calculus result:

Theorem 3.3. *Let $g \in C^4([0, T])$ and $0 \le a \le b \le T$, then*

$$g(B_b^H) - g(B_b^H) = \int_a^b g'(B_t^H) d^\circ B_t^H, \tag{3.31}$$

when $\frac{1}{6} < H < \frac{1}{2}$, and the integral does not exist if $H \le \frac{1}{6}$.

We must now fine some method of comparing our new path-wise integration with the divergence operator.

3.3 Extending the Divergence Operator

We have explored an interesting framework but the standard Malliavin calculus has a slight shortcoming: even simple stochastic integrals like $\int_0^T B_t^H dB_t^H$ is undefined when $H \le \frac{1}{4}$; see Proposition 3.2 of [CN05]. We know what it would be if it could be. The only answer which can possibly make sense is $H_2(B_T^H, T^H) = \frac{1}{2}((B_T^H)^2 - T^{2H})$. However, the process B_t^H is not in $\mathrm{Dom}\,\delta$ for such small H. So our algebraic method of integration currently has a more restricted domain than our path-wise method introduced in the previous section.

There has been much work on extending the divergence operator. The introduction to [CN05] surveys some attempts to extend the Malliavin calculus. [CN05] builds upon [AMN00] [AMN01]. This extension is the basis for a Clark-Ocone-Haussmann theorem for random variables generated by fBm with $H < \frac{1}{2}$. It is also a fairly natural extension based on the same ideas of duality as the normal divergence operator. Also, it is not an operator on Hida distributions. If we can get away with extending integration without entering the realm of functional calculus on spaces of distributions, we probably should. This approach is a more specific instance to the technique Lèon and Nualart use in order to extend the divergence operator for arbitrary isonormal Gaussian processes [LN05].

Pedants, i.e. anyone who would read a mathematics dissertation, would note the extension in [CN05] uses a different set of fractional derivative operators. Our characterization of the RKHS used the Riemann-Liouville fractional derivative, while the paper uses the Marchaud fractional derivatives. Cheridito and Nualart consider fBm as a two-sided process on the entire real line, while we restrict our attention to the interval $[0, T]$. The Marchaud derivative better handles functions rapidly growing functions but that concern is moot when a maturity date is fixed from the start. [LN06] use these results but have the same RKHS Λ_T^H as us and the same fractional calculus operators as we do. So we are keeping kosher at least as rigorously as Lèon and Nualart. We can go through [CN05] and convince ourselves the results translate. The analogous subspace inclusions, and integration by parts formulae hold for the Riemann-Liouville fractional derivatives. Ultimately, the Hilbert space they consider is the $\Lambda^H = I_-^\alpha(L^2(\mathbb{R})) = I_+^\alpha(L^2(\mathbb{R}))$; the image of $L^2(\mathbb{R})$ under the fractional integral under the entire real-axis, and $\Lambda_T^H = \{h \in \Lambda_H; h1_{[0,T]}(\cdot)\}$. So elements of Λ_T^H are functions on the real-line supported on the interval $[0, T]$. Then if we consider the Marchaud derivatives in Equation ()3.16), we have the relation

$$\langle g, h\rangle_{\Lambda^H} = c_H^2 \langle \mathcal{D}_-^\alpha g, \mathcal{D}_-^\alpha h\rangle_{L^2(\mathbb{R})}, \tag{3.32}$$

where c_H is a normalization constant. Pipiras and Taqqu [PT00] have done the analytical verification for us. We interchangeably consider g and h as both functions on $[0, T]$ and as functions on $\mathbb{R}$

supported on $[0,T]$. As a sanity check we verify this alternative representation when f and g are indicator functions on $[0,T]$. Let $g = 1_{[0,t]}$, then the Marchaud derivative of g is

$$\Gamma(1-\alpha)(\mathcal{D}_-^\alpha 1_{[0,t]})(x) = \alpha \int_0^\infty \frac{1_{[0,t]}(x) - 1_{[0,t]}(x+u)}{u^{1+\alpha}} du. \tag{3.33}$$

This is obviously zero when $x \geq t$. If $x \leq 0$ then

$$\begin{aligned}\Gamma(1-\alpha)(\mathcal{D}_-^\alpha 1_{[0,t]})(x) &= -\alpha \int_0^\infty \frac{1_{[0,t]}(x+u)}{u^{1+\alpha}} du \\ &= -\alpha \int_{-x}^{t-x} \frac{1}{u^{1+\alpha}} du \\ &= \left((t-x)^{-\alpha} - (-x)^{-\alpha}\right).\end{aligned} \tag{3.34}$$

Finally, if $0 \leq x \leq t$ then

$$\begin{aligned}\Gamma(1-\alpha)(\mathcal{D}_-^\alpha 1_{[0,t]})(x) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{1 - 1_{[0,t]}(x+u)}{u^{1+\alpha}} du \\ &= \alpha \int_{t-x}^\infty \frac{du}{u^{1+\alpha}} \\ &= (t-x)^{-\alpha}.\end{aligned} \tag{3.35}$$

If we let denote $\max(a,0)$ as $(a)_+$ then we may write

$$(\mathcal{D}_-^\alpha 1_{[0,t]})(x) = \frac{1}{\Gamma(1-\alpha)} \left((t-x)_+^{-\alpha} - (-x)_+^{-\alpha}\right). \tag{3.36}$$

To find the normalizing constant, consider

$$\begin{aligned}&\langle 1_{[0,T]}, 1_{[0,T]} \rangle_\Lambda \\ =& c_H^2 \langle (\mathcal{D}_-^\alpha 1_{[0,T]})(x), (\mathcal{D}_-^\alpha 1_{[0,T]})(x) \rangle_{L^2(\mathbb{R})} \\ =& \frac{c_H^2}{\Gamma(1-\alpha)^2} \int_{\mathbb{R}} \left((T-u)_+^{-\alpha} - (-u)_+^{-\alpha}\right)^2 du.\end{aligned} \tag{3.37}$$

First make the substitution $u = Tv$, then

$$\begin{aligned}
&\langle 1_{[0,T]}, 1_{[0,T]}\rangle_\Lambda \\
=&\frac{c_H^2}{\Gamma(1-\alpha)^2}\int_{\mathbb{R}}\left((T-Tv)_+^{-\alpha}-(-Tv)_+^{-\alpha}\right)^2 T dv \\
=&\frac{c_H^2}{\Gamma(1-\alpha)}T^{1-2\alpha}\int_{\mathbb{R}}\left((1-v)_+^{-\alpha}-(-v)_+^{-\alpha}\right)^2 dv \\
=&\frac{c_H^2}{\Gamma(1-\alpha)}\int_{\mathbb{R}}\left((1-v)_+^{-\alpha}-(-v)_+^{-\alpha}\right)^2 dv T^{2H}
\end{aligned} \tag{3.38}$$

That is promising. Another substitution of $w = -v$ leads us to

$$c_H^2 = \frac{\Gamma(1-\alpha)}{\int_{\mathbb{R}}\left((1+w)_+^{-\alpha}-(w)_+^{-\alpha}\right)^2 dw}. \tag{3.39}$$

Definition 3.4 (Extended Divergence Operator)**.** *First, let $\Lambda^{H,*}$ be the image of the elementary functions $\mathcal{E}$ in $L^2(\mathbb{R})$ under the fractional integration operator $\mathcal{I}_-^\alpha$ where α is our customary quantity $\frac{1}{2} - H$. Let u be a measurable process. We say that u is in the extended domain $\operatorname{Dom}^*\delta$ if and only if there exists a random variable, denoted $\delta(u)$, such that $\delta(u) \in \cup_{p>1} L^p(\Omega)$ and for all natural n and $h \in \Lambda^{H,*}$ with unit length $\|h\|_{\Lambda^H} = 1$, the following conditions hold*

- $u_t H_{n-1}(B^H(h))$ *is in* $L^1(\Omega)$ *for almost all* $t \in \mathbb{R}$
- $\mathbb{E}[uH_{n-1}(B^H(\phi))]\mathcal{D}_+^\alpha\mathcal{D}_-^\alpha h$ *is in* $L^1(\mathbb{R})$
- $c_H^2\int_0^T \mathbb{E}[u_t H_{n-1}(B^H(h))]\mathcal{D}_+^\alpha\mathcal{D}_-^\alpha h(t)dt = \mathbb{E}[\delta(u)H_n(B^H(h))]$

The trade-off for the simpler form is the Marchaud derivative of a function in Λ_T^H is not necessarily supported within $[0, T]$. The Marchaud derivative does smear.

The end result of this work is an Itô formula connecting the Russo-Valois symmetric integral and the extended divergence operator. There is a regularity requirement and a growth condition. The growth condition is fairly generous. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the growth condition on an interval (a, b) is there exist a positive c and λ such that λ is bounded from above by $\frac{1}{4}(|a| \vee |b|)^{-2H}$ and $|f(x)| \le ce^{\lambda x^2}$ for all real x. This ensures $f(B_t^H)$ is in $L^2(\Omega)$ for any t in the interval (a, b).

Theorem 3.5 (Itô formula for extended divergence)**.** *If f satisfies the above growth condition, and*

$f \in C^2(\mathbb{R})$ then the Itô formula for the extended divergence operator is

$$\begin{aligned}\delta\left[f'(B_t^H)1_{(a,b]}(t)\right] =& f(B_b^H) - f(B_a^H) - H\int_a^b f''(B_t^H)t^{2H-1}dt \\ =& \int_a^b f'(B_t^H)d^\circ B_t^H - H\int_a^b f''(B_t^H)t^{2H-1}dt.\end{aligned} \tag{3.40}$$

We reserve the right to rewrite $2Ht^{2H-1}dt$ as the differential $d|t|^{2H}$, and the above result as

$$\begin{aligned}\delta\left[f'(B_t^H)1_{(a,b]}(t)\right] =& f(B_b^H) - f(B_a^H) - \frac{1}{2}\int_a^b f''(B_t^H)d|t|^{2H} \\ =& \int_a^b f'(B_t^H)d^\circ B_t^H - \frac{1}{2}\int_a^b f''(B_t^H)d|t|^{2H}.\end{aligned} \tag{3.41}$$

When we develop exponential formulae, we will need to know how the freezing operator acts on the divergence of a process. The above equation gives us a way forward for a large class of problems. We will extend this further by developing an Itô for processes of the form $t \mapsto f(t, B_t^H)$. The Clark-Ocone-Haussmann formula for fBm with $H < \frac{1}{2}$ will provide a path forward for even more general processes.

3.4 The Clark-Ocone formula for fBm with $H < \frac{1}{2}$

Given a random variable F generated by a fractional Brownian motion B^H with Hurst index $H < \frac{1}{2}$, there is a chaos decomposition of f

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{3.42}$$

where each f_n is an element in the symmetric space $(\Lambda_T^H)^{\odot n}$.

Ideally, we would have a suitable Clark-Ocone formula for representing F as $F = \mathbb{E}[F] + \int_0^T u_s \delta B_s^H$ for some suitable process u_s.

In the case of classical Brownian motion, the process u_s is the conditional expectation of the Malliavin derivative at time s, given the history up to time s. But the conditional expectation of fractional Brownian motion is not pleasant [GN96]. However, there is a more tractable quantity to utilize. [LN06] introduces the fractional conditional expectation of a random variable F generated by an fBm with $H < \frac{1}{2}$. The fractional conditional expectation is denoted either as $\tilde{\mathbb{E}}[F|\mathcal{F}_s]$ (to remind the reader of the concept's origins), and is defined in terms of the chaos expansion of F

$$\tilde{\mathbb{E}}\,[F|\mathcal{F}_s] := \sum_{n=0}^{\infty} I_n(f_1 1_{[0,s]^n}). \tag{3.43}$$

Then $\tilde{\mathbb{E}}[D_s F|\mathcal{F}_s]$ replaces the customary integrand in the Clark-Ocone formula. The main result of that paper is the following Clark-Ocone formula for random variables generated by fractional Brownian motion with $H < \frac{1}{2}$:

$$F = \mathbb{E}[F] + \int_0^T \tilde{\mathbb{E}}[D_s F|\mathcal{F}_s]\delta B_s^H. \tag{3.44}$$

This quantity $\tilde{\mathbb{E}}[D_s F|\mathcal{F}_s]$ is also written as $D_s^{(p)} F$ and should be understood as the predictable projection of the Malliavin derivative of F at time s.

Chapter 4

Exponential Formulae

4.1 A Sketch and Intuition for an Exponential Formula

Remark 2.3 of [JPS15a] provides a simple sketch of the exponential formula by considering exponential martingales. The proof is then made rigorous in [JS16], but there is value in returning to the sketch to guide us on what the exponential formula should be. This does not qualify as a proof of the exponential formula since, as we have remarked repeatedly, the frozen operator is not closed as an operator on $L^2(\Omega)$. Given a function $f \in L^2([0,T])$ (we will assume f is continuous for the sake of our argument), consider its image under the exponential $\epsilon(f) = \exp(\int_0^T f(s)dB_s)$. Now $\exp(\int_0^t f(s)dB_s - \frac{1}{2}\int_0^t f(s)^2 ds)$ is a Martingale, so $\mathbb{E}[\epsilon(f)] = \exp(\frac{1}{2}\int_0^T f(s)^2 ds)$. Note the Malliavin derivative of $\epsilon(f)$ is

$$\begin{aligned} D_t\epsilon(f) =& D_t \exp\left(\int_0^T f(s)dB_s\right) \\ =& \exp\left(\int_0^T f(s)dB_s\right) D_t \int_0^T f(s)dB_s \\ =& \epsilon(f)\left(f(t) + \int_0^T D_t f(s)dB_s\right) \\ =& f(s)\epsilon(f). \end{aligned} \tag{4.1}$$

Freezing the exponent results in

$$\begin{aligned}\omega^0 \int_0^T f(s)dB_s =& -\frac{1}{2}\omega^0 \langle \int_0^{\cdot} f dB_s, B_{\cdot} \rangle_T \\ =& -\frac{1}{2}\omega^0 \int_0^T f(s) d\langle f(\cdot), B_{\cdot} \rangle_s \\ =& 0,\end{aligned} \tag{4.2}$$

since f has bounded variation. Now consider the expression

$$\begin{aligned}&\sum_{i=0}^{\infty} \frac{1}{2^i i!} \omega^0 \left(\Lambda^i \epsilon(f) \right) \\ =& \sum_{i=0}^{\infty} \frac{1}{2^i i!} \int_{[0,T]^k} \omega^0 \left(D_{s_i}^2 \cdots D_{s_1}^2 \epsilon(f) \right) ds_1 \cdots ds_i \\ =& \sum_{i=0}^{\infty} \frac{1}{2^i i!} \int_{[0,T]^k} f(s_1)^2 \cdots f(s_i)^2 \omega^0 \left(\epsilon(f) \right) ds_1 \cdots ds_i \\ =& \sum_{i=0}^{\infty} \omega^0 \exp \left(\int_0^T f(s) dB_s \right) \frac{1}{2^i i!} \int_{[0,T]^k} f(s_1)^2 f(s_i)^2 ds_1 \cdots ds_i \\ =& \exp(\omega^0 \int_0^T f(s) ds) \sum_{i=0}^{\infty} \frac{1}{2^i i!} \left(\int_0^T f(s)^2 ds \right)^i \\ =& \exp(0) \sum_{i=0}^{i} \frac{1}{i!} \left(\frac{1}{2} \int_0^T f(s)^2 ds \right)^i \\ =& \exp \left(\frac{1}{2} \int_0^T d(s)^2 ds \right).\end{aligned} \tag{4.3}$$

Therefore

$$\mathbb{E}\left[\epsilon(f)\right] = \sum_{i=0}^{\infty} \frac{1}{2^i i!} \omega^0 \left(\Lambda^i \epsilon(f) \right), \tag{4.4}$$

for all continuous f in $L^2([0,T])$.

$\epsilon(f)$ was chosen because linear combinations of such random variables is dense in $L^2(\Omega)$. The corresponding family of exponentials when considering random variables generated by two independent copies of Brownian motion is

$$\epsilon(f,g) = \exp \left(\int_0^T f dB_1(s) + \int_0^T g dB_2(s) \right), \tag{4.5}$$

where f and g are each in $L^2([0,T])$. We write $\epsilon(f,g) = FG$ where $F = \exp(\int_0^T f dB_1(s))$ and $G = \exp(\int_0^T g dB_2(s))$. Linear combinations of such exponentials is dense in the family of $L^2(\Omega)$ random variables generated by B_1 and B_2. We can leverage independence to see

$$\mathbb{E}\left[\epsilon(f,g)\right] = \mathbb{E}\left[F\right]\mathbb{E}\left[G\right] = \exp\left(\frac{1}{2}\int_0^T f(s)^2 ds + \frac{1}{2}\int_0^T g(s)^2 ds\right), \tag{4.6}$$

and we reuse the univariate sketch to prove

$$\begin{aligned}
&\mathbb{E}\left[F\right]\mathbb{E}\left[G\right] \\
&=\left(\sum_{i=0}^{\infty}\frac{1}{2^i i!}\omega^0 \circ \Delta_1^i F\right)\left(\sum_{i=0}^{\infty}\frac{1}{2^j j!}\omega^0 \circ \Delta_2^j G\right) \\
&=\sum_{i,j=0}^{\infty}\frac{1}{2^{i+j} i! j!}\left(\omega^0 \circ \Delta_1^i F\right)\left(\omega^0 \circ \Delta_2^j G\right) \\
&=\sum_{i,j=0}^{\infty}\frac{1}{2^{i+j} i! j!}\omega^0 \circ \left(\left(\Delta_1^i \Delta_2^j\right)(FG)\right) \\
&=\sum_{k=0}^{\infty}\frac{1}{2^k k!}\omega^0 \circ \left(\left(\sum_{i+j=k; i,j\geq 0}\frac{k!}{i!j!}\Delta_1^i\Delta_2^j\right)\epsilon(f,g)\right) \\
&=\sum_{k=0}^{\infty}\frac{1}{2^k k!}\omega^0 \circ \left(\Delta^k \epsilon(f,g)\right).
\end{aligned} \tag{4.7}$$

The same sketch applies in the multivariate case as in the univariate case, we should expect the form of the exponential formula to remain the same, and hopefully we are able to let the univariate version of the theorem to do the actual heavy lifting.

4.2 The Freezing operator

To be able to apply the frozen operator in the wild, we need to know how it interacts with stochastic integrals. The simplest counter-example to show the frozen operator and stochastic integration does not generally commute is

$$\omega^0 \int_0^T B_t dB_t = \omega^0\left(\frac{1}{2}\left(B_T^2 - T\right)\right) = -\frac{T}{2}, \tag{4.8}$$

while

$$\int_0^T \omega^0(B_t)dB_t = \int_0^T 0dB_t = 0. \tag{4.9}$$

So in general we cannot have

$$\omega^t \int_0^T u_s dB_s = \int_0^t u_s dB_s, \tag{4.10}$$

for adapted processes.

We have the right operator, but are considering the wrong integral. Suppose we were to define the operator ω^t in terms of the Stratonovich integral. That is, if we have an adapted process u, then we define

$$\omega^t \left(\int_0^T u_s d^\circ B_s \right) := \int_0^t u_s d^\circ B_s. \tag{4.11}$$

We then claim that this new definition coincides with our old definition of the frozen operator, at least for large classes of random variables and processes of interest. Intuitively, the frozen path operator is best visualized as an operator on paths. It makes more sense that it would behave nicely with the integral which emphasizes the geometric nature of stochastic processes (Stratonovich) than the integral which emphasizes the probabilistic nature (Itô).

We can verify their coincidence for the usual classes of random variables of interest. It is a simple exercise to show that this claim holds for geometric Brownian motion. We have to be careful about any argument which leverages density since we have shown than the freezing path operator is not closed in the L^2 sense. However, we can show that the claim holds true for smooth cylindrical random variables via a direct argument. First consider a smooth cylindrical random variable of the form $F = f(B_{t_1}, \ldots, B_{t_M})$ where $0 = t_0 \leq t_1 \leq \cdots \leq t_M = T$ is a partition of $[0, T]$. Without loss of generality, we can rewrite F as

$$F = f\left(B_{t_1} - B_{t_0}, \cdots, B_{t_M} - B_{t_{M-1}}\right). \tag{4.12}$$

We claim $F = f(0, \ldots, 0) + \int_0^T u_s d^\circ B_s$ where

$$u_s = \sum_{i=1}^{M} 1_{[t_{i-1}, t_i]} \frac{\partial f}{\partial x_i} \left(B_{t_1} - B_{t_0}, \ldots, B_{t_{i-1}} - B_{t_{i-2}}, B_s - B_{t_{i-1}}, 0, \ldots, 0\right). \tag{4.13}$$

Then we integrate u_s over the interval $[0,t]$ with respect to the Stratonovich integral. There exists a K such that $t_{K-1} \le t \le t_K$, and

$$\begin{aligned}
&\int_0^t u_s d^\circ B_s \\
=&\int_{t_{K-1}}^{t_K} \frac{\partial f}{\partial x_K}\left(B_{t_1}-B_{t_0},\ldots,B_{t_{i-1}}-B_{t_{i-2}},B_s-B_{t_{i-1}},0,\ldots,0\right) d^\circ B_s \\
&+\sum_{i=1}^{K-1}\int_{t_{i-1}}^{t_i} \frac{\partial f}{\partial x_i}\left(B_{t_1}-B_{t_0},\ldots,B_{t_{i-1}}-B_{t_{i-2}},B_s-B_{t_{i-1}},0,\ldots,0\right) d^\circ B_s \\
=&\left(f\left(B_{t_1}-B_{t_0},\ldots,B_{t_{i-1}}-B_{t_{i-2}},B_s-B_{t_{i-1}},0,\ldots,0\right)\right. \\
&\quad\left.-f\left(B_{t_1}-B_{t_0},\ldots,B_{t_{i-1}}-B_{t_{i-2}},0,0,\ldots,0\right)\right) \\
&+\sum_{i=1}^{K-1}\left(f\left(B_{t_1}-B_{t_0},\ldots,B_{t_{i-1}}-B_{t_{i-2}},B_{t_i}-B_{t_{i-1}},0,\ldots,0\right)\right. \\
&\quad\left.-f\left(B_{t_1}-B_{t_0},\ldots,B_{t_{i-1}}-B_{t_{i-2}},B_{t_{i-1}}-B_{t_{i-2}},0,\ldots,0\right)\right) \\
=&\left(f\left(B_{t_1}-B_{t_0},\ldots,B_{t_{i-1}}-B_{t_{i-2}},B_s-B_{t_{i-1}},0,\ldots,0\right)\right. \\
&\quad\left.-f\left(B_{t_1}-B_{t_0},\ldots,B_{t_{i-1}}-B_{t_{i-2}},0,0,\ldots,0\right)\right) \\
&+\left(f\left(B_{t_1}-B_{t_0},\ldots,B_{t_{i-1}}-B_{t_{i-2}},0,\ldots,0\right)\right. \\
&\quad -f(0,\ldots,0)) \\
=&f\left(B_{t_1}-B_{t_0},\ldots,B_{t_{i-1}}-B_{t_{i-2}},B_s-B_{t_{i-1}},0,\ldots,0\right) f(0,\ldots,0).
\end{aligned} \tag{4.14}$$

Thus

$$\begin{aligned}
\omega^t(F) =&f(0,\ldots,0)+\omega^t\int_0^T u_s d^\circ B_s \\
=&f(0,\ldots,0)+\int_0^t u_s d^\circ B_s \\
=&f\left(B_{t_1}-B_{t_0},\ldots,B_{t_{K-1}}-B_{t_{K-2}},B_t-B_{t_{K-1}},0,\ldots,0\right),
\end{aligned} \tag{4.15}$$

which precisely coincides with the old definition of the frozen operator.

Now consider an adapted process u_s which is a continuous semi-martingale, the Skorokhod integral coincides with the Itô integral, and we can exploit the connection between the two integrals to see

$$\begin{aligned}\omega^t \int_0^T u_s dB_s &= \int_0^t u_s d^\circ B_s - \frac{1}{2}\omega^t \langle u., B.\rangle_T \\ &= \int_0^t u_s dB_s + \frac{1}{2}\langle u., B.\rangle_t - \frac{1}{2}\omega^t \langle u., B.\rangle_T.\end{aligned} \tag{4.16}$$

In our instance, we will only consider $t = 0$, which further simplifies this equation to

$$\omega^0 \int_0^T u_s \delta B_s = -\frac{1}{2}\omega^0 \langle u., B.\rangle_T. \tag{4.17}$$

If we look at the involved calculations in [JS16] and squint, we can see some of the more involved calculations are essentially trying to find a closed expression to iteratively applying the above equation to $I_n(f_n)$. Of course, this is all well and good, but we need to consider the multidimensional case. The reader should refresh themselves on Section 2.7 on the multivariate Malliavin calculus. We consider a d-dimensional Brownian motion

$$\boldsymbol{B}(t) = \begin{bmatrix} B_1(s) \\ \vdots \\ B_1(d) \end{bmatrix}, \tag{4.18}$$

and the random variables and processes generated by it. The best way to state our definition is to first consider a continuous integrable adapted $\mathbb{R}^d$-valued process

$$\boldsymbol{u}(s) = \begin{bmatrix} u_1(s) \\ \vdots \\ u_d(s) \end{bmatrix}, \tag{4.19}$$

so each component u_i is a continuous integrable adapted real-valued process. Then the Skorokhod integral of $\boldsymbol{u}(s)$ is

$$\begin{aligned}&\delta(\boldsymbol{u}(s))\\&=\sum_{i=1}^{d}\delta_i(u_i)\\&=\sum_{i=1}^{d}\int_0^T u_i(s)\delta B_i(s),\end{aligned}\tag{4.20}$$

and the Stratonovich integral is

$$\int_0^T \boldsymbol{u}(s)\cdot d^0\boldsymbol{B}(s)=\sum_{i=1}^{d}\int_0^T u_i(s)d^0B_i(s).\tag{4.21}$$

We define the operator ω^t as

$$\omega^t\left(\int_0^T \boldsymbol{u}(s)\cdot d^0\boldsymbol{B}(s)\right):=\sum_{i=1}^{d}\int_0^t u_i(s)d^0B_i(s).\tag{4.22}$$

Since our assumption was that $\boldsymbol{u}_s$ is adapted and continuous, the Skorokhod integral corresponds to the Itô integral. The translation between the Itô and Stratonovich integral remains the same in higher dimensions; see any standard text on rough paths [FV10]. This allows us to take the frozen path of multidimensional stochastic Itô integrals.

$$\begin{aligned}&\omega^t\circ\delta(\boldsymbol{u})\\&=\omega^t\int_0^T \boldsymbol{u}(s)\cdot d\boldsymbol{B}(s)\\&=\omega^t\left(\sum_{i=1}^{d}\int_0^T u_i(s)dB_i(s)\right)\\&=\omega^t\left(\sum_{i=1}^{d}\int_0^T u_i(s)d^0B_i(s)-\frac{1}{2}\langle u_i(\cdot),B_i(\cdot)\rangle_T\right)\\&=\sum_{i=0}^{d}\int_0^t u_i(s)d^0B_i(s)-\omega^0\frac{1}{2}\langle u_i(\cdot),B_i(\cdot)\rangle_T\\&=\sum_{i=0}^{d}\left(\int_0^t u_i(s)d^0B_i(s)+\frac{1}{2}\langle u_i(\cdot),B_i(\cdot)\rangle_t-\omega^0\frac{1}{2}\langle u_i(\cdot),B_i(\cdot)\rangle_T\right).\end{aligned}\tag{4.23}$$

Now that is a formula with which we can work. Let us this formula and examine a simple exercise which will show us how we can apply the frozen operator in the multidimensional case, and show it is compatible with the conjectured exponential formula given in the preceding section.

Our chosen example is $F = (B_1^2(T)-T)(B_2^2(T)-T)$ because it is a martingale with nice analytical properties, and therefore we can readily check if the answer we produce after cranking the wheels of our machinery is correct. It is also the simplest example which has a non-trivial second-order Gross Laplacian. So it is a nice choice for an example of how these freezing operators and Gross Laplacians work in practice. We would desire $\omega^t(F) = (B_1^2(t) - T)(B_2^2(t) - T)$, which we will prove from Equation (4.23). The Clark-Ocone formula in the multivariate Malliavin calculus is

$$\begin{aligned} F =& \mathbb{E}[F] + \int_0^T \mathbb{E}\left[D_s F | \mathcal{F}_s\right] \cdot \delta \boldsymbol{B}(s) \\ =& \mathbb{E}[F] + \sum_{i=1}^d \int_0^T \left[D_{i,s} F | \mathcal{F}_s\right] dB_i(s). \end{aligned} \tag{4.24}$$

For our chosen F we have $\mathbb{E}[F] = 0$. And the Malliavin derivative in our case

$$\begin{aligned} D_s F =& \begin{bmatrix} D_{1,s}\left((B_1^2(T) - T)(B_2^2(T) - T)\right) \\ D_{2,s}\left((B_1^2(T) - T)(B_2^2(T) - T)\right) \end{bmatrix} \\ =& \begin{bmatrix} 2B_1(T)(B_2^2(T) - T) \\ 2B_2(T)(B_1^2(T) - T) \end{bmatrix}. \end{aligned} \tag{4.25}$$

Each component is the product of independent martingales, so we see the conditional expectation of the Malliavin derivative is

$$\begin{aligned} \mathbb{E}\left[D_s F | \mathcal{F}_s\right] =& \begin{bmatrix} \mathbb{E}\left[2B_1(T)(B_2^2(T) - T) | \mathcal{F}_s\right] \\ \mathbb{E}\left[2B_1(T)(B_2^2(T) - T) | \mathcal{F}_s\right] \end{bmatrix} \\ =& 2\begin{bmatrix} B_1(s)(B_2^2(s) - s) \\ B_2(s)(B_1^2(s) - s) \end{bmatrix}. \end{aligned} \tag{4.26}$$

Therefore

$$\omega^t F = 2\left(\omega^0 \circ \int_0^T B_1(s)(B_2^2(s) - s)dB_1(s) + \omega^0 \int_0^T B_2(s)(B_1^2(s) - s)dB_2(s)\right). \tag{4.27}$$

Let us calculate that first summand.

$$\begin{aligned}
&2\omega^0 \int_0^T B_1(s)(B_2^2(s) - s)dB_1(s) \\
=&2\int_0^t B_1(s)(B_2^2(s) - s)dB_1(s) + \left\langle B_1(s)(B_2^2(s) - s), B_1(s)\right\rangle_t \\
&- \omega^0 \left\langle B_1(s)(B_2^2(s) - s), B_1(s)\right\rangle_T .
\end{aligned} \tag{4.28}$$

To calculate the quadratic variation, it is best to have a martingale representation of $B_1(s)(B_2^2(s)-s)$. Again, we use Clark-Ocone.

$$\begin{aligned}
&B_1(s)(B_2^2(s) - s) \\
=&\mathbb{E}\left[B_1(s)(B_2^2(s) - s)\right] + \int_0^s \mathbb{E}\left[D_s\left(B_1(s)(B_2^2(s) - s)\right)|\mathcal{F}_r\right] \cdot d\boldsymbol{B}(r) \\
=&0 + \int_0^s \mathbb{E}\left[D_{1,s}\left(B_1(s)(B_2^2(s) - s)\right)|\mathcal{F}_r\right] dB_1(r) \\
&+ \int_0^s \mathbb{E}\left[D_{2,s}\left(B_1(s)(B_2^2(s) - s)\right)|\mathcal{F}_r\right] dB_2(r) \\
=&\int_0^s \mathbb{E}\left[B_2^2(s) - s|\mathcal{F}_r\right] dB_1(r) + 2\int_0^s \mathbb{E}\left[B_1(s)B_2(s)|\mathcal{F}_r\right] dB_2(r) \\
=&\int_0^s \left(B_2^2(r) - r\right) dB_1(r) + 2\int_0^s B_1(r)B_2(r)dB_2(r)
\end{aligned} \tag{4.29}$$

Then

$$\begin{aligned}
&\left\langle B_1(s)(B_2^2(s)-s), B_1(s)\right\rangle_t \\
=&\left\langle \int_0^s B_2^2(r)-rdB_1(r)+2\int_0^s B_1(r)B_2(r)dB_2(r), B_1(s)\right\rangle_t \\
=&\left\langle \int_0^s B_2^2(r)-rdB_1(r), B_1(s)\right\rangle_t + 2\left\langle \int_0^s B_1(r)B_2(r)dB_2(r), B_1(s)\right\rangle_t \\
=&\int_0^t B_2^2(r)-rd\langle B_1(\cdot),B_1(\cdot)\rangle_r + 2\int_0^t B_1(r)B_2(r)d\langle B_2(\cdot),B_1(\cdot)\rangle_r \\
=&\int_0^t B_2^2(r)-rdr+2\int_0^t B_1(r)B_2(r)d0 \\
=&\int_0^t B_2^2(r)dr-\frac{t^2}{2},
\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}
&2\omega^0\int_0^T B_1(s)(B_2^2(s)-s)dB_1(s) \\
=&2\int_0^t B_1(s)(B_2^2(s)-s)dB_1(s)+\int_0^t B_2^2(r)dr-\frac{t^2}{2} \\
&-\omega^t\left(\int_0^T B_2^2(r)dr-\frac{T^2}{2}\right) \\
=&2\int_0^t B_1(s)(B_2^2(s)-s)dB_1(s)+\int_0^t B_2^2(r)dr-\frac{t^2}{2} \\
&-\int_0^t \omega^t B_2^2(r)dr+\frac{T^2}{2} \\
=&2\int_0^t B_1(s)(B_2^2(s)-s)dB_1(s)+\int_0^t B_2^2(r)dr-\frac{t^2}{2} \\
&-\int_0^t B_2^2(r)dr-\int_t^T B_2^2(t)dr+\frac{T^2}{2} \\
=&2\int_0^t B_1(s)(B_2^2(s)-s)dB_1(s)-B_2^2(t)(T-t)-\frac{t^2}{2}-\frac{T^2}{2}.
\end{aligned} \tag{4.31}$$

Similarly

$$\begin{aligned}
&2\omega^0\int_0^T B_2(s)(B_1^2(s)-s)dB_2(s) \\
=&2\int_0^t B_2(s)(B_1^2(s)-s)dB_2(s)-B_1^2(t)(T-t)-\frac{t^2}{2}+\frac{T^2}{2}.
\end{aligned} \tag{4.32}$$

Combining these two results

$$
\begin{aligned}
&\omega^t F \\
=&2\int_0^t B_1(s)(B_2^2(s)-s)dB_1(s) - B_2^2(t)(T-t) - \frac{t^2}{2} + \frac{T^2}{2} \\
&2\int_0^t B_2(s)(B_1^2(s)-s)dB_2(s) - B_1^2(t)(T-t) - \frac{t^2}{2} + \frac{T^2}{2} \\
=&\int_0^t \begin{bmatrix} 2B_1(s)(B_2^2(s)-s)dB_1(s) \\ 2B_2(s)(B_1^2(s)-s)dB_2(s) \end{bmatrix} \cdot d\boldsymbol{B}(s) - (B_1^2(t) - B_2^2(t))(T-t) - t^2 + T^2 \\
=&\int_0^t \mathbb{E}\left[D_s(B_1^2(t)-t)(B_2^2(t)-t)|\mathcal{F}_s\right] \cdot d\boldsymbol{B}(s) - (B_1^2(t) - B_2^2(t))(T-t) - t^2 + T^2 \\
=&(B_1^2(t)-t)(B_2^2(t)-t) - (B_1^2(t) - B_2^2(t))(T-t) - t^2 - T^2 \\
=&B_1^2(t)B_2^2(t) - t(B_1^2(t)+B_2^2(t)) + t^2 - (B_1^2(t) - B_2^2(t))(T-t) - t^2 + T^2 \\
=&B_1^2(t)B_2^2(t) - T(B_1^2(t)+B_2^2(t)) + T^2 \\
=&(B_1^2(t)-T)(B_2^2(t)-T),
\end{aligned}
\tag{4.33}
$$

which is positive evidence towards our approach to freezing. To corroborate our conjecture of the exponential formula, let us calculate all the non-trivial the Gross Laplacians:

$$
\begin{aligned}
\Delta_1 F =& \int_0^T D_{1,u}^2 \left((B_1^2(T) - T)(B_2^2(T) - T)\right) du \\
=& 2 \int_0^T (B_2^2(T) - T) du \\
=& 2T(B_2^2(T) - T) \\
\Delta_2 F =& 2T(B_1^2(T) - T) \\
\Delta F =& \Delta_1 F + \Delta_2 F \\
=& 2TB_1^2(T) + 2TB_2^2(T) - 4T^2 \\
\Delta_1^2 F =& \int_0^T D_{1,u}^2 \left(2T(B_2^2(T) - T)\right) du \\
=& 0 \\
\Delta_2^2 F =& 0 \\
\Delta_2 \Delta_1 F =& \int_0^T D_{2,u}^2 \left(2T(B_2^2 - T)\right) du \\
=& 4T \int_0^T du \\
=& 4T^2 \\
\Delta^2 F =& \Delta_1^2 F + 2\Delta_2 \Delta_1 F + \Delta_2^2 F \\
=& 8T^2,
\end{aligned}
\tag{4.34}
$$

and all higher-order Laplacians are zero. Then plugging this into the exponential formula

$$
\begin{aligned}
& \sum_{k=0}^{\infty} \frac{1}{2^k k!} \omega^0 \circ \Delta^k F \\
=& \omega^0 F + \frac{1}{2} \omega^0 \Delta F + \frac{1}{8} \omega^0 \Delta^2 F \\
=& \omega^0 \left(B_1^2(T) - T)(B_2^2(T) - T)\right) + \frac{1}{2} \omega^0 \left(2TB_1^2(T) + 2TB_2^2(T) - 4T^2\right) + \frac{1}{8} \omega^0 \left(8T^2\right) \\
=& T^2 + \frac{1}{2}(-4T^2) + T^2 \\
=& 0,
\end{aligned}
\tag{4.35}
$$

which is precisely $\mathbb{E}[F]$.

4.3 An Exponential Formula for 2-Dimensional Brownian Motion

To prove our analogue, consider Wiener chaos decomposition $\sum_{n=0}^{\infty} I_n(f_n)$ of a random variable F. If we return to our previous exposition on the Malliavin calculus on d-dimensional Brownian Motion (Section 2.7), then f_n is a tensor-valued function of the form

$$f_n(s_1, \ldots, s_n) = \sum_{1 \leq i_1, \ldots, i_n \leq 2} f^{i_1 \cdots i_n}(s_1, \ldots, s_n) \boldsymbol{e}_{i_1} \otimes \cdots \boldsymbol{e}_{i_n}. \tag{4.36}$$

In which case, we have

$$I_n(f_n) = \sum_{1 \leq i_1, \ldots, i_n \leq 2} \delta_{i_n} \cdots \delta_{i_1} f^{i_1 \cdots i_n}. \tag{4.37}$$

From Section 2.7 we know that for any permutation σ on $\{1, \ldots, n\}$ we have

$$\delta_{i_{\sigma(n)}} \cdots \delta_{i_{\sigma(1)}} f^{i_{\sigma(1)} \cdots i_{\sigma(n)}} = \delta_{i_n} \cdots \delta_{i_1} f^{i_1 \cdots i_n}. \tag{4.38}$$

To refresh, we exploit this symmetry and consider what we refer to as a canonical component. Consider the summand of the form

$$f^{\overbrace{1 \cdots 1}^{\ell 1s}\overbrace{2 \cdots 2}^{m2s}} \tag{4.39}$$

where of course $\ell + m = n$. We can denote this summand by $f^{(\ell,m)}$. From the above reasoning, we can rewrite $I_n(f_n)$ as

$$I_n(f_n) = \sum_{\ell + m = n} \binom{n}{\ell} \delta_2^m \delta_1^\ell f^{(\ell,m)}. \tag{4.40}$$

The advantage of writing $I_n(f_n)$ in this form is each summand is unique. Furthermore, each term

$$f^{(\ell,m)}(r_1, \ldots, r_\ell, s_1, \ldots, s_m), \tag{4.41}$$

is invariant under any permutation of the first ℓ parameters, and is also equal under any permutation under the last m parameters which simplifies some analysis. Lastly, when writing in this format it feels "obvious" how to proceed in the d-dimensional case.

Let us show our exponential formula directly. We have our symmetric kernel

$$f(r_1, \ldots, r_\ell, s_1, \ldots, s_m), \tag{4.42}$$

then

$$\begin{aligned}
&\omega^0 \circ \delta_2^m \delta_1^\ell f \\
&= m!\omega^0 \circ \int_0^T \int_0^{s_m} \cdots \int_0^{s_2} \delta_1^\ell f(\cdot, s_1, \ldots, s_m) dB_2(s_1) \cdots dB_2(s_m) \\
&= -\frac{m!}{2}\omega^0 \circ \left\langle \int_0^v \int_{\Delta_{m-2}(s_{m-1})} \delta_1^\ell f(\cdot, s_1, \ldots, s_{m-1}, v) dB_2(s)^{\otimes(m-1)}, B_2(v) \right\rangle_T,
\end{aligned} \tag{4.43}$$

where $\Delta_{m-2}(s_{m-1})$ is the simplex $\{(s_1, \ldots, s_{m-2}); 0 \leq s_1 \leq \cdots \leq s_{m-2} \leq s_{m-1}\}$, and not the Gross Laplacian. Then, if we take a partition $\Pi = \{0 = v_0 \leq v_1 \leq v_M = T\}$, the quadratic variation is the limit of all approximations $\langle \cdot, \cdot \rangle_\Pi$ as $|\Pi| = \sup_i |v_{i+1} - v_i|$ tends towards zero. The value of this approximation would be

$$\begin{aligned}
&\left\langle \int_0^v \int_{\Delta_{m-2}(s_{m-1})} \delta_1^\ell f(\cdot, s_1, \ldots, s_{m-1}, v) dB_2(s)^{\otimes(m-1)}, B_2(v) \right\rangle_\Pi \\
&= \sum_{k=0}^{M-1} (B_2(v_{i+1}) - B_2(v_i)) \left(\int_0^{v_{i+1}} \int_{\Delta_{m-2}(s_{m-1})} \delta_1^\ell f(\cdot, s_1, \ldots, s_{m-1}, v_{i+1}) dB_2(s)^{\otimes(m-1)} \right. \\
&\qquad \left. - \int_0^{v_i} \int_{\Delta_{m-2}(s_{m-1})} \delta_1^\ell f(\cdot, s_1, \ldots, s_{m-1}, v_i) dB_2(s)^{\otimes(m-1)} \right) \\
&= \sum_{k=0}^{M-1} (B_2(v_{i+1}) - B_2(v_i)) \left(\int_{v_i}^{v_{i+1}} \int_{\Delta_{m-2}(s_{m-1})} \delta_1^\ell f(\cdot, s_1, \ldots, s_{m-1}, v_{i+1}) dB_2(s)^{\otimes(m-1)} \right) \\
&= \sum_{k=0}^{M-1} (B_2(v_{i+1}) - B_2(v_i)) \left(\int_{v_i}^{v_{i+1}} \int_{\Delta_{m-2}(s_{m-1})} \delta_1^\ell f(\cdot, s_1, \ldots, s_{m-1,i+1}) dB_2(s)^{\otimes(m-1)} \right) \\
&\quad + \sum_{k=0}^{M-1} (B_2(v_{i+1}) - B_2(v_i)) \left(\int_0^{v_i} \int_{\Delta_{m-2}(s_{m-1})} \left(\delta_1^\ell f(\cdot, s_1, \ldots, s_{m-1}, v_{i+1}) \right.\right. \\
&\qquad \left.\left. - \delta_1^\ell f(\cdot, s_1, \ldots, s_{m-1}, v_i) dB_2(s)^{\otimes(m-1)} \right)\right).
\end{aligned} \tag{4.44}$$

Let us consider the first sum. It has a natural approximation via

$$
\begin{aligned}
&\sum_{k=0}^{M-1}(B_2(v_{i+1})-B_2(v_i))\left(\int_{v_i}^{v_{i+1}}\int_{\Delta_{m-2}(s_{m-1})}\delta_1^\ell f(\cdot,s_1,\ldots,s_{m-1},v_{i+1})dB_2(s)^{\otimes(m-1)}\right)\\
&\approx\sum_{k=0}^{M-1}(B_2(v_{i+1})-B_2(v_i))^2\int_{\Delta_{m-2}(v_i)}\delta_1^\ell f(\cdot,s_1,\ldots,s_{m-2},v_i,v_{i+1})dB_2(s)^{\otimes(m-2)}\\
&=\sum_{k=0}^{M-1}(B_2(v_{i+1})-B_2(v_i))^2\int_{\Delta_{m-2}(v_i)}\delta_1^\ell f(\cdot,s_1,\ldots,s_{m-2},v_i,v_i)dB_2(s)^{\otimes(m-2)}\\
&\quad+\sum_{k=0}^{M-1}(B_2(v_{i+1})-B_2(v_i))^2\int_{\Delta_{m-2}(v_i)}\big(\delta_1^\ell f(\cdot,s_1,\ldots,s_{m-2},v_i,v_{i+1})\\
&\qquad\qquad-\delta_1^\ell f(\cdot,s_1,\ldots,s_{m-2},v_i,v_i)\big)\,dB_2(s)^{\otimes(m-2)}.
\end{aligned}
\tag{4.45}
$$

Then using the differentiability of f, we have

$$
\begin{aligned}
&\sum_{k=0}^{M-1}(B_2(v_{i+1})-B_2(v_i))^2\int_{\Delta_{m-2}(v_i)}\delta_1^\ell f(\cdot,s_1,\ldots,s_{m-2},v_i,v_i)dB_2(s)^{\otimes(m-2)}\\
&\quad+\sum_{k=0}^{M-1}(B_2(v_{i+1})-B_2(v_i))^2\int_{\Delta_{m-2}(v_i)}\big(\delta_1^\ell f(\cdot,s_1,\ldots,s_{m-2},v_i,v_{i+1})\\
&\qquad\qquad-\delta_1^\ell f(\cdot,s_1,\ldots,s_{m-2},v_i,v_i)\big)\,dB_2(s)^{\otimes(m-2)}\\
&\approx\sum_{k=0}^{M-1}(B_2(v_{i+1})-B_2(v_i))^2\int_{\Delta_{m-2}(v_i)}\delta_1^\ell f(\cdot,s_1,\ldots,s_{m-2},v_i,v_i)dB_2(s)^{\otimes(m-2)}\\
&\quad+\sum_{k=0}^{M-1}(B_2(v_{i+1})-B_2(v_i))^2(v_{i+1}-v_i)\\
&\qquad\cdot\int_{\Delta_{m-2}(v_i)}\delta_1^\ell\left(\frac{\partial f}{\partial s_m}(\cdot,s_1,\ldots,s_{m-2},v_i,v_i)\right)dB_2(s)^{\otimes(m-2)}.
\end{aligned}
\tag{4.46}
$$

If we consider the L^2 limit of the first series, it is precisely

$$
\begin{aligned}
&\sum_{k=0}^{M-1}(B_2(v_{i+1})-B_2(v_i))^2\int_{\Delta_{m-2}(v_i)}\delta_1^\ell f(\cdot,s_1,\ldots,s_{m-2},v_i,v_i)dB_2(s)^{\otimes(m-2)}\\
&\to_{L^2}\int_0^T\int_{\Delta_{m-2}(v)}\delta_1^\ell f(\cdot,s_1,\ldots,s_{m-2},v,v)dB_2(s)^{\otimes(m-2)}dv.
\end{aligned}
\tag{4.47}
$$

As for the second series, each term $(v_{i+1}-v_i)(B_2(v_{i+1})-B_2(v_i))^2$ has variance $3(v_{i+1}-v_i)^4$, and the iterated integral has bounded convergence. So overall, the second series has variance on the order of $O(|\Pi|^3)$. It is a mean zero process since the mean of each iterated integral is zero. Therefore, the limit of this second series is simply zero.

We have determined the limit of the first series of Equation (4.44). We must determine the limit of the second series. Once again, we exploit the assumption that f is differentiable.

$$\begin{aligned}&\sum_{k=0}^{M-1}(B_2(v_{i+1})-B_2(v_i))\left(\int_0^{v_i}\int_{\Delta_{m-2}(s_{m-1})}\Big(\delta_1^\ell f(\cdot,s_1,\dots,s_{m-1},v_{i+1})\right.\\&\qquad\qquad\left.-\delta_1^\ell f(\cdot,s_1,\dots,s_{m-1},v_i)dB_2(s)^{\otimes(m-1)}\Big)\right)\\&\approx\sum_{k=0}^{M-1}(v_{i+1}-v_i)(B_2(v_{i+1})-B_2(v_i))\\&\qquad\cdot\int_{\Delta_{m-2}(s_{m-1})}\delta_1^\ell\left(\frac{\partial f}{\partial s_m}(\cdot,s_1,\dots,s_{m-2},v_i,v_i)\right)dB_2(s)^{\otimes(m-2)}.\end{aligned}\tag{4.48}$$

By a similar argument to before, the $(v_{i+1}-v_i)(B_2(v_{i+1})-B_2(v_i))$ are independent of their adjoining iterated integrals, and have variance $(v_{i+1}-v_i)^3$, and therefore the variance of the entire series is $O(|\Pi|^2)$. As before, it is also a mean zero series and therefore the limit of this series is identically zero. From this argument we see

$$\begin{aligned}&\left\langle\int_0^v\int_{\Delta_{m-2}(s_{m-1})}\delta_1^\ell f(\cdot,s_1,\dots,s_{m-1},v)dB_2(s)^{\otimes(m-1)},B_2(v)\right\rangle\\&=\int_0^T\int_{\Delta_{m-2}(v)}\delta_1^\ell f(\cdot,s_1,\dots,v,v)dB_2(s)^{\otimes(m-2)}dv\\&=\frac{1}{(m-2)!}\int_0^T\int_{[0,v]^{m-2}}\delta_1^\ell f(\cdot,v,v)dB_2(s)^{\otimes(m-2)}dv\end{aligned}\tag{4.49}$$

Of course, our choice to first deal with integrals with respect to B_2 was completely arbitrarily, and a similar result applies to B_1. Consequently, we have

$$\begin{aligned}&\omega^0\circ\delta_2^m\delta_1^\ell f\\&=-\frac{m(m-1)}{2}\int_0^T\omega^0\int_{[0,v]^{m-2}}\delta_1^\ell f(\cdot,v,v)dB_2(s)^{\otimes(m-2)}dv.\end{aligned}\tag{4.50}$$

By running through the proof with the example $\delta_2^1\delta_1^\ell f$, we see that random variables frozen image is zero. So by repeated applications of Equation (4.50) we see

$$\begin{aligned}\omega^0 \circ \delta_2^m \delta_1^\ell f =&(-1)^{\frac{m}{2}} \frac{m!}{2^{\frac{m}{2}}} \int_0^T \int_0^{v_{\frac{m}{2}}} \cdots \int_0^{v_2} \omega^0 \delta_1^\ell f(\cdot, v_1, v_1, \dots, v_{\frac{m}{2}}, v_{\frac{m}{2}}) dv^{\otimes \frac{m}{2}} \\ =&(-1)^{\frac{m}{2}} \frac{m!}{2^{\frac{m}{2}} (\frac{m}{2})!} \int_{[0,T]^{\frac{m}{2}}} \omega^0 \circ \delta_1^\ell f(\cdot, v_1, v_1, \dots, v_{\frac{m}{2}}, v_{\frac{m}{2}}) dv^{\otimes \frac{m}{2}}.\end{aligned} \tag{4.51}$$

when m is even and zero otherwise. This perfectly coincides with Proposition 2 of [JS16]. Of course, we can repeat the same procedure for B_1, which leads us to our foundational result for verifying the exponential formula.

Lemma 4.1. *Given a differentiable symmetric kernel f in $\hat{L}^2([0,T]^{2\ell}) \otimes \hat{L}^2([0,T]^{2m})$*

$$\begin{aligned}&\omega^0 \circ \delta_2^{2m} \delta_1^{2\ell} f \\ =&(-1)^{\ell+m} \frac{(2\ell!)(2m!)}{2^{\ell+m} m! \ell!} \int_{[0,T]^\ell \times [0,T]^m} f(u_1, u_1, \dots, u_{\frac{\ell}{2}}, u_{\frac{\ell}{2}}, v_1, v_1, \dots, v_{\frac{m}{2}}, v_{\frac{m}{2}}) du^\ell dv^m\end{aligned} \tag{4.52}$$

and is zero when either 2ℓ or $2m$ is replaced by an odd natural number.

To verify the exponential formula, we first need to check the formula holds for a component $\delta_2^m \delta_1^\ell f$ where f is a symmetric kernel on the product $\hat{L}^2([0,T]^\ell)$ and $\hat{L}^2([0,T]^m)$. We also write the exponential formula differently. We first saw this alternative expression in Equation (4.7)

$$\mathbb{E}\left[\delta_2^m \delta_1^\ell f\right] = \sum_{i,j=0}^{\infty} \frac{1}{2^{i+j} i! j!} \omega^0 \circ \left(\Delta_1^1 \Delta_2^j \left(\delta_2^m \delta_1^\ell f\right)\right). \tag{4.53}$$

When $\ell = m = 0$, this identity holds trivially. When ℓ or m is non-zero, then the left-hand side of the equation is zero. First, let us consider the non-trivial case of Lemma 4.1 and examine $\delta_2^{2m}\delta_1^{2\ell}f$.

$$\begin{aligned}
&\sum_{i,j=0}^{\infty}\frac{1}{2^{i+j}i!j!}\omega^0\circ\left(\Delta_1^1\Delta_2^j\left(\delta_2^{2m}\delta_1^{2\ell}f\right)\right)\\
=&\sum_{0\le i\le\ell;0\le j\le m}\frac{1}{2^{i+j}i!j!}\omega^0\circ\left(\Delta_1^1\Delta_2^j\left(\delta_2^{2m}\delta_1^{2\ell}f\right)\right)\\
=&\sum_{0\le i\le\ell;0\le j\le m}\frac{1}{2^{i+j}i!j!}\omega^0\circ\left(\int_{[0,T]^i}\int_{[0,T]^j}D_{2,v_j}^2\cdot D_{2,v_1}^2D_{1,u_i}^2\cdot D_{1,u_1}^2\delta_2^{2m}\delta_1^{2\ell}fdu^{\otimes i}dv^{\otimes j}\right)\\
=&\sum_{0\le i\le\ell;0\le j\le m}\frac{(2\ell!)(2m!)}{2^{i+j}i!j!(2\ell-2i)!(2m-2j)!}\\
&\quad\omega^0\circ\int_{[0,T]^i}\int_{[0,T]^j}\delta_2^{2m-2j}\delta_1^{2\ell-2i}f(\cdot,u_1,u_1,\dots,u_i,u_i,\cdot,v_1,v_1,\dots,v_j,v_d)du^{\otimes i}dv^{\otimes j}.
\end{aligned}\tag{4.54}$$

Then use Lemma 4.1 to evaluate the frozen images.

$$\begin{aligned}
=&\sum_{0\le i\le\ell;0\le j\le m}\frac{(2\ell!)(2m!)}{2^{i+j}i!j!(2\ell-2i)!(2m-2j)!}\\
&\cdot(-1)^{\ell-i+m-j}\frac{(2\ell-2i)!(2m-2j)!}{2^{\ell-i+m+j}(\ell-i)!(m-j)!}\\
&\cdot\int_{[0,T]^\ell}\int_{[0,T]^m}f(u_1,u_1,\dots,u_\ell,u_\ell,v_1,v_1,\dots,v_m,v_m)du^{\otimes(\ell)}dv^{\otimes(m)}\\
=&(2\ell)!(2m)!C\sum_{0\le i\le\ell;0\le j\le m}(-1)^{\ell-i+m-j}\frac{1}{2^{\ell+m}i!j!(\ell-i)!(m-j)!},
\end{aligned}\tag{4.55}$$

where C is the integral which appears on the penultimate in Equation (4.55). Continuing

$$\begin{aligned}
=&\frac{(2\ell)!(2m)!C}{\ell!m!2^{\ell+m}}\sum_{0\le i\le\ell;0\le j\le m}(-1)^{\ell-i+m-j}\frac{\ell!m!}{i!j!(\ell-i)!(m-j)!}\\
=&\frac{(2\ell)!(2m)!C}{\ell!m!2^{\ell+m}}\left(\sum_{i=0}^{\ell}(-1)^{\ell-i}\frac{\ell!}{(\ell-i)}\right)\left(\sum_{j=0}^{m}(-1)^{m-j}\frac{m!}{(m-j)}\right)\\
=&\frac{(2\ell)!(2m)!C}{\ell!m!2^{\ell+m}}(1-1)^{\ell}(1-1)^m\\
=&0.
\end{aligned}\tag{4.56}$$

Therefore, by a direct calculation, we have shown

Lemma 4.2. *Given a differentiable symmetric kernel f in the product space of $\hat{L}^2([0,T]^\ell)$ and*

$\hat{L}^2([0,T]^2)$, the exponential formula holds for $\delta_2^m \delta_1^\ell f$. The proof follows since there are only finitely many terms in each infinite series,

$$\begin{aligned}
&\mathbb{E}\left[\omega^0 \circ \delta_2^m \delta_1^\ell f\right] \\
=&\sum_{i,j=0}^{\infty} \frac{1}{2^{i+j} i! j!} \omega^0 \circ \left(\Delta_1^1 \Delta_2^j \left(\delta_2^{2m} \delta_1^{2\ell} f\right)\right) \\
=&\sum_{k=0}^{\infty} \frac{1}{2^k k!} \omega^0 \circ \Delta^k \left(\delta_2^{2m} \delta_1^{2\ell} f\right) \\
=&e^{\frac{1}{2}\omega^0 \circ \Delta} \delta_2^m \delta_1^\ell f.
\end{aligned} \tag{4.57}$$

So we have at least proven the exponential formula is true within each chaos, and is therefore at least true for any random variable whose chaos decomposition contains finitely many terms. As is usual in analysis, the difficulty is taking the argument in the limit, and proving the relation holds for a random variable whose chaos decomposition is an infinite sum.

We start by defining a series of linear transformations ω_n^0. Each ω_n^0 will map a subset of the n-th Wiener chaos $\mathcal{H}_n$ to $\mathbb{R}$ by working backwards from the result in Lemma 4.1. So ω_n^0 is the linear map induced by the relation

$$\begin{aligned}
&\omega_n^0 \left(\delta_2^{2m} \delta_1^{2\ell} f\right) \\
:=&(-1)^{\ell+m} \frac{(2\ell!)(2m!)}{2^{\ell+m} m! \ell!} \int_{[0,T]^\ell \times [0,T]^m} f(u_1, u_1, \ldots, u_\ell, u_\ell, v_1, v_1, \ldots, v_m, v_m) du^\ell dv^m,
\end{aligned} \tag{4.58}$$

where $2\ell + 2m = n$, and f is a differentiable element of $\hat{L}^2([0,T]^{2\ell}) \otimes \hat{L}^2([0,T]^{2m})$. We set $\omega^0(\delta_2^{m'} \delta_1^{\ell'} f)$ to zero when $\ell' + m' = n$, either ℓ or m is odd, and f is a differentiable element of $\hat{L}^2([0,T]^{\ell'}) \otimes \hat{L}^2([0,T]^{m'})$. We let $\operatorname{Dom} \omega_n^0$ be the subset of $\mathcal{H}_n$ for which this mapping is finite.

Then our hands our tied. Given a random variable F generated by two independent Brownian motions F, which has a chaos decomposition $F = \sum_{n=0}^{\infty} I_n(f_n)$ where $f_n \in \operatorname{Dom} \tilde{\omega}^0$ for each n, and the series

$$\sum_{n=0}^{\infty} \omega_n^0 (I_n(f_n)), \tag{4.59}$$

converges absolute, we say $F \in \mathrm{Dom}\,\tilde{w}^0$ and define

$$\tilde{w}^0 F = \sum_{n=0}^{\infty} \omega_n^0(I_n(f_n)). \tag{4.60}$$

Since $\tilde{\omega^0}$ agrees with ω_n^0 on the n-th chaos, we can write

$$\tilde{\omega}^0(F) = \sum_{n=0}^{\infty} \tilde{\omega}^0(I_n(f_n)), \tag{4.61}$$

for an $F = \sum_n I_n(f_n)$ in $\mathrm{Dom}\,\tilde{\omega}^0$.

We then define a series of space beginning with $\mathrm{Dom}^0\,\tilde{\omega}^0 := \mathrm{Dom}\,\tilde{\omega}^0$, and say a random variable F is in $\mathrm{Dom}^{k+1}\,\tilde{\omega}^0$ if F is in $\mathrm{Dom}\,\tilde{\omega}^0$, and ΔF exists and is in $\mathrm{Dom}^k\,\tilde{\omega}^0$. Then we define $\mathrm{Dom}^\infty\,\tilde{\omega}^0$ as the intersection of all these spaces. So if we have an $F \in \mathrm{Dom}^\infty\,\tilde{\omega}^0$ with the usual chaos expansion $F = \sum_n I_n(f_n)$, we can safely say each series

$$\tilde{\omega}^0(\Delta^k F) = \sum_{n=0}^{\infty} \omega^0(\Delta^k I_n(f_n)), \tag{4.62}$$

converges absolutely. We simply plug F into the exponential formula and remember Lemma 4.2 implies the exponential formula holds for each chaos $I_n(f_n)$.

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1}{2^k k!}\omega^0(\Delta^k F) \\
=&\sum_{k=0}^{\infty} \frac{1}{2^k k!}\sum_{n=0}^{\infty} \omega^0(\Delta^k I_n(f_n)) \\
=&\sum_{n=0}\sum_{k=0}^{\infty} \frac{1}{2^k k!}\omega^0(\Delta^k I_n(f_n)) \\
=&\sum_{n=0} \mathbb{E}\left[I_n(f_n)\right] \\
=&\mathbb{E}\left[F\right].
\end{aligned} \tag{4.63}$$

There was a lot of scaffolding to erect in order to swap summation signs.

Theorem 4.3. *For a random variable F generated by two independent Brownian such that F lies* *with* $\operatorname{Dom}^\infty \tilde{\omega}^0$, *$F$ satisfies the exponential formula*

$$\begin{aligned} \mathbb{E}[F] =& \sum_{k=0}^{\infty} \frac{1}{2^k k!} \omega^0 \circ \Delta^k(F) \\ =& e^{\frac{1}{2}\omega^0 \circ \Delta} F. \end{aligned} \tag{4.64}$$

Future work should characterize the space $\operatorname{Dom}^\infty \tilde{\omega}^0$, but there are robust families of random variables already known to lie within $\operatorname{Dom}^\infty \tilde{\omega}^0$. Polynomials in B_1 and B_2 are automatically within the space, as are exponentials.

4.4 An Exponential Formula for d-dimensional Brownian Motion

The notation is somewhat scarier, but the logic is the same as the two-dimensional case. We start with a random variable F generated by d-dimensional Brownian motion and its decomposition

$$F = \sum_{\boldsymbol{\mu} \in \mathbb{N}^d} \frac{|\boldsymbol{\mu}|!}{\boldsymbol{\mu}!} \delta^{\boldsymbol{\mu}} f^{\boldsymbol{\mu}}. \tag{4.65}$$

Where $|\boldsymbol{\mu}| = \mu_1 + \cdot + \mu_d$, $\boldsymbol{\mu}! = \mu_1! \cdots \mu_d!$, and $\delta^{\boldsymbol{\mu}} = \delta_d^{\mu_d} \cdots \delta_1^{\mu_1}$.

Each symmetric kernel is of the form

$$f^{\boldsymbol{\mu}}(s_{1,1}, \dots, s_{1,\mu_1}, s_{2,1}, \dots, s_{s,\mu_2}, \cdots, s_{d,1}, s_{d,\mu_d}), \tag{4.66}$$

and is square integrable. Here, $f^{\boldsymbol{\mu}}$ is invariant under any permutation which maps each set of $\{s_{i,1}, \dots, s_{i,\nu_i}\}$ to themselves. Furthermore, we make the restriction that f is differentiable $f^{\boldsymbol{\mu}}$.

We follow the same path as in the proceeding section, and first try to obtain a closed-form expression for $\delta^{\boldsymbol{\mu}} f^{\boldsymbol{\mu}}$. If we repeat the reasoning behind Lemma 4.1 we obtain the following formula.

Lemma 4.4. *Given a differentiable symmetric kernel f in $\otimes_{i=1}^{d}\hat{L}^2([0,T]^{2\nu_i})$*

$$\begin{aligned}&\omega^0 \circ \delta^{2\boldsymbol{\nu}} f\\ =&(-1)^{|\nu|}\frac{(2\boldsymbol{\nu})!}{2^{|\boldsymbol{\nu}|}\boldsymbol{\nu}!}\int_{[0,T]^{|\nu|}} f(u_{1,1},u_{1,1},\dots,u_{1,\nu_1},u_{1,\nu_1},\dots,u_{d,1},u_{d,1},\dots,u_{d,\nu_d},u_{d,\nu_d})du_1^{\otimes\nu_i}du_d^{\otimes\nu_d},\end{aligned} \tag{4.67}$$

and is zero when either 2ℓ or $2m$ is replaced by an odd natural number.

An alternative way to express this relation is

$$\omega^0 \circ \delta^{2\boldsymbol{\nu}} f = (-1)^{|\nu|}\frac{1}{2^{|\boldsymbol{\nu}|}\boldsymbol{\nu}!}\Delta^{\boldsymbol{\nu}}\delta^{2\boldsymbol{\nu}} f. \tag{4.68}$$

So if we have a multi-index $\boldsymbol{\mu} \leq \boldsymbol{\nu}$ ($\mu_1 \leq \nu_1, \dots, \mu_d \leq \nu_d$) and write the iterated partial Gross Laplacian $\Delta^{\boldsymbol{\mu}} = \Delta_d^{\mu_d}\cdots\Delta_1^{\mu_1}$, then we have the identity

$$\begin{aligned}&\omega^0\Delta^{\boldsymbol{\mu}}\delta^{2\boldsymbol{\nu}} f\\ =&\frac{2\boldsymbol{\nu}!}{(2\boldsymbol{\nu}-2\boldsymbol{\mu})!}\int_{[0,T]^{|\mu|}}\omega^0\delta^{2\boldsymbol{\nu}-2\boldsymbol{\mu}} f(\cdot)du_1^{\otimes\mu_1}du_d^{\otimes\mu_d}\\ =&\frac{2\boldsymbol{\nu}!}{(2\boldsymbol{\nu}-2\boldsymbol{\mu})!}\frac{(-1)^{|\boldsymbol{\nu}-\boldsymbol{\mu}|}(2\boldsymbol{\nu}-2\boldsymbol{\mu})!}{2^{|\boldsymbol{\nu}-\boldsymbol{\mu}|}(\boldsymbol{\nu}-\boldsymbol{\mu})!}\int_{[0,T]^{|\nu|}} f(\cdot)du_1^{\otimes\nu_1}du_d^{\otimes\nu_d}\\ =&\frac{(-1)^{|\boldsymbol{\nu}-\boldsymbol{\mu}|}}{2^{|\boldsymbol{\nu}-\boldsymbol{\mu}|}(\boldsymbol{\nu}-\boldsymbol{\mu})!}\Delta^{\boldsymbol{\nu}}\delta^{2\boldsymbol{\nu}} f,\end{aligned} \tag{4.69}$$

and note if there exists some index μ_i that is greater than ν_i, then $\Delta^{\boldsymbol{\mu}}\delta^{2\boldsymbol{\nu}} f$ is identically zero. We trust if the reader has made it this far, then they know what arguments are being integrated without having to bleed into the margins by fully writing out the arguments to f. We were not explicit, but the above identity is what we used in the proof for the two-dimensional case.

Then we apply the appropriate analogue of the exponential formula for such an f. For d-

dimensional Brownian motion it is

$$
\begin{aligned}
&\sum_{\boldsymbol{\mu}\in\mathbb{N}^d} \frac{1}{2^{|\boldsymbol{\mu}|}\boldsymbol{\mu}!}\omega^0\Delta^{\boldsymbol{\mu}}\delta^{2\boldsymbol{\nu}}f \\
=&\sum_{\boldsymbol{\mu}\leq\boldsymbol{\nu}} \frac{1}{2^{|\boldsymbol{\mu}|}\boldsymbol{\mu}!}\omega^0\Delta^{\boldsymbol{\mu}}\delta^{2\boldsymbol{\nu}}f \\
=&\sum_{\boldsymbol{\mu}\leq\boldsymbol{\nu}} \frac{1}{2^{|\boldsymbol{\mu}|}\boldsymbol{\mu}!}\frac{(-1)^{|\boldsymbol{\nu}-\boldsymbol{\mu}|}}{2^{|\boldsymbol{\nu}-\boldsymbol{\mu}|}(\boldsymbol{\nu}-\boldsymbol{\mu})!}\Delta^{\boldsymbol{\nu}}\delta^{2\boldsymbol{\nu}}f \\
=&\Delta^{\boldsymbol{\nu}}\delta^{2\boldsymbol{\nu}}f\sum_{\boldsymbol{\mu}\leq\boldsymbol{\nu}} \frac{1}{2^{|\boldsymbol{\nu}|}\boldsymbol{\mu}!}\frac{(-1)^{|\boldsymbol{\nu}-\boldsymbol{\mu}|}}{(\boldsymbol{\nu}-\boldsymbol{\mu})!} \\
=&\frac{\Delta^{\boldsymbol{\nu}}\delta^{2\boldsymbol{\nu}}}{2^{\boldsymbol{\nu}}\boldsymbol{\nu}!}\sum_{\boldsymbol{\mu}\leq\boldsymbol{\nu}} \frac{(-1)^{|\boldsymbol{\nu}-\boldsymbol{\mu}|}\boldsymbol{\nu}!}{\boldsymbol{\mu}!(\boldsymbol{\nu}-\boldsymbol{\mu})!} \\
=&\frac{\Delta^{\boldsymbol{\nu}}\delta^{2\boldsymbol{\nu}}}{2^{\boldsymbol{\nu}}\boldsymbol{\nu}!}\sum_{\mu_1,\ldots,\mu_d;0\leq\mu_i\leq\nu_i}\prod_{i=1}^d \frac{(-1)^{\nu_i-\mu_i}\nu_i!}{\mu_i!(\nu_i-\mu_i)!} \\
=&\frac{\Delta^{\boldsymbol{\nu}}\delta^{2\boldsymbol{\nu}}}{2^{\boldsymbol{\nu}}\boldsymbol{\nu}!}\prod_{i=1}^d(1-1)^{\nu_i} \\
=&\frac{\Delta^{\boldsymbol{\nu}}\delta^{2\boldsymbol{\nu}}}{2^{\boldsymbol{\nu}}\boldsymbol{\nu}!}0^{|\boldsymbol{\nu}|} \\
=&0.
\end{aligned}
\tag{4.70}
$$

A formal manipulation of series will show

$$
\sum_{k=0}^{\infty}\frac{1}{2^k k!}\Delta^k = \sum_{\boldsymbol{\mu}\in\mathbb{N}^d}\frac{1}{2^{|\boldsymbol{\mu}|}\boldsymbol{\mu}!}\Delta^{\boldsymbol{\mu}}, \tag{4.71}
$$

where we have the total Gross Laplacian $\Delta = \Delta_1 + \cdots + \Delta_d$. When each of these series is applied to $\delta^{2\boldsymbol{\nu}}f$, there are only finitely many non-zero terms, so we do not have any problems of convergence, and $\delta^{2\boldsymbol{\nu}}f$ is mean zero. And the case where there exists an odd component somewhere in the multi-index is trivial, since freezing such a random variable or any of its Gross Laplacians is always zero. Therefore, we can say

Lemma 4.5. *Given a differentiable symmetric kernel f in $\otimes_{i=1}^d \hat{L}^2([0,T]^{\nu_i})$*

$$
\mathbb{E}\left[\delta^{\boldsymbol{\nu}}f\right] = \sum_{k=0}^{\infty}\frac{1}{2^k k!}\omega^0\circ\Delta^k(\delta^{\boldsymbol{\nu}}f). \tag{4.72}
$$

From here, we proceed with constructing a freezing operator on a subset of $L^2(\Omega)$ in the same fashion as in the previous section. So we construct a space $\operatorname{Dom}\tilde{\omega}^0$ such that the relation

$$\tilde{\omega}^0\,(F) = \sum_{n=0}^{\infty} \omega^0(I_n(f_n)), \tag{4.73}$$

holds for any $F \in \operatorname{Dom}\tilde{\omega}^0$ with chaos expansion $\sum_n I_n(f_n)$. From there we construct the d-dimensional version $\operatorname{Dom}^\infty \tilde{\omega}^0$. Thus, we can extend the final result by linearity to

Theorem 4.6. *For a random variable $F \in \operatorname{Dom}^\infty \tilde{\omega}^0$ generated by d independent Brownian motions with a chaos decomposition*

$$F = \sum_{\boldsymbol{\mu}\in\mathbb{N}^d} \frac{|\boldsymbol{\mu}|!}{\boldsymbol{\mu}!}\delta^{\boldsymbol{\mu}} f^{\boldsymbol{\mu}}, \tag{4.74}$$

F satisfies the exponential formula

$$\mathbb{E}\,[F] = \sum_{k=0}^{\infty} \frac{1}{2^k k!}\omega^0 \circ \Delta^k(F) = e^{\frac{1}{2}\omega^0\circ\Delta} F. \tag{4.75}$$

4.5 SABR model with $\beta = 1$

We would like to apply our developed analytical tools to the SABR model with $\beta = 1$; continuing the work done in [GS19]. We break with the traditional notation of the domain to be consistent with the notations within this work. The SABR model [Hag+02] is traditionally expressed as the following SDEs driven by the Brownian motions W and Z

$$\begin{aligned} dF_t =& \sigma_t F_j^\beta dZ_t \\ d\sigma_t =& \alpha\sigma_t dW_t \\ dW_t dZ_t =& \rho dt, \end{aligned} \tag{4.76}$$

with β in $[0,1]$. F should be interpreted as the forward rate, while σ_t is stochastic volatility. We take $\beta = 1$ because we wish to eventually graduate. We will write

$$\begin{aligned} dF_t =& \sigma_t F_t(\sqrt{1-\rho^2}dB_1(t) + \rho dB_2(t)) \\ d\sigma_t =& \alpha\sigma_t dB_2(t), \end{aligned} \tag{4.77}$$

with β in $[0,1]$ and $F_0 = \sigma_0 = 1$, and we set $X_t = \log F_t$. The volatility has the closed-form expression

$$\sigma_t = \sigma_0 e^{\alpha B_2(t) - \frac{\alpha^2}{2}t}, \tag{4.78}$$

and X_t satisfies the SDE

$$\begin{aligned} dX_t =& \sigma_t(\sqrt{1-\rho^2}dB_1(t) + \rho dB_2(t)) - \frac{1}{2}\sigma_t^2 dt \\ =& \sigma_t dZ_t - \frac{1}{2}\sigma_t^2 dt. \end{aligned} \tag{4.79}$$

Our goal is to calculate the price at time 0 of a European call option on F and maturity T. For the sake of simplicity we assume the call is "at the money" and normalize so $K = F_0 = 1$, and thus $X_0 = 0$. To do so, we apply Theorem 6 of [Alò06]. We introduce the following quantities from [Alò06]:

$$\begin{aligned} d_{\pm}(t,x,\sigma) =& \frac{x \pm \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \\ N(z) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{u^2}{2}} du \\ \mathrm{BS}(t,x,\sigma) =& e^x N(d_+) - N(d_-) \\ v_s^2 =& \frac{1}{T-s}\int_s^T \sigma_r^2 dr \\ \Xi(s,X_s,v_s) =& \left((\frac{\partial^3}{\partial x^3} - \frac{\partial}{\partial x}) \mathrm{BS} \right)(s,X_s,v_s) \\ \Lambda_s =& \sigma_s \int_s^T D_{2,s}\sigma_r^2 dr = 2\alpha\sigma_s \int_s^T \sigma_r^2 dr \\ V_0 =& \mathbb{E}\left[\mathrm{BS}(0,X_0,v_0)\right] + \frac{\rho}{2}\int_0^T \mathbb{E}\left[\Xi_s\Lambda_s\right] ds. \end{aligned} \tag{4.80}$$

The application of our results is to approximate V_0 using the exponential formula for random variables generated by multiple Brownian motions. For a random variable G, the identity revolving around the exponential formula is

$$\mathbb{E}\left[G\right] = \sum_{n=0}^{\infty} \frac{1}{n!2^n} \omega^0 \circ \Delta^n G = \omega^0 \circ (G) + \frac{1}{2}\omega^0 \circ (\Delta G) + \frac{1}{8}\omega^0 \circ \left(\Delta^2 G\right) + \cdots \tag{4.81}$$

The resulting approximation of V_0 is the expression

$$\begin{aligned} &\omega^0 \circ (BS(0,0,v_0)) + \frac{1}{2}\omega^0 \circ (\Delta BS(0,0,v_0)) \\ &+ \frac{\rho}{2}\int_0^T \left(\omega^0 \circ (\Xi_s \Lambda_s) + \frac{1}{2}\omega^0 \circ \Delta\left(\Xi_s \Lambda_s\right)\right) ds. \end{aligned} \tag{4.82}$$

To break up the calculation, we introduce the following variables for the terms in the approximation

$$\begin{aligned} \mathbf{A} =& \omega^0 \circ (BS(0,0,v_0)) \\ \mathbf{B} =& \omega^0 \circ (\Delta BS(0,0,v_0)) \\ \mathbf{C} =& \omega^0 \circ (\Xi_s \Lambda_s) \\ \mathbf{D} =& \omega^0 \circ \Delta\left(\Xi_s \Lambda_s\right), \end{aligned} \tag{4.83}$$

and to finally calculate

$$\mathbf{A} + \frac{1}{2}\mathbf{B} + \frac{\rho}{2}\int_0^T \left(\mathbf{C} + \frac{1}{2}\mathbf{D}\right) ds. \tag{4.84}$$

It will behoove us to further break down the calculation of $\mathbf{D}$. $\Delta(\Xi_s \Lambda_s)$ is the sum of $\Delta_1(\Xi_s \Lambda_s)$ and $\Delta_2(\Xi_s \Lambda_s)$; the Gross-Laplacian with respect to B_1 plus the Gross-Laplacian with respect to B_2. So we further write

$$\begin{aligned} \mathbf{E} =& \omega^0 \circ \Delta_1\left(\Xi_s \Lambda_s\right) \\ \mathbf{F} =& \omega^0 \circ \Delta_2\left(\Xi_s \Lambda_s\right) \\ \mathbf{D} =& \mathbf{E} + \mathbf{F}. \end{aligned} \tag{4.85}$$

4.5.1 Preliminaries: The Greeks

We will make common use of the identities

$$\begin{aligned} N'(z) =& \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \\ N''(z) =& -zN'(z) \\ \frac{\partial d_\pm(t,x,\sigma)}{\partial x} =& \frac{1}{\sigma\sqrt{T-t}}. \end{aligned} \tag{4.86}$$

We will also need to know the partial derivative of $d_\pm$ with respect to σ

$$\begin{aligned} \frac{\partial d_\pm}{\partial \sigma} &= -\frac{x}{\sigma^2\sqrt{T-t}} \pm \frac{1}{2}\sqrt{T-t} \\ &= -\frac{1}{\sigma}\left(\frac{x \mp + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\ &= -\frac{d_\mp}{\sigma}. \end{aligned} \tag{4.87}$$

Our approximation will necessitate calculating the various derivatives of the Black-Scholes function BS; the Greeks. For better or worse, only the Greeks up to the third order have names. Though it seems the taxonomic effort started waning half-way through the second-order Greeks. Most tables express the Greeks in terms of the price F and not the rate X, so we will proceed slowly and verify and translation. Tables will be used more as a sanity check on our algebra. Consulting [Hau07], we see the stochastic process Ξ_s is the difference between the speed and the delta. We leave it to our betters to explain any possible interpretation. The first derivative with respect to the rate (delta) is

$$\begin{aligned} \frac{\partial BS}{\partial x} =& e^x N(d_+) + e^x N'(d_+)\frac{\partial d_+}{\partial x} - N'(d_-)\frac{\partial d_-}{\partial x} \\ =& e^x N(d_+) + \frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}\left(e^{x-\frac{d_+^2}{2}} - e^{-\frac{d_-^2}{2}}\right). \end{aligned} \tag{4.88}$$

We recall the basic identity $d_+ - d_- = \sigma\sqrt{T-t}$, and therefore

$$\begin{aligned} x - \frac{d_+^2}{2} =& \frac{2x - (d_- + \sigma\sqrt{T-t})^2}{2} \\ =& \frac{2x - (d_-^2 + 2d_-\sigma\sqrt{T-t} + \sigma^2(T-t))}{2} \\ =& \frac{2x - \left(d_-^2 + 2(x - \frac{1}{2}\sigma^2(T-t)) + \sigma^2(T-t)\right)}{2} \\ =& -\frac{d_-^2}{2}. \end{aligned} \tag{4.89}$$

Thus, $e^x N'(d_+)$ is $N'(d_-)$ and therefore the delta is

$$\frac{\partial BS}{\partial x} = e^x N(d_+). \tag{4.90}$$

Using the previous calculation we find the second derivative with respect to the rate (gamma) is

$$\begin{aligned} \frac{\partial^2 BS}{\partial x^2} =& e^x N(d_+) + \frac{e^x N'(d_+)}{\sigma\sqrt{T-t}} \\ =& e^x N(d_+) + \frac{N'(d_-)}{\sigma\sqrt{T-t}}. \end{aligned} \tag{4.91}$$

Finally, the third derivative with respect to the rate (speed) is

$$\begin{aligned} \frac{\partial^3 BS}{\partial x^3} =& e^x N(d_+) + \frac{N'(d_-)}{\sigma\sqrt{T-t}} - d_- \frac{N'(d_-)}{\sigma\sqrt{T-t}} \frac{\partial d_-}{\partial x} \\ =& e^x N(d_+) + \frac{N'(d_-)}{\sigma\sqrt{T-t}} \left(1 - \frac{d_-}{\sigma\sqrt{T-t}}\right). \end{aligned} \tag{4.92}$$

So if we express the process Ξ_s as $b(s, X_s, v_s)$ where $b(t, x, \sigma)$ is a deterministic function, then

$$\begin{aligned} b(t, x, \sigma) =& \left(\frac{\partial^3}{\partial x^3} - \frac{\partial}{\partial x}\right) BS(t, x, \sigma) \\ =& \frac{N'(d_-)}{\sigma\sqrt{T-t}} \left(1 - \frac{d_-}{\sigma\sqrt{T-t}}\right). \end{aligned} \tag{4.93}$$

We will also need the derivatives of BS with respect to volatility σ. The first derivative of BS with respect to σ (vega) is

$$\begin{aligned}\frac{\partial BS}{\partial \sigma} =& e^x N'(d_+)\frac{\partial d_+}{\partial \sigma} - N'(d_-)\frac{\partial d_-}{\partial \sigma}\\ =& -\frac{d_- N'(d_-)}{\sigma} + \frac{d_+ N'(d_-)}{\sigma}\\ =& \frac{N'(d_-)}{\sigma}(d_+ - d_-)\\ =& \sqrt{T-t}N'(d_-).\end{aligned} \tag{4.94}$$

The second derivative with respect to the volatility (vega) is the vomma multiplied by $\frac{d_+ d_-}{\sigma}$

$$\begin{aligned}\frac{\partial^2 BS}{\partial \sigma^2} =& -\sqrt{T-t}d_- N'(d_-)\frac{\partial d_-}{\partial \sigma}\\ =& \sqrt{T-t}\frac{d_- d_+}{\sigma}N'(d_-).\end{aligned} \tag{4.95}$$

There will be higher derivatives which will be relevant for specific calculations. However, these are the derivatives of BS which are common to multiple calculations.

4.5.2 Preliminaries: The RMS volatility

The stochastic volatility σ_s is ubiquitous throughout every calculation; usually as a sub-term of a more complicated process. The frozen path of σ_t is

$$\omega^0(\sigma_t) = \sigma_0 e^{-\frac{\alpha^2}{2}t}, \tag{4.96}$$

and of course $\omega^0(\sigma_t^2) = e^{-\alpha^2 t}$.

The RMS volatility appears in all calculations. In general, v_s^2 is an easier term to manipulate than v_s itself. Freezing v_s^2 results in the deterministic process

$$\begin{aligned}\omega^0(v_s^2) =& \frac{\sigma_0}{T-s}\int_s^T e^{-\alpha^2 r}dr\\ =& \frac{\sigma_0^2}{\alpha^2(T-s)}\left(e^{-\alpha^2 s} - e^{-\alpha^2 T}\right)\\ \omega^0(v_s) =& \frac{\sigma_0}{\alpha\sqrt{T-s}}\sqrt{e^{-\alpha^2 s} - e^{-\alpha^2 T}}.\end{aligned} \tag{4.97}$$

We will also need to know the first and second-order Malliavin derivatives of v_s. v_s itself is a function of the stochastic volatility σ_s which is entirely generated by the second Brownian motion. So $D_{1,u}v_s$ is identically zero. To calculate $D_{2,u}v_s$ we first calculate $D_{2,u}v_s^2$. What is important to note is how u compares to s.

$$\begin{aligned} D_{2,u}v_s^2 =& \frac{1}{T-s}\int_s^T D_{2,u}\sigma_r^2 dr \\ =& \frac{2\alpha}{T-s}\int_s^T \sigma_r^2 1_{[0,r]}(u)dr \\ =& \frac{2\alpha}{T-s}\left(1_{[0,s]}(u)\int_s^T \sigma_r^2 dr + 1_{[s,T]}(u)\int_u^T \sigma_r^2 dr\right) \\ =& 2\alpha\left(1_{[0,s]}(u)v_s^2 + 1_{[s,T]}(u)\frac{T-u}{T-s}v_u^2\right), \end{aligned} \tag{4.98}$$

which is a sort of interpolation, or bridge, between v_s^2 and v_u^2. Freezing this quantity results in the expression

$$\begin{aligned} &\omega^0 \circ D_{2,u}v_s^2 \\ =& 2\alpha\left(1_{[0,s]}(u)\omega^0(v_s^2) + 1_{[s,T]}(u)\frac{T-u}{T-s}\omega^0(v_u^2)\right) \\ =& \frac{2\alpha\sigma_0^2}{\alpha^2}\left(1_{[0,s]}(u)\frac{e^{-\alpha^2 s}-e^{-\alpha^2 T}}{T-s} + 1_{[s,T]}(u)\frac{T-u}{T-s}\frac{e^{-\alpha^2 u}-e^{-\alpha^2 T}}{T-u}\right) \\ =& \frac{2\sigma_0^2}{\alpha(T-s)}\left(1_{[0,s]}(u)\left(e^{-\alpha^2 s}-e^{-\alpha^2 T}\right) + 1_{[s,T]}(u)\left(e^{-\alpha^2 u}-e^{-\alpha^2 T}\right).\right) \end{aligned} \tag{4.99}$$

To calculate $D_{2,u}v_s$ itself we express v_s as $\sqrt{v_s}^2$. Then since the Malliavin calculus is a first-order calculus we have

$$D_{2,u}v_s = D_{2,u}\sqrt{v_s^2} = \frac{D_{2,u}v_2^2}{2v_s}. \tag{4.100}$$

Then freezing this would result

$$\begin{aligned}\omega^0 \circ D_{2,u} v_s =& \frac{\sigma_0^2}{\alpha(T-s)\omega^0(v_s)} \left(1_{[0,s]}(u)\left(e^{-\alpha^2 s} - e^{-\alpha^2 T}\right)\right. \\ &\left. + 1_{[s,T]}(u)\left(e^{-\alpha^2 u} - e^{-\alpha^2 T}\right)\right) \\ =& \frac{\sigma_0^2}{\alpha(T-s)} \frac{\alpha\sqrt{T-s}}{\sigma_0\sqrt{e^{-\alpha^2 s} - e^{-\alpha^2 T}}} \left(1_{[0,s]}(u)\left(e^{-\alpha^2 s} - e^{-\alpha^2 T}\right)\right. \\ &\left. + 1_{[s,T]}(u)\left(e^{-\alpha^2 u} - e^{-\alpha^2 T}\right)\right) \\ =& \frac{\sigma_0}{\sqrt{T-s}\sqrt{e^{-\alpha^2 s} - e^{-\alpha^2 T}}} \left(1_{[0,s]}(u)\left(e^{-\alpha^2 s} - e^{-\alpha^2 T}\right)\right. \\ &\left. + 1_{[s,T]}(u)\left(e^{-\alpha^2 u} - e^{-\alpha^2 T}\right)\right).\end{aligned} \tag{4.101}$$

It is a value judgement on how far to elaborate these expressions. We could have just left the first line in the above calculation. However, by finishing the calculation we do see some results of interest. $\omega^0 \circ D_{2,u} v_s$ has a linear dependence on σ_0, while the initial leading factor depending on a power of α has disappeared.

The second Malliavin derivative of v_s^2 is

$$D_{2,u}^2 v_s^2 = 2\alpha \left(1_{[0,s]}(u) D_{2,u} v_s^2 + 1_{[s,T]} \frac{T-u}{T-s} D_{2,u} v_u^2\right). \tag{4.102}$$

We get to exploit the short-circuit $1_{[0,s]}(u)1_{[s,T]}(u) = 0$ in the first summand. And in the second summand we have $D_{2,u} v_u^2 = 2\alpha v_u^2$. Therefore,

$$D_{2,u}^2 v_s^2 = 4\alpha^2 \left(1_{[0,s]}(u) v_s^2 + 1_{[s,T]} \frac{T-u}{T-s} v_u^2\right). \tag{4.103}$$

That is, $D_{2,u}^2 v_s^2 = 2\alpha D_{2,u} v_s^2$, so $\omega^0 \circ D_{2,u}^2 v_s^2 = 2\alpha\omega^0 \circ D_{2,u} v_s^2$.

For the second Malliavin derivative of v_s itself

$$
\begin{aligned}
D^2_{2,u} v_s =& D_{2,u}\left(\frac{D_{2,u} v_2^2}{2v_s}\right) \\
=& \frac{D^2_{2,u} v_s^2}{2v_s} - \frac{(D_{2,u} v_s^2) D_{2,u} v_s}{2v_s^2} \\
=& \alpha \frac{D_{2,u} v_s^2}{v_s} - \frac{(D_{2,u} v_s^2)^2}{4v_s^3}.
\end{aligned}
\tag{4.104}
$$

Again, we exploit the short-circuit $1_{[0,s]}(u)1_{[s,T]}(u) = 0$. Using equation (4.98) for $D_{2,u} v_s^2$ we see

$$
\begin{aligned}
D^2_{2,u} v_s =& \frac{2\alpha^2}{v_s}\left(1_{[0,s]}(u) v_s^2 + 1_{[s,T]}(u)\frac{T-u}{T-s} v_u^2\right) \\
&- \frac{4\alpha^2}{4v_s^3}\left(1_{[0,s]}(u) v_s^4 + 1_{[s,T]}(u)\frac{(T-u)^2}{(T-s)^2} v_u^4\right) \\
=& \alpha^2 v_s 1_{[0,s]}(u) + \alpha^2\left(2 - \frac{T-u}{T-s}\frac{v_u^2}{v_s^2}\right)\frac{(T-u)v_u^2}{(T-s)v_s} 1_{[s,T]}(u),
\end{aligned}
\tag{4.105}
$$

which when frozen is

$$
\begin{aligned}
\omega^0 \circ D^2_{2,u} v_s =& \alpha^2 \omega^0(v_s) 1_{[0,s]}(u) \\
&+ \alpha^2\left(2 - \frac{T-u}{T-s}\frac{\omega^0(v_u^2)}{\omega^0(v_s^2)}\right)\frac{(T-u)\omega^0(v_u^2)}{(T-s)\omega^0(v_s)} 1_{[s,T]}(u).
\end{aligned}
\tag{4.106}
$$

4.5.3 Calculating $\mathbf{A} = \omega^0 \circ BS(0, 0, v_0)$

By far, **A** is our most trivial term.

$$
\begin{aligned}
\mathbf{A} =& \omega^0 \circ BS(0, 0, v_0) \\
=& N(d_+(0, 0, \omega^0(v_s)) - N(d_+(0, 0, \omega^0(v_s)).
\end{aligned}
\tag{4.107}
$$

Now $d_\pm(0, 0, \sigma) = \pm\frac{1}{2}\sigma\sqrt{T}$, so

$$
\begin{aligned}
\mathbf{A} =& N(d_+(0, 0, \omega^0(v_s)) - N(d_+(0, 0, \omega^0(v_s)) \\
=& N\left(\frac{1}{2}\omega^0(v_0)\sqrt{T}\right) - N\left(-\frac{1}{2}\omega^0(v_0)\sqrt{T}\right) \\
=& N\left(\frac{\sigma_0}{2\alpha}\sqrt{1 - e^{-\alpha^2 T}}\right) - N\left(-\frac{\sigma_0}{2\alpha}\sqrt{1 - e^{-\alpha^2 T}}\right).
\end{aligned}
\tag{4.108}
$$

4.5.4 Calculating $\mathbf{B} = \omega^0 \circ \Delta BS(0, 0, v_0)$

The Gross-Laplacian Δ is the sum, $\Delta_1+\Delta_2$, of the Gross-Laplacian with respect to the first Brownian motion and the Gross-Laplacian with respect to the second Brownian motion. The key to saving ourselves a lot of effort is to realize the stochasticity of $\mathbf{B}$ is entirely dependent on v_0. Since v_0 is entirely generated by B_2, $BS(0, 0, v_0)$ is entirely generated by B_2. The Gross-Laplacian satisfies the identity $\Delta_i G = \int_0^T D^2_{i,u} G du$, and from the above reasoning, we have $D_{1,u}BS(0, 0, v_0)$ is identically zero. Therefore, $\Delta_1 \Delta BS(0, 0, v_0)$ is zero, and thus $\Delta BS(0, 0, v_0)$ coincides with the Gross-Laplacian of $BS(0, 0, v_0)$ with respect to B_2 only. So $\Delta BS(0, 0, v_0)$ is just $\Delta_2 BS(0, 0, v_0)$. We must consider

$$\begin{aligned} D^2_{2,u}BS(0,0,v_0) =& D_{2,u}\left(\frac{\partial BS}{\partial \sigma}(0,0,v_0)D_{2,u}v_0\right) \\ =& \frac{\partial BS}{\partial \sigma}(0,0,v_0)D^2_{2,u}v_0 + \frac{\partial^2 BS}{\partial \sigma^2}(0,0,v_0)\left(D_{2,u}v_0\right)^2 . \end{aligned} \tag{4.109}$$

The freezing operator and deterministic integration commute, so we have

$$\begin{aligned} \omega^0\left(\Delta BS(0,0,v_0)\right) = \int_0^T & \omega^0\left(\frac{\partial BS}{\partial \sigma}(0,0,v_0)\right)\omega^0\left(D^2_{2,u}v_0\right) \\ & + \omega^0\left(\frac{\partial^2 BS}{\partial \sigma^2}(0,0,v_0)\right)\omega^0\left(D_{2,u}v_0\right)^2 du. \end{aligned} \tag{4.110}$$

Effectively, we are allowed to simplify before integrating. Now we have already calculated $\omega^0\left(D_{2,u}v_0\right)$ and $\omega^0\left(D^2_{2,u}v_0\right)$ in Equations (4.101) and (4.106). In this instance where we are concerned with v_0 specifically, we have

$$\omega^0 \circ \left(D_{2,u}v_0\right)^2 = \frac{\sigma_0^2\left(e^{-\alpha^2 u} - e^{-\alpha^2 T}\right)^2}{T(1-e^{-\alpha^2 T})}. \tag{4.111}$$

We calculate $D_{2,u}v_0$ using Equation (4.97)

$$\begin{aligned}\omega^0 \circ (D^2_{2,u}v_0) =&\alpha^2 \left(2 - \frac{(T-u)\omega^0(v_u^2)}{T\omega^0(v_0^2)}\right) \frac{(T-u)\omega^0(v_u^2)}{T\omega^0(v_0)} \\ =&\frac{\sigma_0^2}{T\omega^0(v_0)} \left(2 - \frac{e^{-\alpha^2 u} - e^{-\alpha^2 T}}{1-e^{-\alpha^2 T}}\right) \left(e^{-\alpha^2 u} - e^{-\alpha^2 T}\right) \\ =&\frac{\alpha\sigma_0}{\sqrt{T}} \frac{(1-e^{-\alpha^2 T}) - (1-e^{-\alpha^2 u})}{\sqrt{1-e^{-\alpha^2 T}}} \frac{(1-e^{-\alpha^2}) + (1-e^{-\alpha^2 u})}{1-e^{-\alpha^2 T}} \\ =&\frac{\alpha\sigma_0}{\sqrt{T}} \frac{\left(1-e^{-\alpha^2 T}\right)^2 - \left(1-e^{-\alpha^2 u}\right)^2}{\left(1-e^{-\alpha^2 T}\right)^{\frac{3}{2}}}.\end{aligned} \tag{4.112}$$

Now we calculate the frozen vega $\omega^0 \circ \frac{\partial BS}{\partial \sigma}(0,0,v_0)$. To calculate this and the frozen vomma we need to freeze $d_\pm(0,0,v_0)$, which we calculated in Equation (4.108)

$$\begin{aligned}\omega^0(d_\pm(0,0,v_0)) =& \pm \frac{\sqrt{T}}{2}\omega^0(v_0) \\ =& \pm \frac{\sigma_0}{2\alpha}\sqrt{1-e^{-\alpha^2 T}}.\end{aligned} \tag{4.113}$$

From the calculation for the vega in Equation (4.94), we have

$$\begin{aligned}\omega^0 \circ \frac{\partial BS}{\partial \sigma}(0,0,v_0) =&\sqrt{T}N'(\omega^0(d_-)) \\ =&\sqrt{T}N'\left(-\frac{\sigma_0}{2\alpha}\sqrt{1-e^{-\alpha^2 T}}\right).\end{aligned} \tag{4.114}$$

And for the vomma, we use Equation (4.95)

$$\begin{aligned}&\omega^0 \circ \frac{\partial^2 BS}{\partial \sigma^2}(0,0,v_0) \\ =&\sqrt{T}\frac{\omega^0(d_+)\omega^0(d_-)}{\omega^0(v_0)}N'\left(\omega^0(d_-)\right) \\ =&-\frac{T^{\frac{3}{2}}}{4}\omega^0(v_0)N'(\omega^0(d_-)) \\ =&-\frac{\sigma_0 T}{4\alpha}\sqrt{1-e^{-\alpha^2 T}}N'\left(-\frac{\sigma_0}{2\alpha}\sqrt{1-e^{-\alpha^2 T}}\right)\end{aligned} \tag{4.115}$$

We have calculated all the terms in the integrand, and thus we have shown how to calculate

$$\mathbf{B} = \int_0^T \left(\omega^0 \left(\frac{\partial BS}{\partial \sigma}(0,0,v_0)\right) \omega^0 \left(D^2_{2,u} v_0\right) + \omega^0 \left(\frac{\partial^2 BS}{\partial \sigma^2}(0,0,v_0)\right) \omega^0 \left(D_{2,u} v_0\right)^2\right) du. \tag{4.116}$$

When calculating approximations, we will see that the integral of $\omega^0 \circ \Delta BS(0,0,v_0)$ actually has a closed-form expression.

4.5.5 The log price X_s

So far we have only been concerned about terms at the initial time $s = 0$, at which the forward rate process X is 0. We now turn our attention to the corrective term $\int_0^T \mathbb{E}[\Xi_s \Lambda_s] ds$. To reason about this quantity, we must first examine the process X: determine a closed form expression for this process, it's various Malliavin derivatives, and the images of each under the frozen operator. X satisfies the SDE given in Equation (4.79); the solution to which is

$$X_s = \sqrt{1-\rho^2} \int_0^s \sigma_r dB_1(r) + \rho \int_0^s \sigma_r dB_2(r) - \frac{1}{2} \int_0^s \sigma_r^2 dr. \tag{4.117}$$

If we have defined the frozen path operators in terms of the Stratonovich, then we have

$$\omega^0 \circ \int_0^T Y_s dZ_s = -\frac{1}{2} \omega^0 \circ \langle Y, Z \rangle_T. \tag{4.118}$$

To freeze, we follow our nose; guided by the properties of quadratic variation as outlined in Sections 2.2 and 2.3 of [Kun90]. Freezing the first integral results in

$$
\begin{aligned}
\omega^0 \circ \int_0^s \sigma_r dB_1(r) =& -\frac{1}{2}\omega^0 \circ \langle \sigma, B_1 \rangle_s \\
=& -\frac{1}{2}\omega^0 \circ \langle \sigma_0 + \alpha \int_0^{\cdot} \sigma_r dB_2(r), B_1 \rangle_s \\
=& -\frac{\alpha}{2}\omega^0 \circ \langle \int_0^{\cdot} \sigma_r dB_2(r), B_1 \rangle_s \\
=& -\frac{\alpha}{2}\omega^0 \circ \int_0^s \sigma_r d\langle B_2, B_1 \rangle_r \\
=& 0,
\end{aligned} \tag{4.119}
$$

since the joint quadratic variation of two independent Brownian motions is zero. The calculation short-circuits before we even have to apply the freezing operator. We follow the same intermediary steps for the second integral

$$
\begin{aligned}
\omega^0 \circ \int_0^s \sigma_r dB_2(r) =& -\frac{\alpha}{2}\omega^0 \circ \int_0^s \sigma_r d\langle B_2, B_2 \rangle_r \\
=& -\frac{\alpha}{2}\omega^0 \circ \int_0^s \sigma_r dr \\
=& -\frac{\alpha\sigma_0}{2} \int_0^s e^{-\frac{1}{2}\alpha^2 r} dr \\
=& -\frac{\alpha\sigma_0}{2} \frac{2}{\alpha^2}(1 - e^{-\frac{1}{2}\alpha^2 s}) \\
=& -\frac{\sigma_0}{\alpha}\left(1 - e^{-\frac{1}{2}\alpha^2 s}\right).
\end{aligned} \tag{4.120}
$$

The third integral has become routine for us

$$
\begin{aligned}
\omega^0 \circ \int_0^s \sigma_r^2 dr =& \sigma_0^2 \int_0^s e^{-\alpha^2 r} dr \\
=& \frac{\sigma_0^2}{\alpha^2}\left(1 - e^{-\alpha^2 s}\right).
\end{aligned} \tag{4.121}
$$

Therefore, X_s when frozen is

$$
\omega^0(X_s) = -\frac{\rho\sigma_0}{\alpha}\left(1 - e^{-\frac{1}{2}\alpha^2 s}\right) - \frac{\sigma_0^2}{2\alpha^2}\left(1 - e^{-\alpha^2 s}\right). \tag{4.122}
$$

The various partial Malliavin derivatives of X are required to calculate our last term **E** and **F**. Calculating $D_{1,u}X_s$ is easy. We remember $D_{1,u}\sigma_r$ is identically zero, and that the partial Malliavin derivative with respect to B_1 commutes with integration against B_2. So $D_{1,u}X_s$ and its frozen image is

$$\begin{aligned} D_{1,u}X_s =& \sqrt{1-\rho^2}\left(1_{[0,s]}(u)\sigma_u + \int_0^T 1_{[0,s]}(r)D_{1,u}\sigma_r dB_1(r)\right) \\ &+ \rho\int_0^s D_{1,u}\sigma_r dB_2(r) - \frac{1}{2}\int_0^s D_{1,u}\sigma_r^2 dr \\ =& \sqrt{1-\rho^2}1_{[0,s]}(u)\sigma_u \\ \omega^0 \circ D_{1,u}X_s =& \sigma_0\sqrt{1-\rho^2}1_{[0,s]}(u)e^{-\frac{\alpha^2}{2}u}. \end{aligned} \tag{4.123}$$

Then $D_{1,u}^2 X_s$ is identically zero, as is $\omega^0 \circ D_{1,u}^2 X_s$. The Malliavin derivative with respect to B_2 is

$$\begin{aligned} D_{2,u}X_s =& \sqrt{1-\rho^2}\int_0^T 1_{[0,s]}(r)D_{2,u}\sigma_r dB_1(r) + \rho 1_{[0,s]}(u)\sigma_u \\ &+ \rho\int_0^T 1_{[0,s]}(r)D_{2,u}\sigma_r dB_2(r) - \frac{1}{2}\int_0^T 1_{[0,s]}(r)D_{2,u}\sigma_r^2 dr \\ =& \alpha\sqrt{1-\rho^2}\int_0^T 1_{[0,r]}(u)1_{[0,s]}(r)\sigma_r dB_1(r) + \rho 1_{[0,s]}(u)\sigma_u \\ &+ \alpha\rho\int_0^T 1_{[0,s]}(r)1_{[0,r]}(u)\sigma_r dB_2(r) - \alpha\int_0^T 1_{[0,s]}(r)1_{[0,r]}(u)\sigma_r^2 dr. \end{aligned} \tag{4.124}$$

The term $1_{[0,s]}(r)1_{[0,r]}(u)$ is non-zero only when $u \le r \le s$, therefore

$$\begin{aligned} D_{2,u}X_s =& 1_{[0,s]}(u)\left(\alpha\sqrt{1-\rho^2}\int_s^u \sigma_r dB_1(r) + \rho\sigma_u\right. \\ & \left. + \alpha\rho\int_u^s \sigma_r dB_2(r) - \alpha\int_u^s \sigma_r^2 dr\right). \end{aligned} \tag{4.125}$$

So when we freeze we have

$$\begin{aligned} \omega^0 \circ D_{2,u}X_s =& 1_{[0,s]}(u)\left(0 + \rho\sigma_0 e^{-\frac{\alpha^2}{2}u}\right. \\ & \left. + \alpha\rho\omega^0 \circ \int_u^s \sigma_r dB_2(r) - \alpha\int_u^s \omega^0(\sigma_r^2)dr\right). \end{aligned} \tag{4.126}$$

From Equation (4.120) we see

$$\omega^0 \circ \int_u^s \sigma_r dB_2(r) = -\frac{\sigma_0}{\alpha}\left(e^{-\frac{\alpha^2}{2}u} - e^{-\frac{\alpha^2}{2}s}\right), \tag{4.127}$$

while from equation (4.121) we have

$$\omega^0 \circ \int_u^s \sigma_r^2 dr = \frac{\sigma_0^2}{\alpha^2}\left(e^{-\alpha^2 u} - e^{-\alpha^2 s}\right). \tag{4.128}$$

Therefore

$$\begin{aligned}\omega^0 \circ D_{2,u}X_s =& 1_{[0,s]}(u)\left(\rho\sigma_0 e^{-\frac{\alpha^2}{2}u} - \rho\alpha\frac{\sigma_0}{\alpha}\left(e^{-\frac{\alpha^2}{2}u} - e^{-\frac{\alpha^2}{2}s}\right)\right.\\ &\left.-\alpha\frac{\sigma_0^2}{\alpha^2}\left(e^{-\alpha^2 u} - e^{-\alpha^2 s}\right)\right)\\ =& 1_{[0,s]}(u)\left(\rho\sigma_0 e^{-\frac{\alpha^2}{2}s} - \frac{\sigma_0^2}{\alpha}\left(e^{-\alpha^2 u} - e^{-\alpha^2 s}\right)\right).\end{aligned} \tag{4.129}$$

Since $D_{2,u}X_s$ has a factor of $1_{[0,s]}(u)$, we can assume $u \leq s$ when calculating $D_{2,u}^2 X_s$ since that same leading factor will be present.

$$\begin{aligned}D_{2,u}^2 X_s =& 1_{[0,s]}(u)\left(\alpha\sqrt{1-\rho^2}\int_s^u D_{2,u}\sigma_r dB_1(r) + \rho D_{2,u}\sigma_u + \alpha\rho\sigma_u\right.\\ &\left.+\alpha\rho\int_u^s D_{2,u}\sigma_r dB_2(r) - \alpha\int_u^s D_{2,u}\sigma_r^2 dr\right)\\ =& 1_{[0,s]}(u)\left(\alpha^2\sqrt{1-\rho^2}\int_s^u \sigma_r dB_1(r) + 2\alpha\rho\sigma_u\right.\\ &\left.+\alpha^2\rho\int_u^s \sigma_r dB_2(r) - 2\alpha^2\int_u^s \sigma_r^2 dr\right).\end{aligned} \tag{4.130}$$

Then we freeze this quantity using the previously calculated intermediary results in Equations (4.127) and (4.128)

$$\begin{aligned}\omega^0 \circ D^2_{2,u} X_s =& 1_{[0,s]} \left(0 + 2\alpha\rho\sigma_0 e^{-\frac{\alpha^2}{2}u} - \alpha^2 \rho \frac{\sigma_0}{\alpha} \left(e^{-\frac{\alpha^2}{2}u} - e^{-\frac{\alpha^2}{2}s}\right)\right. \\ &\left. -2\alpha^2 \frac{\sigma_0^2}{\alpha^2} \left(e^{-\alpha^2 u} - e^{-\alpha^2 s}\right)\right) \\ =& 1_{[0,s]}(u) \left(\alpha\rho\sigma_0 \left(e^{-\frac{\alpha^2}{2}u} + e^{-\frac{\alpha^2}{2}s}\right)\right. \\ &\left. -2\sigma_0^2 \left(e^{-\alpha^2 u} - e^{-\alpha^2 s}\right)\right).\end{aligned} \tag{4.131}$$

We are now able to tackle the corrective term.

4.5.6 Calculating $\mathbf{C} = \omega^0(\Xi_s \Lambda_s)$

Of course $\omega^0(\Xi_s\Lambda_s) = \omega^0(\Xi_s)\omega^0(\Lambda_s)$. Remember $\Xi_s = b(s, X_s, v_s)$ where

$$b(t, x, \sigma) = \frac{N'(d_-(t, x, \sigma))}{\sigma\sqrt{T-t}} \left(1 - \frac{d_-(t, x, \sigma)}{\sigma\sqrt{T-t}}\right), \tag{4.132}$$

where we rely on the reader's mathematical maturity to know when we are referring to the stochastic process σ_r and the parameter σ which is standard in the financial literature. Then

$$\omega^0(\Xi_s) = \frac{N'(d_-(s, \omega^0(X_s), \omega^0(v_s)))}{\omega^0(v_s)\sqrt{T-s}} \left(1 - \frac{d_-(s, \omega^0(X_s), \omega^0(v_s))}{\omega^0(v_s)\sqrt{T-s}}\right), \tag{4.133}$$

and we have already calculated every subterm of this expression in Equations (4.122) and (4.97). To freeze Λ_s is also simple

$$\begin{aligned}\omega^0 \circ \Lambda_s =& 2\alpha\omega^0(\sigma_s) \int_s^T \omega^0(\sigma_r^2) dr \\ =& 2\alpha^2 \sigma_0 e^{-\frac{\alpha^2}{2}s} \int_s^t \sigma_0^2 e^{-\alpha^2 r} dr \\ =& 2\sigma_0^3 e^{-\frac{\alpha^2}{2}s} \left(e^{-\alpha^2 s} - e^{-\alpha^2 T}\right).\end{aligned} \tag{4.134}$$

And thus we have the factors of $\mathbf{C}$.

4.5.7 The derivatives of b

The last term that remains is **D**. As we calculate the terms **E** and **F**, which **D** comprises, we will need the partial derivatives (up to order two) of our function $b(t,x,\sigma)$ with respect to x and σ. Thankfully, we will not need to also consider the derivatives with respect to t since we were not so masochistic as to make the time parameter of model a stochastic process. We will make liberal use of the identities in equations (4.86) and (4.86). To begin

$$\begin{aligned}\frac{\partial b}{\partial x} =&\frac{\partial}{\partial x}\left(N'(d_-)\left(\frac{1}{\sigma\sqrt{T-t}}-\frac{d_-}{\sigma^2\sqrt{T-t}^2}\right)\right)\\ =&-\frac{d_-}{\sigma\sqrt{T-t}}(N'(d_-)\left(\frac{1}{\sigma\sqrt{T-t}}-\frac{d_-}{\sigma^2\sqrt{T-t}^2}\right)\\ &+N'(d_-)\left(0-\frac{1}{\sigma^3\sqrt{T-t^3}}\right)\\ =&N'(d_-)\left(-\frac{d_-}{\sigma^2\sqrt{T-t}^2}+\frac{-1+d_-^2}{\sigma^3\sqrt{T-t}^3}\right),\end{aligned} \tag{4.135}$$

and

$$\begin{aligned}&\frac{\partial^2 b}{\partial x^2}\\ =&-\frac{d_-}{\sigma\sqrt{T-t}}N'(d_-)\left(-\frac{d_-}{\sigma^2\sqrt{T-t}^2}+\frac{-1+d_-^2}{\sigma^3\sqrt{T-t}^3}\right)\\ &+N'(d_-)\left(-\frac{1}{\sigma^3\sqrt{T-t}^3}+\frac{0+2d_-}{\sigma^4\sqrt{T-t}^4}\right)\\ =&N'(d_-)\left(\frac{-1+d_-^2}{\sigma^3\sqrt{T-t}^3}+\frac{3d_--d_-^3}{\sigma^4\sqrt{T-t}^4}\right).\end{aligned} \tag{4.136}$$

There is some pattern dealing with polynomials with the parameters d_- and $\frac{1}{\sigma\sqrt{T-t}}$, but such basic algebra is beyond us. We will also need the derivatives with respect to σ. We can use the following formulas to aid in the subsequent calculations

$$\begin{aligned}\frac{\partial N(d_-)}{\partial\sigma} =&\frac{d_-d_+}{\sigma}N'(d_-)\\ \frac{\partial}{\partial\sigma}(d_-^m d_+^n) =&-\frac{md_-^{m-1}d_+^{n+1}+nd_-^{m+1}d_+^{n-1}}{\sigma}.\end{aligned} \tag{4.137}$$

Then the first derivative with respect to σ would be.

$$
\begin{aligned}
\frac{\partial b}{\partial \sigma} =& \frac{d_- d_+}{\sigma} N'(d_-) \left(\frac{1}{\sigma\sqrt{T-t}} - \frac{d_-}{\sigma^2 \sqrt{T-t}^2} \right) \\
&+ N'(d_-) \left(-\frac{1}{\sigma^2\sqrt{T-t}} + 2\frac{d_-}{\sigma^3\sqrt{T-t}^2} + \frac{d_+}{\sigma^3\sqrt{T-t}^2} \right) \\
=& N'(d_-) \left(\frac{d_- d_+}{\sigma^2\sqrt{T-t}} - \frac{d_-^2 d_+}{\sigma^3\sqrt{T-t}^2} \right) \\
&+ N'(d_-) \left(-\frac{1}{\sigma^2\sqrt{T-t}} + \frac{2d_- + d_+}{\sigma^3\sqrt{T-t}^2} \right) \\
=& N'(d_-) \left(\frac{-1 + d_- d_+}{\sigma^2\sqrt{T-t}} + \frac{2d_- + d_+ - d_-^2 d_+}{\sigma^3\sqrt{T-t}^2} \right).
\end{aligned} \tag{4.138}
$$

Then the second derivative with respect to σ is

$$
\begin{aligned}
\frac{\partial^2 b}{\partial \sigma^2} =& \frac{d_- d_+}{\sigma} N'(d_-) \left(\frac{-1 + d_- d_+}{\sigma^2\sqrt{T-t}} + \frac{2d_- + d_+ - d_-^2 d_+}{\sigma^3\sqrt{T-t}^2} \right) \\
&+ N'(d_-) \left(-2\frac{-1 + d_- d_+}{\sigma^3\sqrt{T-t}} - \frac{d_-^2 + d_+^2}{\sigma^3\sqrt{T-t}} \right. \\
&\left. -3\frac{2d_- + d_+ - d_-^2 d_+}{\sigma^4\sqrt{T-t}^2} - \frac{2d_+ + d_- - d_-^3 - 2d_- d_+^2}{\sigma^4\sqrt{T-t}^2} \right) \\
=& N'(d_-) \left(\frac{-d_- d_+ + d_-^2 d_+^2}{\sigma^3\sqrt{T-t}} + \frac{2d_-^2 d_+ + d_- d_+^2 - d_-^3 d_+^2}{\sigma^4\sqrt{T-t}^2} \right) \\
&+ N'(d_-) \left(\frac{2 - 2d_- d_+ - d_-^2 - d_+^2}{\sigma^3\sqrt{T-t}} \right. \\
&\left. + \frac{-7d_- - 5d_+ + d_-^3 + 3d_-^2 d_+ + 2d_- d_+^2}{\sigma^4\sqrt{T-t}^2} \right) \\
=& N'(d_-) \left(\frac{2 - d_-^2 - 3d_- d_+ - d_+^2 + d_-^2 d_+^2}{\sigma^3\sqrt{T-t}} \right. \\
&\left. + \frac{-7d_- - 5d_+ + d_-^3 + 5d_-^2 d_+ + 3d_- d_+^2 - d_-^3 d_+^2}{\sigma^4\sqrt{T-t}^2} \right).
\end{aligned} \tag{4.139}
$$

Then finally the cross term

$$
\begin{aligned}
\frac{\partial^2 b}{\partial x \partial \sigma} =& \frac{\partial}{\partial x}\left(N'(d_-)\left(\frac{-1+d_-d_+}{\sigma^2\sqrt{T-t}}+\frac{2d_-+d_+-d_-^2d_+}{\sigma^3\sqrt{T-t}^2}\right)\right)\\
=& -\frac{d_-}{\sigma\sqrt{T-t}}N'(d_-)\left(\frac{-1+d_-d_+}{\sigma^2\sqrt{T-t}}+\frac{2d_-+d_+-d_-^2d_+}{\sigma^3\sqrt{T-t}^2}\right)\\
&+N'(d_-)\left(\frac{0+d_-+d_+}{\sigma^3\sqrt{T-t}^2}+\frac{2+1-2d_-d_+-d_-^2}{\sigma^4\sqrt{T-t}^3}\right)\\
=& N'(d_-)\left(\frac{d_--d_-^2d_+}{\sigma^3\sqrt{T-t}^2}+\frac{-2d_-^2-d_-d_++d_-^3d_+}{\sigma^4\sqrt{T-t}^3}\right)\\
&+N'(d_-)\left(\frac{d_-+d_+}{\sigma^3\sqrt{T-t}^2}+\frac{3-d_-^2-2d_-d_+}{\sigma^4\sqrt{T-t}^3}\right)\\
=& N'(d_-)\left(\frac{2d_-+d_+-d_-^2d_+}{\sigma^3\sqrt{T-t}^2}+\frac{3-3d_-^2-3d_-d_++d_-^3d_+}{\sigma^4\sqrt{T-t}^3}\right).
\end{aligned}
\tag{4.140}
$$

With these results in hand, we are ready to proceed onto calculating $\mathbf{D} = \mathbf{E} + \mathbf{F}$.

4.5.8 Calculating $\mathbf{E} = \omega^0(\Delta_1(\Xi_s\Lambda_s))$

We have the following identity

$$
\Delta_1(\Xi_s\Lambda_s) = \int_0^T D^2_{1,u}\left(\Xi_s\Lambda_s\right) du. \tag{4.141}
$$

However, Λ_s is entirely generated by B_2, and therefore $D_{1,u}\Lambda_s$ is identically zero. So we can further simplify and see

$$
\begin{aligned}
\mathbf{E} =& \omega^0 \circ \int_0^T D^2_{1,u}\left(\Xi_s\Lambda_s\right) du\\
=& \omega^0(\Lambda_s)\int_0^T \omega^0 \circ D^2_{1,u}\Xi_s du.
\end{aligned}
\tag{4.142}
$$

We have already calculated the leading term $\omega^0(\Lambda_s)$, and for the integrand

$$\begin{aligned}\omega^0 \circ D^2_{1,u}\Xi_s =&\omega^0 \circ D^2_{1,u} b(s, X_s, v_s)\\ =&\omega^0 \circ D_{1,u}\left(\frac{\partial b}{\partial x}(s, X_s, v_s) D_{1,u} D_{1,u} X_s\right)\\ =&\omega^0\left(\frac{\partial^2 b}{\partial x^2}(s, X_s, v_s)\left(D_{1,u}X_s\right)^2 + \frac{\partial b}{\partial x}(s, X_s, v_s) D^2_{1,u} X_s\right)\\ =&\frac{\partial^2 b}{\partial x^2}(s, \omega^0(X_s), \omega^0(v_s))\omega^0\left(D_{1,u}X_s\right)^2,\end{aligned} \tag{4.143}$$

since $D^2_{1,u}X_s$ is zero. Thus, using Equations (4.123) and (4.134), we see

$$\begin{aligned}\mathbf{E} =&\omega^0(\Lambda_s)\int_0^T \frac{\partial^2 b}{\partial x^2}(s, \omega^0(X_s), \omega^0(v_s))\omega^0\left(D_{1,u}X_s\right)^2 du\\ =&2\sigma_0^3 e^{-\frac{\alpha^2}{2}s}\left(e^{-\alpha^2 s} - e^{-\alpha^2 T}\right)\\ &\cdot \int_0^T \frac{\partial^2 b}{\partial x^2}(s, \omega^0(X_s), \omega^0(v_s))\sigma_0\sqrt{1-\rho^2}1_{[0,s]}(u) e^{-\frac{\alpha^2}{2}u} du\\ =&2\sigma_0^4\sqrt{1-\rho^2} e^{-\frac{\alpha^2}{2}s}\left(e^{-\alpha^2 s} - e^{-\alpha^2 T}\right)\int_0^s e^{-\frac{\alpha^2}{2}u}\frac{\partial^2 b}{\partial x^2}(s, \omega^0(X_s), \omega^0(v_s)) du.\end{aligned} \tag{4.144}$$

4.5.9 Calculating $\mathbf{F} = \omega^0(\Delta_2(\Xi_s\Lambda_s))$

For our last calculation, there are no real shortcuts, or quick reductions. The term is as general as it can be. We break **F** into three summands

$$\begin{aligned}&\mathbf{F}\\ =&\omega^0\int_0^T D^2_{2,u}\left(\Xi_s\Lambda_s\right) du\\ =&\omega^0(\Xi_s)\int_0^T \omega^0 \circ D^2_{2,u}\Lambda_s du + 2\int_0^T \omega^0 \circ D_{2,u}\Xi_s \omega^0 \circ D_{2,u}\Lambda_s du\\ &+ \omega^0(\Lambda_s)\int_0^T \omega^0 \circ D^2_{2,u}\Xi_s du\\ =&\mathbf{F1} + 2\mathbf{F2} + \mathbf{F3}.\end{aligned} \tag{4.145}$$

We have already calculated $\omega^0(\Lambda_s)$ in equation (4.134). Let us then calculate $\omega^0 \circ D_{2,u}\Lambda_s$

$$\begin{aligned}
D_{2,u}\Lambda_s =& 2\alpha D_{2,u}\left(\sigma_s \int_s^T \sigma_r^2 dr\right)\\
=& 2\alpha^2\sigma_s 1_{[0,s]}(u)\int_s^T \sigma_r^2 dr + 2\alpha\sigma_s \int_0^T 1_{[s,T]}(r) D_{2,u}\sigma_r^2 dr\\
=& \alpha\Lambda_s 1_{[0,s]}(u) + 4\alpha^2\sigma_s \int_0^T 1_{[s,T]}(r)1_{[0,r]}(u)\sigma_r^2 dr\\
=& \alpha\Lambda_s 1_{[0,s]}(u) + 4\alpha^2\sigma_s\left(1_{[0,s]}(u)\int_s^T \sigma_r^2 dr + 1_{[s,T]}(u)\int_u^T \sigma_r^2 dr\right)\\
=& 3\alpha\Lambda_s 1_{[0,s]}(u) + 4\alpha^2\sigma_s \int_u^T \sigma_r^2 dr \cdot 1_{[s,T]}(u),
\end{aligned} \tag{4.146}$$

and

$$\begin{aligned}
\omega^0 \circ D_{2,u}\Lambda_s =& 3\alpha\omega^0(\Lambda_s)1_{[0,s]}(u) + 4\alpha^2\sigma_0 e^{-\frac{\alpha^2}{2}s}\int_u^T \sigma_0^2 e^{-\alpha^2 r} dr 1_{[s,T]}(u)\\
=& 3\alpha\omega^0(\Lambda_s)1_{[0,s]}(u) + 4\sigma_0^3 e^{-\frac{\alpha^2}{2}s}\left(e^{-\alpha^2 u} - e^{-\alpha^2 T}\right)1_{[s,T]}(u).
\end{aligned} \tag{4.147}$$

For the second Malliavin derivative of Λ_s with respect to B_2 we compute

$$\begin{aligned}
D_{2,u}^2\Lambda_s =& 3\alpha\left(3\alpha\Lambda_s 1_{[0,s]}(u) + 4\alpha^2\sigma_s\int_u^T \sigma_r^2 dr \cdot 1_{[s,T]}(u)\right)1_{[0,s]}(u)\\
&+ 4\alpha^2\left(\alpha\sigma_s 1_{[0,s]}(u)\int_u^T \sigma_r^2 dr + 2\alpha\sigma_s\int_u^T \sigma_r^2 dr\right)1_{[s,T]}(u)\\
=& 9\alpha^2\Lambda_s 1_{[0,s]}(u) + 8\alpha^3\int_u^T \sigma_r^2 dr 1_{[s,T]}(u),
\end{aligned} \tag{4.148}$$

using the identity $1_{[0,s]}(u)1_{[s,T]}(u) = 0$. The frozen image is

$$\omega^0 \circ D_{2,u}^2\Lambda_s = 9\alpha^2\omega^0(\Lambda_s)1_{[0,s]}(u) + 8\alpha\sigma_0^3 e^{-\frac{\alpha^2}{2}s}\left(e^{-\alpha^2 u} - e^{-\alpha^2 T}\right)1_{[s,T]}(u). \tag{4.149}$$

Now onto the Malliavin derivatives of Ξ_s. First

$$\begin{aligned}
&\omega^0 \circ D_{2,u}\Xi_s\\
=&\frac{\partial b}{\partial x}(s,\omega^0(X_s),\omega^0(v_s))\omega^0 \circ D_{2,u}X_s + \frac{\partial b}{\partial \sigma}(s,\omega^0(X_s),\omega^0(v_s))\omega^0 \circ D_{2,u}v_s.
\end{aligned} \tag{4.150}$$

The subterms of this expression were previously calculated in the following equations: $\frac{\partial b}{\partial x}$ in Equation (4.135), $\frac{\partial b}{\partial \sigma}$ in Equation (4.138), $\omega^0(D_{2,u}X_s)$ in Equation (4.129), and $\omega^0(D_{2,u}v_s)$ in Equation (4.101). Therefore

$$\begin{aligned}
&D_{2,u}\Xi_s \\
=&D_{2,u}\left(\frac{\partial b}{\partial x}(s,X_s,v_s)D_{2,u}X_s + \frac{\partial b}{\partial \sigma}(s,X_s,v_s)D_{2,u}v_s\right) \\
=&\frac{\partial^2 b}{\partial x^2}(s,X_s,v_s)(D_{2,u}X_s)^2 + 2\frac{\partial^2 b}{\partial x \partial \sigma}(s,X_s,v_s)(D_{2,u}X_s)(D_{2,u}v_s) \\
&+\frac{\partial^2 b}{\partial \sigma^2}(s,X_s,v_s)(D_{2,u}v_s)^2 \\
&+\frac{\partial b}{\partial x}(s,X_s,v_s)D_{2,u}^2X_s + \frac{\partial b}{\partial \sigma}(s,X_s,v_s)D_{2,u}^2 v_s,
\end{aligned} \tag{4.151}$$

and freezing this results in

$$\begin{aligned}
&\omega^0 \circ D_{2,u}\Xi_s \\
=&\frac{\partial^2 b}{\partial x^2}(s,\omega^0(X_s),\omega^0(v_s))\omega^0(D_{2,u}X_s)^2 + +\frac{\partial^2 b}{\partial \sigma^2}(s,\omega^0(X_s),\omega^0(v_s))\omega^0(D_{2,u}v_s)^2 \\
&+2\frac{\partial^2 b}{\partial x \partial \sigma}(s,\omega^0(X_s),\omega^0(v_s))\omega^0(D_{2,u}X_s)\omega^0(D_{2,u}v_s) \\
&+\frac{\partial b}{\partial x}(s,\omega^0(X_s),\omega^0(v_s))\omega \circ D_{2,u}^2X_s + \frac{\partial b}{\partial \sigma}(s,\omega^0(X_s),\omega^0(v_s))\omega^0 \circ D_{2,u}^2 v_s.
\end{aligned} \tag{4.152}$$

In addition to utilizing the aforementioned equations, we calculate this result using Equation (4.136) for $\frac{\partial^2 b}{\partial x^2}$, Equation (4.139) for $\frac{\partial^2 b}{\partial \sigma^2}$, Equation (4.140) for $\frac{\partial^2 b}{\partial x \partial \sigma}$, Equation (4.106) for $\omega^0(D_{2,u}^2 v_s)$, and Equation (4.131) for $\omega^0(D_{2,u}^2 X_s)$.

Then

$$\mathbf{F1} = \omega^0(\Xi_s)\int_0^T \omega^0 \circ D_{2,u}^2\Lambda_s du, \tag{4.153}$$

is calculated via Equations (4.133) and (4.149).

$$\mathbf{F2} = \int_0^T \omega^0 \circ D_{2,u}\Xi_s \omega^0 \circ D_{2,u}\Lambda_s du, \tag{4.154}$$

is calculated via equations (4.150) and (4.147), and

$$\mathbf{F3} = \omega^0(\Lambda_s) \int_0^T \omega^0 \circ D^2_{2,u} \Xi_s du, \tag{4.155}$$

is calculated via Equations (4.152) and (4.134).

Then we combine these results to finally calculate

$$\omega^0 \circ \Delta_2(\Xi_s \Lambda_s) = \mathbf{F} = \mathbf{F1} + 2\mathbf{F2} + \mathbf{F3}. \tag{4.156}$$

4.5.10 Final Assembly

Now we can finally calculate

$$\omega^0 \circ \Delta(\Xi_s \Lambda_s) = \mathbf{D} = \mathbf{E} + \mathbf{F}. \tag{4.157}$$

With our four intermediary results $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ we can then approximate the value of the call option by calculating

$$\begin{aligned} &\omega^0 \circ (BS(0,0,v_0)) + \omega^0 \circ (\Delta BS(0,0,v_0)) \\ &+ \frac{\rho}{2} \int_0^T \omega^0 \circ (\Xi_s \Lambda_s) + \omega^0 \circ \Delta\left(\Xi_s \Lambda_s\right) ds \\ =&\mathbf{A} + \frac{1}{2}\mathbf{B} + \frac{\rho}{2} \int_0^T \left(\mathbf{C} + \frac{1}{2}\mathbf{D}\right) ds. \end{aligned} \tag{4.158}$$

4.5.11 Empirical Results

What we are most interested in is the corrective term $\frac{\rho}{2}\int_0^T (\mathbf{C} + \frac{1}{2}\mathbf{D}) ds$, and the relative contribution it makes to value of the call option. Define $V_{\text{Orig}}(\alpha, \sigma_0, T)$ to be $\mathbf{A} + \mathrm{B}$; which is an approximation of $\mathbb{E}[BS(0, X_0, v_0)]$ and what we consider the "traditional" value of the call option. Let $V_{\text{Corr}}(\alpha, \sigma_0, T, \rho)$ to be $\frac{\rho}{2}\int_0^T (\mathbf{C} + \frac{1}{2}\mathbf{D}) ds$; which is an approximation of the (corrective or correlated) term $\frac{\rho}{2}\int_0^T \mathbb{E}[\Xi_s \Lambda_s] ds$ introduced by Alòs. Once we fix α, σ_0 and T, we can look at the statistic

$$S = \frac{V_{\text{Corr}}}{|V_{\text{Orig}}| + |V_{\text{Corr}}|}, \tag{4.159}$$

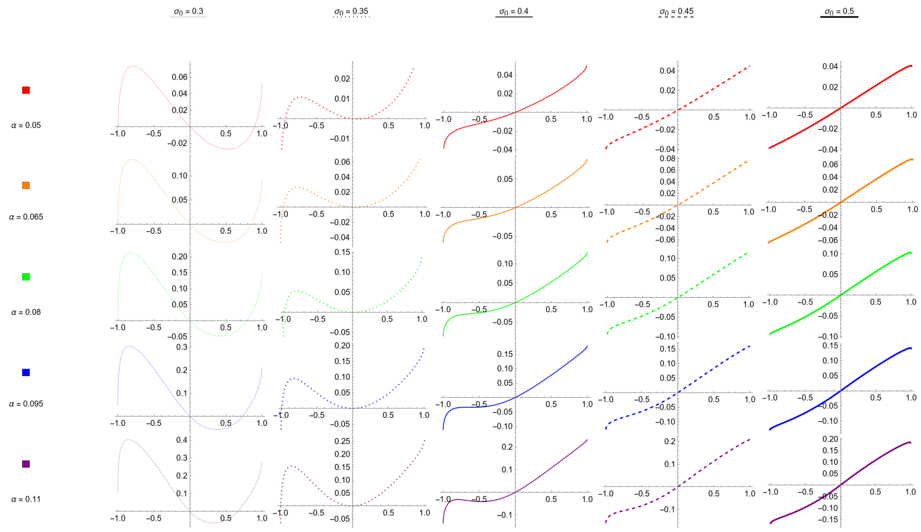

Figure 4.1: Plots of relative contribution of the corrective term S as a function of ρ for different values of α and σ_0. Time to maturity is fixed at $T = 10$.

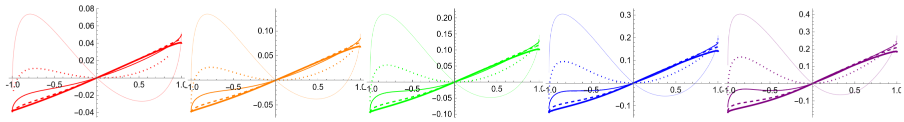

Figure 4.2: Simulations grouped by α as σ_0 varies.

to approximate the relative impact of Alòs's term to the overall price as ρ varies. There is no absolute sign in the numerator, so the score can reflect whether the correction increases or decreases the option price.

The longer the time to maturity, the more pronounced the corrective term. Hence, we chose a fixed time of $T = 10$ across all simulations in order to get a better intuition of our score S. Then, we ran twenty-five simulations for S. One for each $\alpha \in \{0.05, 0.065, 0.08, 0.095, 0.11\}$ and $\sigma_0 \in \{0.30, 0.35, 0.40, 0.45, 0.50\}$. These choices are fairly realistic values for α and σ_0. With every other parameter fixed, S is plotted as a function of ρ over the domain $[-1, 1]$. The results are displayed in Figure 4.1.

We can group these plots by α and overlay them in order to see the effect of σ_0 on the profile of the score.

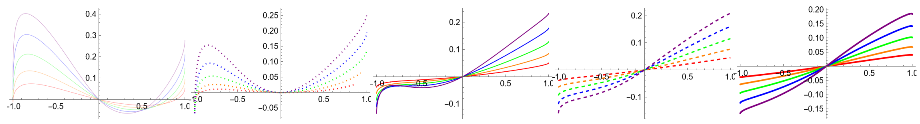

Figure 4.3: Simulations grouped by σ_0 as α varies.

Similarly, we can group these plots by σ_0 and overlay them in order to see the effect of α on the profile of the score.

And finally, it is a bit chaotic, but there is some intuition to gain by overlaying all plots.

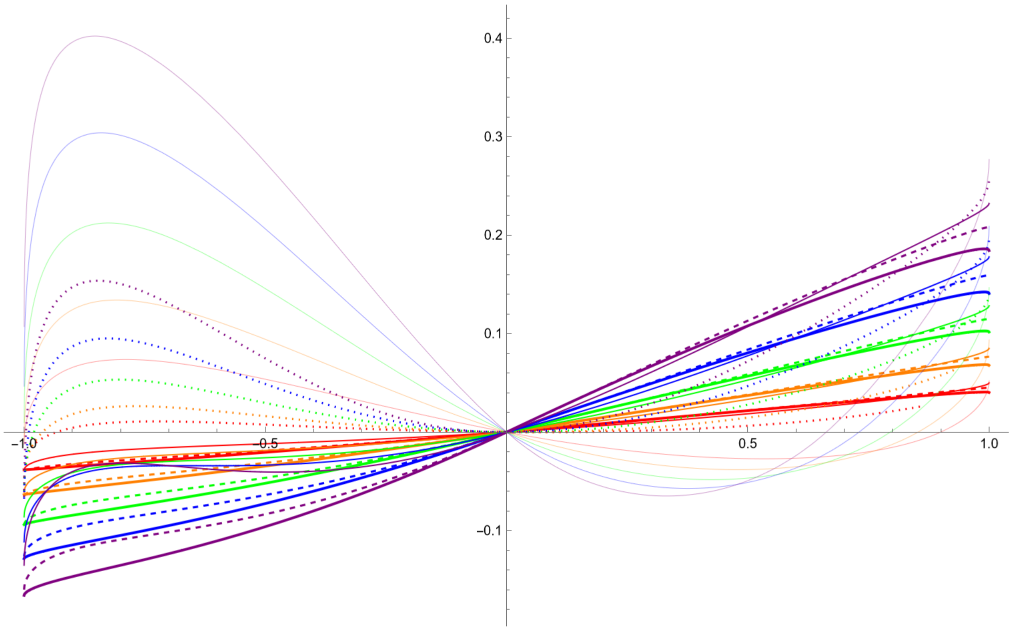

Figure 4.4: All simulation overlaid.

We see as σ_0 varies, there is a qualitative difference in the “smile” of S, while increasing α increases the weight of the contribution, but the shape of the curve remains intact. When all are overlaid, one can see curves grouped by σ_0 with relatively little difference seen as α is varied. This suggests the score is more sensitive to changes in σ_0 than in α. We can infer from Figure 4.4 that the score S is well-behaved statistics that variously smoothly with changes in α and σ_0, and should therefore be a good metric for measuring the contribution of Alòs’s corrective term on the overall price.

This has been a lot of work, but it is worth stopping a moment to appreciate what we have done. We have successfully simulated the values of a call option for a highly non-trivial stochastic model using only deterministic techniques. We do not need to generate any paths; our implementation never makes a single call to a random number generator. Only second-order integrals are used, so we have thus escaped the tyranny of higher-dimensions. It is somewhat modest to describe the calculation as a first-order approximation since we do have second-order Malliavin derivatives present. Our approach is also not ad hoc. The same technique can be used whenever there is a closed-form expression for the pricing formula and forward rate.

To make a comparison, we estimated the call option value via Monte Carlo simulation. For each simulation, we fixed parameters α and σ_0, choose an expiration time T, and set the time increment to $\Delta t = \frac{T}{100}$. We realized a hundred paths for the underlying processes σ_s and X_s, by using the built-in SDE solver in Mathematica [Inc]. From these, the paths for v_s, Λ_s and Xi_s were computed. When deterministic integration was required, we did a simple first-order Eulerian scheme. Simulations were run over the inputs:

$$\begin{aligned} T &\in \{1, 10\}, \\ \alpha &\in \{0.05, 0.065, 0.08, 0.095, 0.11\}, \\ \sigma_0 &\in \{0.30, 0.35, 0.40, 0.45, 0.50\} \cup \{0.1, 0.2, 0.3, 0.4, 0.5\}, \\ \rho &\in \{-0.5, -0.25, 0, 0.25, 0.5\}. \end{aligned} \tag{4.160}$$

The results were mixed. For some configuration of parameters, our frozen approximation and Monte Carlo approximations coincide. In general, frozen approximations with $T = 1$ and $\sigma_0 \geq 0.3$ were within ten percent of the Monte Carlo simulations. α and ρ were not great predictors of agreement. When the initial volatility was low ($\sigma_0 \leq 0.2$) and the maturity time large ($T = 10$), there was usually a divergence between the frozen path approximation and the Monte Carlo simulation. The closest approximations were when ρ was 0, which is promising. This may hint that errors may be more due to accumulated inaccuracies in our (rather simplistic) integration methods, and not in the exponential formula itself. When we consider classical term $\mathbb{E}[BS(0, 0, v_0)]$, the two methods never differed more than four percent, regardless of the other parameter values. For this term, the only integration is a single deterministic integration where the integrand has highly regular. For the Monte Carlo simulation of the term $\int_0^T \mathbb{E}[\Xi_s \Lambda_s] ds$, we have to numerically integrate paths which have Hölder regularity less than $\frac{1}{2}$, since they are driven by classical Brownian motion. The exact integral is ensured by Young, but it would not be surprising if some observed discrepancies were due to lower order of the truncation error of our first-order Eulerian scheme applied to low regularity paths.

4.6 The Road Forward

The hardest decision we faced when writing this dissertation was how wide should its scope be. There is plenty of more exposition to write, theory to extend, and applications to tackle. In terms of the literature review, we wish we could have done more presenting the theory of rough paths and its applications to integration for fractional Brownian motion less than $\frac{1}{2}$. Path integration is the natural home for the freezing operator. Delving more into that subject might be fruitful for extending the notion to more general processes.

Application-wise, it is lengthy calculation but the SABR model we chose was the simplest non-trivial instance. We chose an at-the-money call option with $\beta = 1$. Among the first advances we would wish to make would be to retry our application using numerical methods with higher-order truncation errors, as well as extending the result by dropping the at-the-money assumption.

Choosing a different β would be another challenge entirely. It might be worth-while to examine $\beta = \frac{1}{2}$ and see how apply our methods in a scenario where the underlying forward rate does not have a closed-form solution. Even our current implementation could be improved for $\beta = 1$. We used a computer algebra system to verify much of our SABR calculations. Now that we have our results in hand, there could be an advantage of reimplementing our code within some performance-focused numerical computing environment.

As for theory, we have hinted many times that we would like to prove an exponential formula for random variables generated by fractional Brownian motion with Hurst index less than $\frac{1}{2}$. d-dimensional fractional Brownian motion should follow that. However, we should be honest and remember we have just partially generalized old results. The original exponential formulae were for the conditional expectation of a random variable; not just the overall expectation. We are confident these results can be extended to conditional expectations, but that bookkeeping still needs to be performed. Finally, sheets are more interesting than motions. Though we have no idea if that mountain is presently scalable.

Appendix A

Hermite Polynomials

We use a Rodrigues formula as our definitional basis.

Definition A.1. *The n-th Hermite Polynomials H_n is defined for any natural n by*

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} \left(e^{-\frac{x^2}{2}} \right). \tag{A.1}$$

By convention, we define $H_{-1}(x) = 0$ in order to ensure certain relations which hold for positive n also hold when n is zero.

The first few Hermite polynomials are

$$\begin{aligned} H_0(x) =& 1 \\ H_1(x) =& x \\ H_2(x) =& \frac{1}{2}\left(x^2 - 1\right) \\ H_3(x) =& \frac{1}{6}\left(x^3 - 3x\right) \\ H_4(x) =& \frac{1}{24}\left(x^4 - 6x^2 + 3\right). \end{aligned} \tag{A.2}$$

An author has the choice of whether to include the $\frac{1}{n!}$ factor in the definition; most choose to omit. When [PT11] introduce the Hermite polynomials in Chapter 8, the authors provide a brief survey of the sides in this debate. [PT11], [NØP09], [KS91], and [Kyo75] all omit the factor. The books and papers of Nualart are noticeable holdouts. We side with Nualart; partially due to our contrarian nature and partially because of switching cost. Nualart's work was our entry point to this domain. To add to the confusion, there are also the physicists' Hermite polynomials (ours are referred to as the probabilists' Hermite polynomials) which is a completely different rescaling than what we are talking about.

Given our minority stance and the Hermite polynomials' ubiquitous presence throughout calculation and proof, we collect and prove various properties of the Hermite polynomials. What proceeds can be considered a translation of the results collected in Appendix C of [Hol+10] and Section 1.4 of [NP12] to our chosen rescaling. Our first translation is of the standard recurrence relation.

Proposition A.2.

$$(n+1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x), \tag{A.3}$$

holds for all natural n.

We proceed by direct calculation with Leibniz's product rule.

$$\begin{aligned}(n+1)H_{n+1}(x) =& -\frac{(-1)^n}{n!}e^{\frac{x^2}{2}}\frac{d^{n+1}}{d^{n+1}}e^{-\frac{x^2}{2}} = \frac{(-1)^n}{n!}e^{\frac{x^2}{2}}\frac{d^n}{d^n}\left(xe^{-\frac{x^2}{2}}\right)\\ =&\frac{(-1)^n}{n!}e^{\frac{x^2}{2}}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{d^k}{dx^k}x\right)\left(\frac{d^{n-k}}{dx^{n-k}}e^{-\frac{x^2}{2}}\right)\\ =&\frac{(-1)^n}{n!}e^{\frac{x^2}{2}}\left(x\frac{d^n}{dx^n}e^{-\frac{x^2}{2}} + n\frac{d^{n-1}}{dx^{n-1}}e^{-\frac{x^2}{2}}\right)\\ =&xH_n(x) - H_{n-1}(x).\end{aligned} \tag{A.4}$$

We also have the pleasant differential recurrence relation

Proposition A.3.

$$H_n'(x) = H_{n-1}'(x), \tag{A.5}$$

for all natural n.

Again, we proceed by direct calculation and Equation (A.3)

$$\begin{aligned} H_n'(x) =& \frac{(-1)^n}{n!}\left(xe^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}} + e^{\frac{x^2}{2}}\frac{d^{n+1}}{dx^{n+1}}e^{-\frac{x^2}{2}} \right) \\ =& xH_n(x) - (n+1)H_{n+1}(x) \\ =& H_{n-1}(x). \end{aligned} \tag{A.6}$$

The generating function of the Hermite polynomials is

Proposition A.4.

$$e^{tx-\frac{t^2}{2}} = \sum_{n=0}^{\infty} t^n H_n(x), \tag{A.7}$$

which may contain the origin of the dispute over the $\frac{1}{n!}$ factor. We begin by rewriting the original expression.

$$\begin{aligned} e^{tx-\frac{t^2}{2}} =& e^{\frac{x^2}{2}}e^{-\frac{(x-t)^2}{2}} \\ =& e^{\frac{x^2}{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^n}{dt^n}\right|_{t=0} e^{-\frac{(x-t)^2}{2}}. \end{aligned} \tag{A.8}$$

The result immediately follows if we prove the n-th derivative of $e^{-\frac{(x-t)^2}{2}}$ with respective to t evaluated at $t=0$ is equal to $(-1)^n\frac{d^n}{dx^n}e^{-\frac{x^2}{2}}$. We proceed by induction. The base case is trivial and the induction step is, once again, handled by Leibniz's product rule. We apply Leibniz's rule with respect differentiation by t and then with respect to differentiation by x.

$$\begin{aligned} \left.\frac{d^{n+1}}{dt^{n+1}}\right|_{t=0} e^{-\frac{(x-t)^2}{2}} =& \left.\frac{d^n}{dt^n}\right|_{t=0}(x-t)e^{-\frac{(x-t)^2}{2}} \\ =& \sum_{k=0}^{n}\binom{n}{k}\left.\frac{d^k}{dt^k}(x-t)\right|_{t=0}\left.\frac{d^{n-k}}{d^{n-k}t}e^{-\frac{(x-t)^2}{2}}\right|_{t=0} \\ =& x(-1)^n\frac{d^n}{dx^n}e^{-\frac{x^2}{2}} - (-1)^n\frac{d^{n-1}}{dx^{n-1}}e^{-\frac{x^2}{2}} \\ =& -(-1)^n\frac{d^n}{dx^n}\left(-xe^{-\frac{x^2}{2}}\right) \\ =& (-1)^{n+1}\frac{d^{n+1}}{dx^{n+1}}e^{-\frac{x^2}{2}}. \end{aligned} \tag{A.9}$$

The equality holds and plugging this result back into Equation (A.8) proves our desired result.

Our notation is from [MT05] We write $\partial f = f'$ and call ∂ the annihilation operator, and refer to its adjoint $(\partial^* f)(x) = -(\partial f)(x) + xf(x)$ as the creation operator. Finally, we let γ be the Gaussian probability measure on $\mathbb{R}$, so $\int f d\gamma = \int f(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$. We will prove some additional properties of the Hermite polynomials using only elementary calculus, but the context deserves explanation. As Chapter 1 of [NP12] makes clear, we will essentially be developing a one-dimensional Malliavin calculus in a deterministic setting. The Malliavin calculus is a variational calculus, and the Malliavin derivative is a Fréchet derivative and thus a generalization of ∂, while the Skorokhod integral/divergence operator is the generalization of ∂^*. We use ∂ and ∂^* to avoid overloading the operators D and δ more than we already do, and also to emphasize there is no stochastic or functional analytic witchery involved; just introductory calculus. And we write ∂_n and $(\partial_n^*$ for their iterates. Our following proofs also form a sketch of how to construct the Malliavin calculus on isonormal Gaussian processes. The idea is analogous to developing the Lebesgue measure on $\mathbb{R}$. Constructing the Lebesgue measure on $\mathbb{R}$ provides a simple and intuitive sketch of how to proceed when extending measure theory beyond $\mathbb{R}$. And in the same way simple functions are the basis of the "standard machine" in real analysis, the Hermite polynomials are the usual stepping stones when reasoning about isonormal Gaussian processes.

Let us first verify our claim that creation is the adjoint of annihilation. For two sufficiently smooth functions f, g in $L^2(\gamma)$ we have

$$\begin{aligned}\langle \partial f, g\rangle &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f'(x)g(x)e^{-\frac{x^2}{2}}dx = -\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\left(g(x)e^{-\frac{x^2}{2}}\right)' dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left(g'(x) - xg(x)\right)e^{-\frac{x^2}{2}}dx = \langle f, \partial^* g\rangle.\end{aligned} \tag{A.10}$$

Our notation is justified and of course

$$\langle \partial_n f, g\rangle = \langle f, \partial_n^* g\rangle, \tag{A.11}$$

follows. Following close behind is the Heisenberg commutativity principle

$$
\begin{aligned}
(\partial\partial^* - \partial^*\partial) =& \frac{d}{dx}(-f' + xf) - (-\frac{d}{dx} + x)f' \\
=& -f'' + f + xf' - (-f'' + xf') \\
=& f,
\end{aligned}
\tag{A.12}
$$

which can generalized by induction.

Proposition A.5.

$$
\partial\partial_n^* - \partial_n^*\partial = n\partial_{n-1}^* \tag{A.13}
$$

for all $n \geq 1$.

As we said, by induction

$$
\begin{aligned}
\partial\partial_{n+1}^* - \partial_{n+1}^*\partial =& (\partial\partial_n^*)\partial^* - \partial_{n+1}^*\partial = (n\partial_{n-1}^* + \partial_n^*\partial)\partial^* - \partial_{n+1}^*\partial \\
=& n\partial_n^* + \partial_n^*(\partial\partial^* - \partial^*\partial) \\
=& (n+1)\partial_n^*.
\end{aligned}
\tag{A.14}
$$

We can start to see how ∂^* is the proper "toy model" of the Skorokhod integral with the following calculation.

Proposition A.6.

$$
\partial_n^* 1 = n! H_n(x), \tag{A.15}
$$

for all natural n.

The base case is obvious and by induction

$$
\begin{aligned}
\partial_{n+1}^* 1 =& n!\partial^* H_n(x) = n!\left(xH_n(x) - H_{n-1}(x)\right) \\
=& (n+1)! H_{n+1}(x).
\end{aligned}
\tag{A.16}
$$

One reason we call Definition A.1 a Rodrigues formula is similar to how the Legendre polynomials are an orthonormal basis for $L^2([-1,1])$, the Hermite polynomials can be made into an orthonormal basis of $L^2(\gamma)$.

Proposition A.7. *$\{\sqrt{n!}H_n\}$ is an orthonormal basis in $L^2(\gamma)$.*

We simply calculate. Without loss of generality, assume $m \geq n$

$$\begin{aligned}\langle H_m(x), H_n(x)\rangle &= \langle \frac{1}{m!}\partial_m^* 1, H_n(x)\rangle \\ &= \frac{1}{m!}\langle 1, \partial_m H_n(x)\rangle.\end{aligned} \tag{A.17}$$

If m is greater than n then this is 0, and when they are equal then $\partial_n H_n(x) = 1$, so $\|H_n\|^2 = \frac{1}{n!}$. All polynomials are in the span of $\{\sqrt{n!}H_n\}$ and the polynomials are dense in $L^2(\gamma)$ and therefore $\{\sqrt{n!}H_n\}$ is also a basis. Consequently, we have the following representation formula.

Proposition A.8. *For any f in $L^2(\gamma)$*

$$f = \sum_{n=0}^{\infty} n! \left(\int f H_n d\gamma\right) H_n. \tag{A.18}$$

For certain well-behaved f, the above can be written as follows.

Proposition A.9. *For a smooth f whose derivatives are all in $L^2(\gamma)$, we have the following Stroock [Str87] decomposition*

$$f = \sum_{n=0}^{\infty} \left(\int \partial_n f d\gamma\right) H_n. \tag{A.19}$$

The proof is the calculation

$$n! \int f H_n d\gamma = \langle f, n! H_n\rangle = \langle f, \partial_n^* 1\rangle = \langle \partial_n f, 1\rangle = \int \partial_n f d\gamma. \tag{A.20}$$

As we've established, the Hermite polynomials are naturally generated by the derivative operator and its adjoint on $L^2(\gamma)$. This preceding web of relations forms a natural foundation for analysis on $L^2(\gamma)$. Consequently, they should be the appropriate building blocks for an analytical toolkit applicable to isonormal Gaussian processes.

For convenience, we define a related family of polynomials $H_n(x, \lambda)$ which we will call the two-parameter Hermite polynomials.

Definition A.10. *The two-parameter polynomials $H_n(x, \lambda)$ on $\mathbb{R} \times \mathbb{R}^+$ are defined as*

$$H_n(x, \lambda) = \lambda^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{\lambda}}\right), \tag{A.21}$$

for any natural n.

We will also make use of the following recurrence and differential relations on $H_n(x, \lambda)$:

Proposition A.11.

$$\begin{aligned} (n+1)H_{n+1}(x,\lambda) =& xH_n(x,\lambda) - \lambda H_{n-1}(x,\lambda) \\ \frac{\partial H_{n+1}}{\partial x}(x,\lambda) =& H_n(x,\lambda) \\ \frac{\partial H_{n+1}}{\partial \lambda}(x,\lambda) =& -\frac{1}{2}H_{n-1}(x,\lambda). \end{aligned} \tag{A.22}$$

Firstly

$$\begin{aligned} (n+1)H_{n+1}(x,\lambda) =& \lambda^{\frac{n+1}{2}}\left(\frac{x}{\sqrt{\lambda}}H_n(x,\lambda) - H_{n-1}(x,\lambda)\right) \\ =& x\left(\lambda^{\frac{n}{2}}H_n(x,\lambda)\right) - \lambda\left(\lambda^{\frac{n-1}{2}}H_{n-1}(x,\lambda)\right) \\ =& xH_n(x,\lambda) - \lambda H_{n-1}(x,\lambda), \end{aligned} \tag{A.23}$$

secondly

$$\frac{\partial}{\partial x}H_n(x,\lambda) = \lambda^{\frac{n}{2}}H_{n-1}(x,\lambda)\frac{1}{\sqrt{\lambda}} = H_{n-1}(x,\lambda), \tag{A.24}$$

and lastly

$$\begin{aligned} \frac{\partial}{\partial \lambda}H_{n+1}(x,\lambda) =& \frac{n+1}{2}\lambda^{\frac{n-1}{2}}H_{n+1}\left(\frac{x}{\sqrt{\lambda}}\right) + \lambda^{\frac{n+1}{2}}H_n\left(\frac{x}{\sqrt{\lambda}}\right)\left(-\frac{x}{2\sqrt{\lambda}^3}\right) \\ =& \frac{1}{2}\lambda^{\frac{n-1}{2}}\left((n+1)H_{n+1}\left(\frac{x}{\sqrt{\lambda}}\right) - \frac{x}{\sqrt{\lambda}}H_n\left(\frac{x}{\sqrt{\lambda}}\right)\right) \\ =& \frac{1}{2}\lambda^{\frac{n-1}{2}}\left(-H\left(\frac{x}{\sqrt{\lambda}}\right)\right) \\ =& -\frac{1}{2}H_{n-1}(x,\lambda). \end{aligned} \tag{A.25}$$

An immediate consequence is the two-parameter Hermite polynomials satisfy the following series of SDEs

$$\begin{aligned} dH_n(t, W_t) =& H_{n-1}(t, W_t) dW_t \\ H_0 =& 1. \end{aligned} \tag{A.26}$$

Appendix B

An Orthonormal System for the n-th Wiener Chaos

We wish to show $\{\boldsymbol{H_a}; |\boldsymbol{a}| = n\}$ is a complete orthonormal system of the n-th Wiener chaos $\mathcal{H}_n$. Ultimately, this requires us to compute the inner product of some $H_n(X(\phi))$ and $\boldsymbol{H_a}$ where $\|\phi\| = 1$ and $|\boldsymbol{a}| = n$.

To calculate, we investigate a familiar-looking expression:

$$\mathbb{E}\left[e^{tZ-\frac{t^2}{2}}e^{s_1Y_1-\frac{s_1^2}{2}}\cdots e^{s_mY_m-\frac{s_m^2}{2}}\right], \tag{B.1}$$

where Z is a standard normal random variable and $\boldsymbol{Y} = (Y_1, \ldots, Y_m)$ is a standard normal random vector. If we evaluate this value by expansion via the generating function $e^{cx-\frac{c^2}{2}} = \sum_k c^k H_k(x)$ then Equation (B.1) becomes

$$\sum_{j=0}^{\infty}\sum_{\boldsymbol{k}\in\mathbb{N}^m} t^j \boldsymbol{s}^{\boldsymbol{k}}\mathbb{E}\left[H_j(Z)H_{\boldsymbol{k}}(\boldsymbol{Y})\right], \tag{B.2}$$

where $\boldsymbol{s}^{\boldsymbol{k}} = s_1^{k_1}\cdots s_m^{k_m}$ and $H_{\boldsymbol{k}}(\boldsymbol{Y}) = H_{k_1}(Y_i)\cdots H_{k_m}(Y_k)$.

There is another other way to calculate Equation (B.1). Recognize $(Z, Y_1, \ldots, Y_m)$ has a multivariate Gaussian distribution with mean zero and covariance matrix

$$\Sigma = \text{Var}\begin{bmatrix} Z \\ Y_1 \\ \vdots \\ Y_m \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{\rho}^T \\ \boldsymbol{\rho} & I_{m\times m} \end{bmatrix}, \tag{B.3}$$

where $\boldsymbol{\rho} = (\rho_1, \dots, \rho_m) = (\mathbb{E}[ZY_1], \dots, \mathbb{E}[ZY_m])$ and I_m is the m-by-m identity matrix. Then $tZ + s_1Y_1 + \cdots + s_mY_m = tZ + \boldsymbol{s}Y$ is another normal random variable with mean zero, and variance equal to

$$\begin{bmatrix} t & \boldsymbol{s}^T \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{\rho}^T \\ \boldsymbol{\rho} & I_m \end{bmatrix} \begin{bmatrix} t \\ \boldsymbol{s} \end{bmatrix} = \begin{bmatrix} t & \boldsymbol{s}^T \end{bmatrix} \begin{bmatrix} t + \boldsymbol{\rho}\cdot\boldsymbol{s} \\ t\boldsymbol{\rho} + \boldsymbol{s} \end{bmatrix} = t^2 + 2t\boldsymbol{\rho}\cdot\boldsymbol{s} + \|s\|^2. \tag{B.4}$$

Then reducing Equation (B.1) we find

$$\begin{aligned} \mathbb{E}\left[e^{tZ+\boldsymbol{s}\cdot\boldsymbol{Y}-\frac{t^2-\|s\|^2}{2}}\right] =& e^{t\boldsymbol{\rho}\cdot\boldsymbol{s}} \\ =& e^{ts_1\rho_1+\cdots+ts_m\rho_m} \\ =& \sum_{\boldsymbol{\ell}\in\mathbb{N}^m} \frac{1}{\boldsymbol{\ell}!} t^{|\boldsymbol{\ell}|} \boldsymbol{s}^{\boldsymbol{\ell}} \boldsymbol{\rho}^{\boldsymbol{\ell}}. \end{aligned} \tag{B.5}$$

We equate this expression with Equation (B.2). They must be equal when considered as analytical expressions in the variables $t, s_1, \dots, s_m$. Therefore, $\mathbb{E}[H_j(Z)H_{\boldsymbol{k}}(\boldsymbol{Y})]$ is zero if j and $|\boldsymbol{k}|$ differ, and when they coincide we see

$$\mathbb{E}\left[H_{|k|}(Z)H_{\boldsymbol{k}}(\boldsymbol{Y})\right] = \frac{1}{\boldsymbol{k}!}\boldsymbol{\rho}^{\boldsymbol{k}} = \prod_{i=1}^{m} \frac{1}{k_i!}\mathbb{E}[ZY_i]^{k_1}. \tag{B.6}$$

Consequently, the n-th Wiener chaos is orthogonal to any family $\{\boldsymbol{H_a}; |\boldsymbol{a}| = m\}$ whenever m differs with n. Now suppose a random variable $H_n(X(\phi))$, where $\|\phi\| = 1$, is orthogonal to each element in $\{\boldsymbol{H_a}; |\boldsymbol{a}| = n\}$. Specifically, $H_n(X(\phi))$ is orthogonal to each $H_n(X(e_i))$. This would imply each $\mathbb{E}[X(\phi)X(e_i)]$, and therefore each $\langle \phi, e_i\rangle_{\mathfrak{H}}$, is zero. The only such ϕ would be zero, but we assumed $\|\phi\| = 1$. From this we can conclude $\{\boldsymbol{H}_{\boldsymbol{a}}; |\boldsymbol{a}| = n\}$ is a complete orthonormal system for the n-th Wiener chaos.

Appendix C

An Itô's Lemma for fBm with $H < \frac{1}{2}$

[CN05] develops an Itô's formula for any Hurst index H in $(0,1)$. However, their result is only for processes of the form $f(B_t)$ (here, we drop the superscript and consider B an fBm with $H < \frac{1}{2}$). We follow the path in section 4 of their paper to construct an Itô's formula for smooth functions. While our result is not terribly novel, it is more useful and provides the flavor of proofs involving the extended divergence operator. To properly extend [CN05] we would also need to prove a Tanaka local-time formula and then use those to prove an Itô formula for C^2 functions.

Lemma C.1. *Let $f \in C^\infty([0,\infty] \times \mathbb{R})$ satisfy the growth conditions in Definition 4.1 of [CN05], and $0 \le a \le b \le T$. Then*

$$\frac{\partial f}{\partial x}(t, B_t) 1_{(a,b]}(t) \in \operatorname{Dom}^* \delta, \tag{C.1}$$

and

$$\begin{aligned} &f(B_b) - f(B_a) \\ &= \int_a^b \frac{\partial f}{\partial t}(t, B_t) dt + \int_a^b \frac{\partial f}{\partial x}(t, B_t) \delta B_t + \frac{1}{2} \int_a^b \frac{\partial^2 f}{\partial x^2}(t, B_t) d|t|^{2H}. \end{aligned} \tag{C.2}$$

To prove this, we need to show the following equation holds.

$$\begin{aligned} &c_H^2 \int_a^b \mathbb{E}\left[\frac{\partial f}{\partial x}(t, B_t) H_{n-1}(B(\phi))\right] (\mathcal{D}_+^\alpha \mathcal{D}_-^\alpha \phi)(t) dt \\ =&\mathbb{E}\left[\left(f(b, B_b) - f(B_a, a) - \int_a^b \frac{\partial f}{\partial t}(t, B_t)\delta dt \right.\right. \\ &\left.\left. -\frac{1}{2}\int_a^b \frac{\partial^2 f}{\partial x^2}(t, B_t) d|t|^{2H}\right) H_n(B(\phi))\right], \end{aligned} \tag{C.3}$$

for all natural n and $\phi \in \Lambda^{H,*}$ with unit length. The hardest part of the argument is remembering that, occasionally, one can actually calculate the expectation of a random variable by integrating the PDF. The growth conditions are what allows us to claim both

$$\frac{\partial f}{\partial x}(t, B_t) 1_{(a,b]}(t), \tag{C.4}$$

and

$$f(B_b) - f(B_a) - \int_a^b \frac{\partial f}{\partial t}(t, B_t) dt - \frac{1}{2}\int_a^b \frac{\partial^2 f}{\partial x^2}(t, B_t) d|t|^{2H}, \tag{C.5}$$

are in the requisite L^2 spaces.

Let $p(\sigma, x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}$ be the usual PDF of a centered Gaussian distribution with variance σ^2. Recall, p satisfies the heat equation $\frac{\partial p}{\partial \sigma} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}$. Then for any natural ℓ, m and $0 < t \leq b$ we see

$$\begin{aligned} &\frac{d}{dt}\mathbb{E}\left[\frac{\partial^{\ell+m} f}{\partial t^\ell \partial x^m}(t, B_t)\right] \\ =&\frac{d}{dt}\int_{\mathbb{R}} p(t^{2H}, x) \frac{\partial^{\ell+m} f}{\partial t^\ell \partial x^m}(t, x) dx \\ =&p(t^{2H}, x) \frac{\partial^{\ell+m+1} f}{\partial t^{\ell+1} \partial x^m}(t, x) dx + 2H \int_{\mathbb{R}} t^{2H-1} \frac{\partial p}{\partial \sigma}(t^{2H}, x) \frac{\partial^{\ell+m} f}{\partial t^\ell \partial x^m}(t, x) dx \\ =&\mathbb{E}\left[\frac{\partial^{\ell+m+1} f}{\partial t^{\ell+1} \partial x^m}(t, B_t)\right] + H t^{2H-1} \int_{\mathbb{R}} \frac{\partial^2 p}{\partial x^2}(t^{2H}, x) \frac{\partial^{\ell+m} f}{\partial t^\ell \partial x^m}(t, x) dx \\ =&\mathbb{E}\left[\frac{\partial^{\ell+m+1} f}{\partial t^{\ell+1} \partial x^m}(t, B_t)\right] + H t^{2H-1} \int_{\mathbb{R}} p(t^{2H}, x) \frac{\partial^{\ell+m+2} f}{\partial t^\ell \partial x^{m+2}}(t, x) dx \\ =&\mathbb{E}\left[\frac{\partial^{\ell+m+1} f}{\partial t^{\ell+1} \partial x^m}(t, B_t)\right] + \mathbb{E}\left[\frac{\partial^{\ell+m+2} f}{\partial t^\ell \partial x^{m+2}}(t, B_t)\right]. \end{aligned} \tag{C.6}$$

Setting $\ell = m = 0$ and integrating both sides from 0 to b, we see the following expression is identically zero

$$\mathbb{E}\left[f(b,B_b)\right]-f(0,0)-\int_0^b \mathbb{E}\left[\frac{\partial f}{\partial t}(t,B_t)\right]dt-\frac{1}{2}\int_0^b \mathbb{E}\left[\frac{\partial^2 f}{\partial x^2}(t,B_t)\right]d|t|^{2H}. \tag{C.7}$$

From this, we can conclude when $n=0$ the right-hand side of Equation (C.3) is identically zero. By convention $H_{-1}(x)$ is identically zero, so the left-hand side of Equation (C.3) is also zero. So we have proven the lemma holds for $n=0$.

We continue by assuming Equation (C.3) holds for all k bounded by some n, and now consider the case $k=n+1$. By the construction of $\Lambda^{H,*}$, and the integration by parts formula for the Marchaud derivatives, we have

$$\begin{aligned}\langle 1_{(0,t]},\phi\rangle_{\Lambda^H} =& c_H^2\langle \mathcal{D}_-^{\alpha}1_{(0,t]},\phi\mathcal{D}_-^{\alpha}1_{(0,t]},\rangle_{L^2(\mathbb{R})}\\ =&\int_{\mathbb{R}}\left(\mathcal{D}_-^{\alpha}1_{(0,t]}\right)(s)\left(\mathcal{D}_-^{\alpha}\phi\right)(s)ds\\ =&\int_{\mathbb{R}}1_{(0,t]}(s)\left(\mathcal{D}_+^{\alpha}\mathcal{D}_-^{\alpha}\phi\right)(s)ds\\ =&\int_0^t\left(\mathcal{D}_+^{\alpha}\mathcal{D}_-^{\alpha}\phi\right)(s)ds.\end{aligned} \tag{C.8}$$

Then

$$\begin{aligned}&\frac{d}{dt}\left(\mathbb{E}\left[\frac{\partial^{n+1}f}{\partial x^{n+1}}(t,B_t)\right]\langle 1_{(0,t]},\phi\rangle_{\Lambda^H}^{n+1}\right)\\ =&\mathbb{E}\left[\frac{\partial^{n+2}f}{\partial t\partial x^{n+1}}(t,B_t)\right]\langle 1_{(0,t]},\phi\rangle_{\Lambda^H}^{n+1}+Ht^{2H-1}\mathbb{E}\left[\frac{\partial^{n+3}f}{\partial x^{n+3}}(t,B_t)\right]\langle 1_{(0,t]},\phi\rangle_{\Lambda^H}^{n+1}\\ &+c_H^2(n+1)\mathbb{E}\left[\frac{\partial^{n+1}f}{\partial x^{n+1}}(t,B_t)\right]\langle 1_{(0,t]},\phi\rangle_{\Lambda^H}^{n}\left(\mathcal{D}_+^{\alpha}\mathcal{D}_-^{\alpha}\phi\right)(t).\end{aligned} \tag{C.9}$$

We do our usual integration from 0 to b while noting $\langle 1_{(0,0]}\phi\rangle_{\Lambda^H}$ is zero

$$
\begin{aligned}
&\mathbb{E}\left[\frac{\partial^{n+1} f}{\partial x^{n+1}}(b, B_b)\right] \langle 1_{(0,b]}, \phi\rangle_{\Lambda^H}^{n+1} \\
&= \int_0^b \mathbb{E}\left[\frac{\partial^{n+2} f}{\partial t \partial x^{n+1}}(t, B_t)\right] \langle 1_{(0,t]}, \phi\rangle_{\Lambda^H}^{n+1} dt \\
&\quad + \frac{1}{2}\int_0^b \mathbb{E}\left[\frac{\partial^{n+3} f}{\partial x^{n+3}}(t, B_t)\right] \langle 1_{(0,t]}, \phi\rangle_{\Lambda^H}^{n+1} d|t|^{2H} \\
&\quad + c_H^2(n+1)\int_0^b \mathbb{E}\left[\frac{\partial^{n+1} f}{\partial x^{n+1}}(t, B_t)\right] \langle 1_{(0,t]}, \phi\rangle_{\Lambda^H}^{n} \left(\mathcal{D}_+^\alpha \mathcal{D}_-^\alpha \phi\right)(t) dt.
\end{aligned} \tag{C.10}
$$

Now $H_n(B(\phi))\phi$ is in $\operatorname{Dom}\delta$ and

$$\delta\left[H_n(B(\phi))\phi\right] = (n+1)H_{n+1}(B(\phi)). \tag{C.11}$$

The quickest way to note this is via the Clark-Ocone-Haussmann formula for fBm with $H < \frac{1}{2}$ [LN06] but can be proved directly via integration by parts and the recurrence formula for Hermite polynomials.

$$
\begin{aligned}
\delta\left[H_n(B(\phi))\phi(\cdot)\right] =& H_n(B(\phi))B(\phi) - \langle DH_n(B(\phi)), \phi(\cdot)\rangle_{\Lambda^H} \\
=& H_n(B(\phi))B(\phi) - \langle H_{n-1}(B(\phi))\phi(\cdot), \phi(\cdot)\rangle_{\Lambda^H} \\
=& H_n(B(\phi))B(\phi) - \langle H_{n-1}(B(\phi)) \\
=& (n+1)H_{n+1}(B(\phi)),
\end{aligned} \tag{C.12}
$$

since $\|\phi\|^2_{\Lambda^H} = 1$. Now to exploit the smoothness of f

$$\begin{aligned} & k!\mathbb{E}\left[\frac{\partial^{\ell+m} f}{\partial t^\ell \partial x^m}(t, B_t) H_k(B(\phi))\right] \\ =& \mathbb{E}\left[\frac{\partial^{\ell+m} f}{\partial t^\ell \partial x^m}(t, B_t)\delta^p(\phi^{\otimes k})\right] \\ =& \mathbb{E}\left[\left\langle D^p\left(\frac{\partial^{\ell+m} f}{\partial t^\ell \partial x^m}(t, B_t)\right), \phi^{\otimes k}\right\rangle_{(\Lambda^H)^{\otimes k}}\right] \\ =& \mathbb{E}\left[\left\langle \frac{\partial^{\ell+m+k} f}{\partial t^\ell \partial x^{m+k}}(t, B_t) 1^{\otimes k}_{(0,t]}, \phi^{\otimes k}\right\rangle_{(\Lambda^H)^{\otimes k}}\right] \\ =& \mathbb{E}\left[\frac{\partial^{\ell+m+k} f}{\partial t^\ell \partial x^{m+k}}(t, B_t)\right] \langle 1_{(0,t]}, \phi\rangle^k_{\Lambda^H}. \end{aligned} \tag{C.13}$$

If we substitute the above result to each term in Equation (C.10) then we obtain

$$\begin{aligned} & (n+1)!\mathbb{E}\left[f(b, B_b) H_{n+1} B(\phi)\right] \\ =& (n+1)! \int_0^b \mathbb{E}\left[\frac{\partial f}{\partial t}(t, B_t) H_{n+1} B(\phi)\right] dt \\ &+ \frac{1}{2}(n+1)! \int_0^b \mathbb{E}\left[\frac{\partial^2 f}{\partial x^2}(t, B_t) H_{n+1} B(\phi)\right] d|t|^{2H} \\ &+ c_H^2 (n+1) n! \int_0^b \mathbb{E}\left[\frac{\partial f}{\partial x}(t, B_t) H_n B(\phi)\right] \left(\mathcal{D}^\alpha_+ \mathcal{D}^\alpha_- \phi\right)(t) dt. \end{aligned} \tag{C.14}$$

We have a similar identity once we integrate from 0 to a. Then we cancel $n!$ from both sides, rearrange terms, and take the difference of the result with a as the endpoint from the result with b as the endpoint to come to our conclusion.

$$\begin{aligned} & c_H^2 \int_a^b \mathbb{E}\left[\frac{\partial f}{\partial x}(t, B_t) H_n(B(\phi))\right] (\mathcal{D}^\alpha_+ \mathcal{D}^\alpha_- \phi)(t) dt \\ =& \mathbb{E}\Bigg[\Bigg(f(b, B_b) - f(B_a, a) - \int_a^b \frac{\partial f}{\partial t}(t, B_t)\delta dt \\ & -\frac{1}{2}\int_a^b \frac{\partial^2 f}{\partial x^2}(t, B_t) d|t|^{2H}\Bigg) H_{n+1}(B(\phi))\Bigg], \end{aligned} \tag{C.15}$$

which is precisely what we need to conclude the induction step, and therefore prove for the lemma.

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