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INFINITELY MANY RADIAL SOLUTIONS FOR A SUB-SUPER CRITICAL DIRICHLET BOUNDARY VALUE PROBLEM IN A BALL

ALFONSO CASTRO, JOHN KWON, CHEE MENG TAN

ABSTRACT. We prove the existence of infinitely many solutions to a semilinear Dirichlet boundary value problem in a ball for a nonlinearity $g(u)$ that grows subcritically for u positive and supercritically for u negative.

1. INTRODUCTION

In this paper we consider the *sub-super critical* boundary-value problem

$$\begin{aligned}\Delta u + g(u(x)) &= 0, & x \in \mathbb{R}^N, \|x\| \leq 1 \\ u(x) &= 0 & \text{for } \|x\| = 1,\end{aligned}\tag{1.1}$$

where

$$g(u) = \begin{cases} u^p, & u \geq 0 \\ |u|^{q-1}u, & u < 0, \end{cases}\tag{1.2}$$

with

$$1 < p < \frac{N+2}{N-2} < q < \infty,\tag{1.3}$$

that is, g has subcritical growth for $u > 0$ and supercritical growth for $u < 0$. Our results hold for more general nonlinearities. For example, it is easy to see that (1.2) may be replaced by $\lim_{u \rightarrow +\infty} g(r, u)/u^p \in (0, \infty)$ and $\lim_{u \rightarrow -\infty} g(r, u)/(|u|^{q-1}u) \in (0, \infty)$, uniformly for $r \in [0, 1]$.

Our main result is as follows.

Theorem 1.1. *Problem (1.1) has infinitely many radial solutions.*

This theorem extends the results of [4] where it was established that if $1 < p < (N+1)/(N-1)$ and $q > 1$, or $p, q \in (1, (N+2)/(N-2))$, or $p \in (1, (N+2)/(N-2))$ and $q = (N+2)/(N-2)$, then (1.1) has infinitely many radial solutions. This result is optimal in the sense that if $p, q \in [(N+2)/(N-2), \infty)$ then $u = 0$ is the only solution to (1.1) (see [12]). For related results for quasilinear equations the reader is referred to [8] and [10]. Studies on positive solutions for sub-super critical problems may be found in [9]. For other studies on the critical case, $p = q = (N+2)/(N-2)$

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and $\lim_{|u| \rightarrow \infty} (u|u|^{p-1}) \in \mathbb{R}$, see [1, 3, 4, 6, 7]. In [2] the reader can find a complete classification of the radial solutions to (1.1) for $1 < p = q < (N+2)/(N-2)$. For a recent survey of radial solutions for elliptic boundary-value problems that includes the case where the Laplacian operator is replaced by the more general k -Hessian operator, see [11].

Radial solutions to (1.1) are the solutions to the singular ordinary differential equation

$$\begin{aligned} u'' + \frac{n}{t}u' + g(u(t)) &= 0 \\ u'(0) = u(1) &= 0, \end{aligned} \quad (1.4)$$

where, and henceforth, $n = N - 1$.

For $d > 0$ let $u(t, d)$ be the solution to the initial-value problem

$$\begin{aligned} u'' + \frac{n}{t}u' + g(u(t)) &= 0 \\ u(0) = d, \quad u'(0) &= 0. \end{aligned} \quad (1.5)$$

We define the *energy* function

$$E(t, d) \equiv \frac{(u'(t, d))^2}{2} + G(u(t, d)), \quad (1.6)$$

where $G(u) = \int_0^u g(s)ds$. For future reference we note that

$$\frac{dE}{dt}(t) = -\frac{n}{t}(u'(t))^2 \leq 0. \quad (1.7)$$

The proof of Theorem 1.1 is based on the properties of the energy and the *argument* function defined below (see (1.9)).

Theorem 1.2. *There exists $D > 0$ such that if $d \geq D$, then*

$$t^{N-1} \left(tE(t) + \frac{N-2}{2}u(t)u'(t) \right) \geq cd^\xi \quad \text{for all } t \geq \sqrt{Nd}^{(1-p)/2}, \quad (1.8)$$

where $\xi = \frac{N+2-p(N-2)}{2}$. Also $u(t) \geq d/2$ for $t \in [0, \sqrt{Nd}^{(1-p)/2}]$.

As a consequence of Theorem 1.2 we see that, for $d \geq D$, $\rho(t, d) \equiv u^2(t, d) + (u'(t, d))^2 > 0$ for all $t \in [0, 1]$. Hence there exists a continuous function $\theta : [0, 1] \times [D, \infty) \rightarrow \mathbb{R}$ such that

$$u(t, d) = \rho(t, d) \cos(\theta(t, d)) \quad \text{and} \quad u'(t, d) = -\rho(t, d) \sin(\theta(t, d)). \quad (1.9)$$

In section 7 we prove that

$$\lim_{d \rightarrow \infty} \theta(1, d) = +\infty, \quad (1.10)$$

see (7.7) below.

2. FIRST ZERO

Let $d > 0$ and $t_0 > 0$ be such that $u(t_0) = d/2$, and $u(t) > d/2$ for $t \in (0, t_0)$. Following the arguments in [4], based on

$$-u'(t) = t^{-n} \int_0^t s^n g(u(s)) ds, \quad (2.1)$$

it is easily seen that

$$\sqrt{Nd}^{(1-p)/2} \leq t_0 \leq \sqrt{2^p Nd}^{(1-p)/2}. \quad (2.2)$$

Multiplying (1.5) by $r^{N-1}u$ and integrating on $[s, t]$, then multiplying the same equation by $r^N u'$ and integrating also on $[s, t]$ one has the following identity, known as Pohozaev's identity,

$$t^n H(t) = s^n H(s) + \int_s^t r^n \left(NG(u(r)) - \frac{N-2}{2} u(r)g(u(r)) \right) dr, \quad (2.3)$$

where $H(x) \equiv xE(x) + \frac{N-2}{2} u'(x)u(x)$. In particular, taking $s = 0$ and $t = t_0$ we have

$$t_0^n H(t_0) \geq \frac{t_0^N \gamma d^{p+1}}{2^{p+1} N} \geq \frac{N^{(N-2)/2} \gamma}{2^{p+1}} d^\xi \equiv c_1 d^\xi, \quad (2.4)$$

where $\gamma = N/(p+1) - (N-2)/2$ and ξ is as in (1.8). Also, from (2.3), if $u(r) \geq 0$ for all $r \in [0, t]$ we have

$$t^N (u'(t))^2 = -(N-2)u \cdot t^n u' - 2t^N \frac{u^{p+1}}{p+1} + 2 \int_0^t \gamma r^n u^{p+1} dr. \quad (2.5)$$

Thus from (2.5) and the fact that $t^{-n} \int_0^t s^n u^{p+1} ds \geq -u(t)u'(t)$ we have

$$\begin{aligned} \left(\frac{-tu'}{u} \right)' &= \frac{(-tu'' - u')u + t(u')^2}{u^2} \\ &= \frac{-t(-\frac{n}{t}u' - u^p)u - uu' + t(u')^2}{u^2} \\ &= \frac{\overbrace{(n-1)}^{N-2} uu' + tu^{p+1} + t(u')^2}{u^2} \\ &= \frac{2t^{-n} \int_0^t s^n \gamma u^{p+1} ds - 2\frac{tu^{p+1}}{p+1} + tu^{p+1}}{u^2} \\ &= \frac{2t^{-n} \int_0^t s^n \gamma u^{p+1} ds + t\left(\frac{p-1}{p+1}\right)u^{p+1}}{u^2} \\ &\geq \frac{2\gamma}{t} \left(\frac{-tu'(t)}{u(t)} \right), \end{aligned} \quad (2.6)$$

provided $u(s) > 0$ for $s \in (0, t)$. Integrating (2.6) on $[t_0, t]$ we have

$$\ln \left(\frac{-tu'(t)/u(t)}{-t_0 u'(t_0)/u(t_0)} \right) \geq \ln \left(\frac{t}{t_0} \right)^\gamma.$$

Letting $\Gamma = -t_0 u'(t_0)/u(t_0)$ we conclude that

$$\frac{-tu'(t)}{u(t)} \geq \Gamma \left(\frac{t}{t_0} \right)^\gamma.$$

For future reference we note that

$$\Gamma \geq 2^{1-p}, \quad (2.7)$$

where we have used (2.2), and $-u'(t_0) \geq t_0 d^p / (2^p N)$ (see (2.1)). Integrating again in $[t_0, t]$ yields

$$\ln \left(\frac{u(t_0)}{u(t)} \right) \geq \frac{\Gamma}{\gamma t_0^\gamma} [t^\gamma - t_0^\gamma].$$

Assuming that $u(t) \geq 0$ for all $t \in [t_0, t_0 \ln^{1/\gamma}(d) \equiv T]$ we have

$$u(T) \leq u(t_0) (ed^{-1})^{\Gamma/\gamma} = \frac{e^{\Gamma/\gamma}}{2} d^{1-\Gamma/\gamma}.$$

Now we estimate E for $t \geq t_0$ with $u(s) \geq 0$ for $s \in (t_0, t]$. Since $E'(t) \geq -2nE(t)/t$,

$$E(t) \geq E(s)(s/t)^{2n} \quad \text{for any } 0 \leq s \leq t \leq 1. \quad (2.8)$$

Thus

$$\begin{aligned} \frac{(u'(T))^2}{2} &\geq E(t_0)\left(\frac{t_0}{T}\right)^{2n} - \frac{u^{p+1}(T)}{p+1} \\ &\geq \frac{d^{p+1}}{(p+1)2^{p+1}\ln^{2n/\gamma}(d)} - \frac{1}{p+1}\left(\frac{e^{\Gamma/\gamma}}{2}d^{1-\Gamma/\gamma}\right)^{p+1} \\ &\geq \frac{d^{p+1}}{(p+1)2^{p+2}\ln^{2n/\gamma}(d)}, \end{aligned} \quad (2.9)$$

for d sufficiently large.

Let us suppose now that $u(t) > 0$, for any $t \in [T, 2T]$. Arguing as in (2.9) we have

$$\begin{aligned} \frac{(u'(t))^2}{2} &\geq E(T)\left(\frac{T}{t}\right)^{2n} - \frac{u^{p+1}(T)}{p+1} \\ &\geq \frac{d^{p+1}}{(p+1)2^{p+1+2n}\ln^{2n/\gamma}(d)} - \frac{1}{p+1}\left(\frac{e^{\Gamma/\gamma}}{2}d^{1-\Gamma/\gamma}\right)^{p+1} \\ &\geq \frac{d^{p+1}}{(p+1)2^{p+2+2n}\ln^{2n/\gamma}(d)}, \end{aligned} \quad (2.10)$$

for d large. Integrating on $[T, t]$ we have

$$\begin{aligned} 0 \leq u(t) &= u(T) + \int_T^t u'(s)ds \\ &\leq \frac{e^{\Gamma/\gamma}}{2}d^{1-\Gamma/\gamma} - (t-T)\frac{\sqrt{2}d^{(p+1)/2}}{2^{1+n+(p/2)}\ln^{n/\gamma}(d)\sqrt{p+1}}. \end{aligned} \quad (2.11)$$

Hence u has a zero in $[d^{(1-p)/2}, T + e^{\Gamma/\gamma}d^{(1-p)/2-\Gamma/\gamma}2^{n+(p/2)}\ln^{n/\gamma}(d)\sqrt{p+1}]$. We summarize the above in the following lemma.

Lemma 2.1. *For $d > 0$ sufficiently large, there exists*

$$t_1 \in (\sqrt{N}d^{(1-p)/2}, 2\sqrt{N}d^{(1-p)/2}\ln^{1/\gamma}(d)) \quad (2.12)$$

such that $u(t_1) = 0$, $u(s) > 0$ for $s \in [0, t_1)$, and

$$\frac{d^{p+1}}{(p+1)2^{p+2+2n}\ln^{2n}(d)} \leq E(t_1) \leq \frac{d^{p+1}}{p+1} \quad (2.13)$$

3. FIRST LOCAL MINIMUM

Let $t \in (t_1, t_1 + (1/2)(2/(q+1))^{q/(q+1)}|u'(t_1)|^{(1-q)/(1+q)} \equiv t_1 + \tau)$. From (2.12) and (2.13),

$$\begin{aligned} \frac{t_1}{t} &\geq 1 - \frac{\tau}{t_1 + \tau} \geq 1 - \frac{\tau}{t_1} \\ &\geq 1 - \frac{(1/2)(2/(q+1))^{q/(q+1)}|u'(t_1)|^{(1-q)/(1+q)}}{\sqrt{Nd}^{(1-p)/2}} \\ &\geq 1 - \frac{(1/2)(2/(q+1))^{q/(q+1)}(d^{(p+1)/2}/\sqrt{p+1})^{(1-q)/(1+q)}}{\sqrt{Nd}^{(1-p)/2}} \\ &\equiv 1 - md^{(p-q)/(1+q)} \geq 0.9^{1/n}. \end{aligned} \quad (3.1)$$

for d large. Hence for d positive and large

$$\begin{aligned} u'(t) &= t^{-n} \left[t_1^n u'(t_1) - \int_{t_1}^t s^n |u(s)|^{q-1} u(s) ds \right] \\ &\leq 0.9u'(t_1) + (t - t_1) \left(\frac{q+1}{2} \right)^{q/(q+1)} |u'(t_1)|^{(2q)/(q+1)} \\ &\leq 0.4u'(t_1), \end{aligned} \quad (3.2)$$

where we have used that, since $E' \leq 0$, $|u(t)|^{q+1} \leq (q+1)(u'(t_1))^2/2$ for $t \geq t_1$ with $u(t) \leq 0$. This and (3.2) yield

$$u(t_1 + \tau) \leq 0.4u'(t_1)\tau \leq -0.2(2/(q+1))^{q/(q+1)}|u'(t_1)|^{2/(1+q)}. \quad (3.3)$$

Now for $t \geq t_1 + \tau$ with $u(s) \leq -0.2(2/(q+1))^{q/(q+1)}|u'(t_1)|^{2/(1+q)}$ for all $s \in (t_1 + \tau, t)$ we have

$$\begin{aligned} u'(t) &= t^{-n} \left[t_1^n u'(t_1) - \int_{t_1}^t s^n |u(s)|^{q-1} u(s) ds \right] \\ &\geq u'(t_1) + t^{-n} (0.2(2/(q+1))^{q/(q+1)})^q |u'(t_1)|^{2q/(1+q)} \int_{t_1+\tau}^t s^n ds \\ &\geq -u'(t_1) \left[-1 + t^{-n} (0.2(2/(q+1))^{q/(q+1)})^q |u'(t_1)|^{(q-1)/(1+q)} \frac{t^N - (t_1 + \tau)^N}{N} \right] \\ &\geq -u'(t_1) \left[-1 + \frac{(0.2(2/(q+1))^{q/(q+1)})^q}{N} |u'(t_1)|^{(q-1)/(1+q)} (t - (t_1 + \tau)) \right]. \end{aligned} \quad (3.4)$$

This and the definition of τ imply the following lemma.

Lemma 3.1. *There exists τ_1 in*

$$\begin{aligned} &\left(t_1, t_1 + \left\{ (1/2)(2/(q+1))^{q/(q+1)} + \frac{N}{(0.2(2/(q+1))^{q/(q+1)})^q} \right\} |u'(t_1)|^{(1-q)/(1+q)} \right] \\ &\equiv (t_1, t_1 + \kappa_1 |u'(t_1)|^{(1-q)/(1+q)}) \end{aligned}$$

such that $u(\tau_1) = 0$.

4. SECOND ZERO

Let $\tau_0 > \tau_1$ be such that $u(s) \leq 0.5u(\tau_1)$ for all $s \in [\tau_1, \tau_0]$. Imitating the arguments leading to (2.2) we see that

$$\tau_1 + |u(\tau_1)|^{(1-q)/2} \leq \tau_0 \leq \tau_1 + \sqrt{2^q N} |u(\tau_1)|^{(1-q)/2}. \quad (4.1)$$

Hence

$$\begin{aligned} u'(\tau_0) &= \tau_0^{-n} \int_{\tau_1}^{\tau_0} s^n |u(s)|^q ds \\ &\geq \frac{|u(\tau_1)|^q (\tau_0^N - \tau_1^N)}{N 2^q \tau_0^n} \\ &\geq \frac{|u(\tau_1)|^q (\tau_0 - \tau_1)}{2^q N} \\ &\geq \frac{|u(\tau_1)|^{(1+q)/2}}{2^q N}, \end{aligned} \quad (4.2)$$

and

$$\tau_0^n \geq .9s^n \quad \text{for any } s \in (\tau_0, \tau_0 + 2^{q+2}N|u(\tau_1)|^{(1-q)/2}), \quad (4.3)$$

for $d > 0$ sufficiently large.

Suppose now that for all $s \in [\tau_0, r \equiv \tau_0 + 2^{q+1}N|u(\tau_1)|^{(1-q)/2}]$ we have $u(s) \leq 0$. Then

$$u'(s) \geq 0.9u'(\tau_0) \quad \text{for all } s \in [\tau_0, r]. \quad (4.4)$$

This and the definition of r give

$$\begin{aligned} 0 \geq u(r) &\geq \frac{u(\tau_1)}{2} + .9(r - \tau_0)u'(r) \\ &\geq \frac{u(\tau_1)}{2} + .9(2^{q+1}N|u(\tau_1)|^{(1-q)/2}) \frac{|u(\tau_1)|^{(1+q)/2}}{2^q N} \\ &= 1.3|u(\tau_1)|, \end{aligned}$$

which is a contradiction. From (3.3), $|u(\tau_0)| \geq 0.2(2/(q+1))^{q/(q+1)}|u'(t_1)|^{2/(1+q)}$. Since also $\tau_0 \leq t_1 + (\kappa_1 + 0.2\sqrt{2^q N}(2/(q+1))^{q/(q+1)})|u'(t_1)|^{(1-q)/(1+q)}$ (see Lemma 3.1 and (4.2)). Thus

$$\begin{aligned} &\tau_0 + 2^{q+1}N|u(\tau_1)|^{(1-q)/2} \\ &\leq t_1 + (\kappa_1 + .2(2^{q+2}N)(2/(q+1))^{q/(q+1)})|u'(t_1)|^{(1-q)/(1+q)} \\ &\equiv t_1 + k_2|u'(t_1)|^{(1-q)/(1+q)}. \end{aligned}$$

Thus we have proven the following lemma.

Lemma 4.1. *There exists $t_2 \in [t_1, t_1 + k_2|u'(t_1)|^{(1-q)/(1+q)}]$ such that $u(t_2) = 0$ and $u(s) < 0$ in (t_1, t_2) .*

5. FIRST POSITIVE MAXIMUM

Let $t > t_2$ be such that $u'(s) > 0$ on $[t_2, t]$. Thus $u'' \leq 0$ in $[t_2, t]$. Hence $u(s) \leq u'(t_2)(s - t_2)$ for all $s \in [t_2, t]$. Integrating (1.5) on $[t_2, s]$, we have

$$\begin{aligned} s^n u'(s) &= t_2^n u'(t_2) - \int_{t_2}^s r^n |u(r)|^{p-1} u(r) dr \\ &\geq t_2^n u'(t_2) - s^n \frac{|u'(t_2)|^p (s - t_2)^{p+1}}{p+1} \\ &\geq u'(t_2) \left(t_2^n - \frac{s^n}{p+1} \right), \end{aligned} \quad (5.1)$$

for $s \leq t_2 + u'(t_2)^{(1-p)/(1+p)}$. Since $t_2^N |u'(t_2)|^2 \geq 2c_1 d^\xi$ (see (2.4)) and $(u'(t_2))^2 \leq 2d^{p+1}/(p+1)$, we have

$$t_2^N \geq 2c_1 \left(\frac{p+1}{2} \right)^{\xi/(p+1)} |u'(t_2)|^{N(1-p)/(1+p)}. \quad (5.2)$$

Now for

$$s \in [t_2, \min \{ 2^{1/n}, 1 + (2c_1)^{-1/N} \left(\frac{2}{p+1} \right)^{\frac{\xi}{N(p+1)}} \} t_2] \equiv \alpha t_2,$$

from (5.1) and (5.2), we have

$$u'(s) \geq u'(t_2) \left(\frac{t_2^n}{s^n} - \frac{1}{p+1} \right) \geq u'(t_2) \frac{p-1}{p+1}. \quad (5.3)$$

Integration on $[t_2, \alpha t_2]$ yields

$$u(\alpha t_2) \geq \frac{p-1}{p+1} \alpha t_2 u'(t_2). \quad (5.4)$$

Therefore, assuming again that $u' > 0$ on $[t_2, t]$, we have

$$\begin{aligned} t^n u'(t) &\leq t_2^n u'(t_2) - \int_{\alpha t_2}^t r^n |u(r)|^{p-1} u(r) dr \\ &\leq t_2^n u'(t_2) - t_2^n (t - \alpha t_2) \left(\frac{p-1}{p+1} \alpha t_2 u'(t_2) \right)^p. \end{aligned} \quad (5.5)$$

This and (5.2) imply

$$\begin{aligned} t - \alpha t_2 &\leq \left(\frac{p-1}{p+1} \alpha \right)^{-p} t_2^{-p} |u'(t_2)|^{1-p} \\ &\leq \left(\frac{p-1}{p+1} \alpha \right)^{-p} (2c_1)^{-p/N} \left(\frac{p+1}{2} \right)^{-p\xi/(N(p+1))} |u'(t_2)|^{(1-p)/(p+1)} \\ &\equiv \kappa_2 |u'(t_2)|^{(1-p)/(p+1)}. \end{aligned} \quad (5.6)$$

This proves the following result.

Lemma 5.1. *There exists $\tau_2 \in [t_2, \alpha t_2 + \kappa_2 |u'(t_2)|^{(1-p)/(p+1)}]$ such that $u'(\tau_2) = 0$ and $u'(s) > 0$ on $[t_2, \tau_2)$.*

6. ENERGY ON THE INTERVAL $[t_0, \tau_2]$

Now we estimate the energy on $[t_0, \tau_2]$.

Lemma 6.1. *For $t \in [t_0, \tau_2]$,*

$$t^n H(t) \geq c_1 d^\xi. \quad (6.1)$$

Proof. Let us prove first that

$$\int_{t_0}^{t_1} t^n \gamma u^{p+1}(t) dt \geq \int_{t_1}^{t_2} t^n \gamma_1 |u(t)^{q+1}| dt, \quad (6.2)$$

where $\gamma_1 = ((q+1)(N-2)-2N)/(2(q+1))$. Let $\hat{t}_0 \in [t_0, t_1]$ be such that $u(\hat{t}_0) = d/4$. Then, for $t \in [t_0, \hat{t}_0]$, we have

$$-u'(t) = t^{-n} \int_0^t s^n u^p(s) ds \leq \frac{td^p}{N}. \quad (6.3)$$

Integrating on $[t_0, \hat{t}_0]$ we have $(d/4) \leq (\hat{t}_0^2 - t_0^2)d^p/(2N)$. This and (2.2) yield

$$\hat{t}_0 \geq \sqrt{\frac{Nd^{1-p}}{2} + t_0^2} = t_0 \sqrt{1 + \frac{Nd^{1-p}}{2t_0^2}} \geq t_0 \sqrt{1 + \frac{1}{2^{p+1}}}, \quad (6.4)$$

which combined with (2.2) gives

$$\begin{aligned} \int_{t_0}^{t_1} t^n \gamma u^{p+1}(t) dt &\geq \int_{t_0}^{\hat{t}_0} t^n \gamma u^{p+1}(t) dt \\ &\geq \gamma (d/4)^{p+1} \frac{\hat{t}_0^N - t_0^N}{N} \\ &\geq \frac{\gamma}{4^{p+1}N} t_0^N \left(\left(1 + \frac{1}{2^{p+1}}\right)^{N/2} - 1 \right) d^{p+1} \\ &\geq \frac{\gamma}{4^{p+1}N} \left(\left(1 + \frac{1}{2^{p+1}}\right)^{N/2} - 1 \right) N^{N/2} d^\xi. \end{aligned} \quad (6.5)$$

Using (1.7), we have $|u(t)|^{q+1} \leq (q+1)d^{p+1}/(p+1)$. Also from (2.3) and (2.4), we have $t_1^N |u'(t_1)|^2/2 = t_1 H(t_1) \geq c_1 d^\xi$. This implies that $k_2 |u'(t_1)|^{(1-q)/(1+q)} < t_1$ for $d > 0$ large. These inequalities and Lemma 4.1 imply

$$\begin{aligned} \int_{t_1}^{t_2} t^n |u(t)|^{q+1} dt &\leq \left(\frac{q+1}{p+1} d^{p+1} \right) \frac{t_2^N - t_1^N}{N} \\ &\leq \left(\frac{q+1}{p+1} d^{p+1} \right) \frac{(t_1 + k_2 |u'(t_1)|^{(1-q)/(1+q)})^N - t_1^N}{N} \\ &\leq \left(\frac{q+1}{p+1} d^{p+1} \right) t_1^n \frac{(2^N - 1) k_2 |u'(t_1)|^{(1-q)/(1+q)}}{N} \\ &\leq \left(\frac{q+1}{p+1} d^{p+1} \right) \frac{(2^N - 1) k_2}{N} t_1^n \left(d^{\xi/2} t_1^{-N/2} \right)^{(1-q)/(1+q)} \\ &= \left(\frac{q+1}{p+1} \right) \frac{(2^N - 1) k_2}{N} d^{p+1 + (\xi(1-q)/(2(1+q)))} t_1^{N-1 - N(1-q)/(2(1+q))} \\ &\leq \left(\frac{q+1}{p+1} \right) \frac{(2^N - 1) k_2}{N} \ln^{M/\gamma}(d) d^\eta, \end{aligned} \quad (6.6)$$

where

$$\begin{aligned} \eta &= p+1 + \frac{\xi(1-q) + (1-p)(2(N-1)(1+q) - N(1-q))}{2(1+q)}, \\ M &= (2(N-1)(1+q) - N(1-q))/(2(1+q)). \end{aligned}$$

An elementary calculation shows that $\xi > \eta$. Thus from (6.5) and (6.6), (6.2) follows. Thus for $t \in [t_1, \tau_2]$,

$$\begin{aligned} t^n H(t) &= t_0^n H(t_0) + \int_{t_0}^t s^n \left(NG(u(s)) - \frac{N-2}{2} u(s)g(u(s)) \right) ds \\ &\geq t_0^n H(t_0) + \int_{t_0}^{t_2} s^n \left(NG(u(s)) - \frac{N-2}{2} u(s)g(u(s)) \right) ds \\ &\geq t_0^n H(t_0) \\ &\geq c_1 d^\xi, \end{aligned} \quad (6.7)$$

which proves the lemma. \square

7. PROOF OF THEOREM 1.1

Arguing as in Lemmas 2.1 and 4.1, we see that for $d > 0$ sufficiently large there exist numbers

$$t_3 < \cdots < t_k \leq 1 \quad (7.1)$$

such that

$$u(t) < 0 \quad \text{in } (t_{2i-1}, t_{2i}), \quad \text{and} \quad u(t) > 0 \quad \text{in } (t_{2i}, t_{2i+1}), \quad i = 1, \dots, \min\left\{\frac{k}{2}, \frac{k+1}{2}\right\}. \quad (7.2)$$

Imitating the arguments leading to (6.2), one sees that

$$\int_{t_{2i}}^{t_{2i+1}} t^n \gamma u^{p+1}(t) dt \geq \int_{t_{2i+1}}^{t_{2i+2}} t^n \gamma_1 |u(t)|^{q+1} dt. \quad (7.3)$$

This in turn (see (6.7)) leads to

$$t^n H(t) \geq c_1 d^\xi \quad \text{for all } t \in [t_0, 1]. \quad (7.4)$$

This, together with Lemma 2.1, proves Theorem 1.2. From (7.4) we see that

$$\rho^2(t) \equiv u^2(t) + (u'(t))^2 \rightarrow \infty \quad \text{as } d \rightarrow +\infty, \quad (7.5)$$

uniformly for $t \in [0, 1]$. Therefore, there exists a continuous *argument* function $\theta(t, d) \equiv \theta(t)$ such that

$$u(t) = \rho(t) \cos(\theta(t)) \quad \text{and} \quad u'(t) = -\rho(t) \sin(\theta(t)). \quad (7.6)$$

From this we see that $\theta'(t) = \{(n/t)u'(t) + g(u(t))u(t) + (u'(t))^2\}/\rho^2(t)$. Thus $\theta'(t) > 0$ for $\theta(t) = j\pi/2$ with $j = 1, \dots$, which implies that if $\theta(t) = j\pi/2$ then $\theta(s) > j\pi/2$ for all $s \in (t, 1]$.

Imitating the arguments of Lemmas 2.1 and 4.1, we see that $t_{2i} - t_{2(i-1)} \leq c_3 \ln^{1/\gamma}(d) d^{(1-p)/2}$. Thus $k \geq c_4 \ln^{-1/\gamma}(d) d^{(p-1)/2}$ (see (7.1)), which implies that

$$\lim_{d \rightarrow +\infty} \theta(1, d) = +\infty. \quad (7.7)$$

By the continuity of θ and the intermediate value theorem we see that there exists a sequence $d_1 < \cdots < d_j < \cdots \rightarrow \infty$ such that $\theta(1, d_j) = j\pi + (\pi/2)$. Hence $u(t, d_j)$ is a solution to (1.1) having exactly j zeroes in $(0, 1)$, which proves Theorem 1.1.

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