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A LOCAL INVERSION PRINCIPLE OF THE NASH–MOSER TYPE*

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Abstract. We prove an inverse function theorem of the Nash–Moser type. The main difference between our method and that of [J. Moser, *Proc. Nat. Acad. Sci. USA,* 47 (1961), pp. 1824–1831] is that we use continuous steepest descent while Moser uses a combination of Newton-type iterations and approximate inverses. We bypass the *loss of derivatives problem* by working on finite dimensional subspaces of infinitely differentiable functions.

Key words. Nash–Moser methods, inverse function theorem, continuous steepest descent

AMS subject classifications. Primary, 34B15; Secondary, 35J65

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1. Introduction.

Inverse function theorems are fundamental tools for the study of solutions to nonlinear equations. Proofs depend on iteration arguments. When a nonlinear equation comes from a partial differential equation, it often happens that the operators under consideration do not have enough *regularity* and the iteration process is defined only for a few steps. This is known as loss of derivatives. In order to overcome this difficulty, iteration techniques have been designed to allow the iteration to lead to a limit. These are known as generalized inverse function theorems. They generally depend on having a *scale* of intermediate spaces between the domain and the range of the nonlinear operator under consideration. Most notable of such results is the one proven by J. Moser in [5] and [6] (see also [2]). The reader is referred to [3] for a sharper version of the results of [5].

Here we prove an inverse function theorem (Theorem 2.1 below) using finite dimensional subspaces of infinitely differentiable functions, which makes the phenomenon of loss of derivatives immaterial. In addition we only assume the operators to have a first order derivative. This is in contrast with the results of [5] and [3] where Newton-like iteration techniques require extensive use of the properties of the *quadratic remainder* (see (10)). In [12] another inverse function theorem is proven without assumptions on the quadratic remainder. Our use of continuous steepest descent has roots in [1], [7], [8], and [9]. For applications of generalized inverse function theorems to elliptic systems the reader is referred to [4] and [11].

2. Main result.

For the sake of simplicity we present our main result in the context of a particular class of Sobolev spaces. However, the general principle applies to many other *scales* of spaces sharing the general properties of the Sobolev spaces that are defined in the next paragraph. The reader is referred to [2] for various other scales of spaces.

Let each of m and n denote a positive integer. For each nonnegative integer ρ let C^ρ denote the set of ρ-differentiable functions $u : R^n \to R^m$ that are 2π-periodic in each of its n independent real variables. The norm in C^ρ is given by

$$|u|_\rho = \max\{|u(x)|; x \in R^n\} + \max\{|D^\alpha u(x)|; x \in R^n, |\alpha| = \rho\}; \tag{1}$$

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here, $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of nonnegative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Similarly for $r \geq 0$ we define H^r as the Sobolev space of functions $u : R^n \to R^m$ of the form

$$u(x) = \sum_{j \in \mathbf{Z}^n} c_j e^{ij\cdot x} \tag{2}$$

such that

$$\|u\|_r^2 \equiv \sum_{j \in \mathbf{Z}^n} |c_j|^2 \; + \; \sum_{j \in \mathbf{Z}^n} |j|^{2r} |c_j|^2 < \infty, \tag{3}$$

where $|(j_1, \dots, j_n)|^2 = j_1^2 + \cdots + j_n^2$. Actually H^r is also defined for $r < 0$ provided that the expression in (2) is understood in the sense of distributions and

$$\|u\|_r^2 \equiv |c_0|^2 \; + \sum_{j \in \mathbf{Z}^n - \{0\}} |j|^{2r} |c_j|^2 < \infty. \tag{4}$$

For $r \geq 0$ the inner product in H^r is given by

$$\left\langle \sum_{j \in \mathbf{Z}^n} c_j e^{ij\cdot x}, \sum_{j \in \mathbf{Z}^n} d_j e^{ij\cdot x} \right\rangle_r = \sum_{j \in \mathbf{Z}^n} (1 + |j|^{2r}) c_j \bar{d}_j. \tag{5}$$

Let $\epsilon > 0$ and $F : \{x \in C^1; |x|_1 < \epsilon\} \to C^0$ be a continuous function such that $F(0) = 0$. Typically F is a first order differential operator. We assume that for $u, v \in C^1$, with $|u|_1 < \epsilon$, the limit

$$\lim_{t \to 0} \frac{F(u + tv) - F(u)}{t} \equiv F'(u)v \tag{6}$$

exists and is a continuous function of v and that $F'(\cdot)v$ defines a continuous function from C^1 into C^0, for each $v \in C^1$. We also assume that there exist $\lambda \in R$, positive constants k_1, k_2, k_3, and $l > (n/2) + 1$ with

$$\langle F'(u)v, v \rangle_0 \geq k_1 \|v\|_0^2 \tag{7}$$

for all $u, v \in C^1$;

$$\langle F'(u)v, v \rangle_l \geq k_2 \|v\|_\lambda^2 - k_3 \|v\|_0^2 \tag{8}$$

for $u \in C^1$, $|u|_l \leq \epsilon$, $v \in H^l$; and for each positive integer ρ there exists M_ρ such that

$$\|F(u)\|_\rho \leq M_\rho \|u\|_{\rho+1} \quad \text{for} \quad |u|_1 < \epsilon. \tag{9}$$

Without loss of generality we may assume that $k_1 < k_2$.

In [6], it is shown that the equation $F(u) = g$ has a solution when the operators $F'(u)$ have *approximate inverses*, and

$$\|F(u + v) - F(u) - F'(u)v\|_0 \leq M \|v\|_0^{2-\beta} \|v\|_l^\beta \;\text{ for }\; |u|_1 < \epsilon,\; |u + v|_1 < \epsilon, \tag{10}$$

where M is a constant independent of u, v and $0 \leq \beta < 1$. In addition, it is shown that (7) and (8) with $l = \lambda$ are sufficient for $F'(u)$ to have an approximate inverse.

Here we prove the following theorem.

THEOREM 2.1. *If* (6)–(9) *hold and* $l > (n/2) + 1$, *then there exist* $\delta > 0$ *such that, if* $\|g\|_l < \delta$, *then the equation*

$$F(y) = g \tag{11}$$

has a solution.

Proof. For each positive integer k, let X_k denote the linear subspace of H^l of functions of the form

$$u(x) = \sum_{\|j\|^2 \le k^2} c_j e^{ij\cdot x}. \tag{12}$$

Let $P_k \equiv P$ denote the orthogonal projection of H^l onto X_k. An elementary Fourier series argument shows that P is also an orthogonal projection of H^0 onto X_k and

$$\langle Pu, v\rangle_0 = \langle u, Pv\rangle_0 \quad \text{for all} \quad u, v \in H^l. \tag{13}$$

Now let $\Lambda \in (n/2+1, l)$. Since X_k is finite dimensional and a subset of C^1, by (6) there exists a bounded differentiable function $\tilde{F} : X_k \to X_k$ such that $PF(u) = \tilde{F}(u)$ if $\|u\|_\Lambda < \epsilon/2$, $u \in X_k$. Hence the initial value problem

$$z'(t) = -\tilde{F}(z(t)) + P(g), \; t \ge 0, \quad z(0) = 0, \tag{14}$$

has a solution defined on $[0, \infty)$.

Let us see that, for $\|g\|_l$ small enough, $|z(t)|_1 < \epsilon/2$ for all $t \ge 0$. In fact, let $w(t) = P(\tilde{F}(z(t)) - g)$. Thus

$$\begin{aligned}(\|w(t)\|_0^2)' &= 2\langle w(t), P\tilde{F}'(z(t))z'(t)\rangle_0 \\ &= -2\langle Pw(t), \tilde{F}'(z(t))(P(w(t))\rangle_0 \\ &\le -2k_1\|w(t)\|_0^2.\end{aligned} \tag{15}$$

In particular we see that the quantity $\|w(t)\|_0$ is a decreasing function of t. In addition, from (15) we have

$$\|w(t)\|_0 \le \|w(0)\|_0 e^{-k_1 t} = \|g\|_0 e^{-k_1 t}. \tag{16}$$

Now we estimate the H^l norm of $w(t)$. In order to do so we observe that for each (k, λ) there exists a positive constant $C(k, \lambda)$ such that

$$\|x\|_\lambda^2 \ge C(k, \lambda)\|x\|_l^2 \quad \text{for all} \quad x \in X_k. \tag{17}$$

We note that $C(k, \lambda) \to 0$ as $k \to \infty$ when $\lambda < l$; otherwise $C(k, \lambda)$ can be taken to be equal to 1. From now on we restrict ourselves to the case $\lambda < l$; the case $\lambda \ge l$ is simpler. Thus we may assume that

$$C(k, \lambda) \to 0 \quad \text{as} \quad k \to \infty. \tag{18}$$

From (8), (13), (16), (17), and (18) we infer that for k sufficiently large

$$\begin{aligned}(\|w(t)\|_l^2)' &= 2\langle w(t), P\tilde{F}'(z(t))z'(t)\rangle_l \\ &= -2\langle Pw(t), \tilde{F}'(z(t))(P(w(t))\rangle_l \\ &\le -2(k_2\|w(t)\|_\lambda^2 - k_3\|w(t)\|_0^2) \\ &\le -2(k_2 C(k, \lambda)\|w(t)\|_l^2 - k_3\|g\|_0^2 e^{-2k_1 t}).\end{aligned} \tag{19}$$

Thus for k sufficiently large

$$\begin{aligned} \|w(t)\|_l^2 &\le (\|g\|_l^2 + k_3\|g\|_0^2/(k_1 - k_2 C(k,\lambda)))e^{-2k_2C(k,\lambda)t} \\ &\le (\|g\|_l^2 + 2k_3\|g\|_0^2/k_1)e^{-2k_2C(k,\lambda)t} \\ &\equiv M(\|g\|_l)e^{-2k_2C(k,\lambda)t}, \end{aligned} \tag{20}$$

where $M(\|g\|_l) \to 0$ as $\|g\|_l \to 0$. By interpolation properties of Sobolev space (see section I.2 in [6]) one has

$$\|w(t)\|_\Lambda \le \|w(t)\|_0^{(1-\Lambda/l)}\|w(t)\|_l^{\Lambda/l} \le M(\|g\|_l)^{\Lambda/(2l)}\|g\|_0^{(1-\Lambda/l)}e^{-k_1(1-\Lambda/l)t}. \tag{21}$$

Integrating (14) by (20) and (21), we see that there exist $\delta > 0$ such that if $\|g\|_l \le \delta$, then

$$\|z(t)\|_\Lambda < \epsilon/3 \quad \text{for all} \quad t \ge 0. \tag{22}$$

Now letting $x_k \in X_k$ be an element in the w-limit set of the orbit defined by z, we see that $\|x_k\|_\Lambda \le \epsilon/3$ and $\tilde{F}(x_k) = PF(x_k) = P(g)$. Therefore by the Sobolev imbedding theorem we may assume that it converges to some element $x \in H^{\lambda_1}$, with $\lambda_1 \in ((N/2)+1, l)$. By (9), $F(x_k)$ is bounded in $H^{\lambda_1 - 1}$. Using again that bounded sequences in H^s have converging subsequences in H^t if $s > t$, we may further assume that $\{F(x_k)\}$ converges in H^{λ_2} with $\lambda_2 \in (n/2, \lambda_1 - 1)$. Recalling that, by Poincaré's inequality,

$$\|z\|_{\lambda_2 - 1} \le C_k\|z\|_{\lambda_2} \quad \text{for all} \quad z \in X_k^\perp, \tag{23}$$

with C_k converging to zero as $k \to \infty$, we conclude that $\|(I-P)F(x_k)\|_{\lambda_2 - 1} \to 0$ as $k \to \infty$. This and the fact that $\{F(x_k)\}$ converges to $F(x)$ imply that $\{P(F(x_k))\}$ converges to $F(x)$ in H^{λ_2}. Hence in H^{λ_2} we have

$$g = \lim P_k(g) = \lim P_k(F(x_k)) = F(x), \tag{24}$$

and this proves the theorem. □

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