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**A SEMILINEAR WAVE EQUATION WITH NONMONOTONE  
NONLINEARITY**

ALFONSO CASTRO AND SUMALEE UNSURANGSIE

# A SEMILINEAR WAVE EQUATION WITH NONMONOTONE NONLINEARITY

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**We prove that a semilinear wave equation in which the range of the derivative of the nonlinearity includes an eigenvalue of infinite multiplicity has a solution. The solution is obtained through an iteration scheme which provides a priori estimates.**

**1. Introduction.** Here we study the nonlinear wave equation

$$(1.1) \quad u_{tt} - u_{xx} + \lambda u = cq(x, t) + r(x, t) + g(u), \quad (x, t) \in [0, \pi] \times \mathbf{R},$$

$$(1.2) \quad u(0, t) = u(\pi, t) = 0, \quad u(x, t) = u(x, t + 2\pi), \\ (x, t) \in [0, \pi] \times \mathbf{R},$$

where  $\lambda \in \mathbf{R} - \{k^2 - j^2: k = 1, 2, 3, \dots, j = 0, 1, 2, \dots\}$  and  $g$  is a function of class  $C^1$  such that

$$(1.3) \quad \lim_{|u| \rightarrow \infty} g'(u) = 0.$$

A main difficulty in studying (1.1)–(1.2) arises when  $-\lambda \in g'(\mathbf{R})$ . This causes compactness arguments to fail because 0 is an eigenvalue of  $u_{tt} - u_{xx}$ , (1.2) of infinite multiplicity. Recent studies on (1.1)–(1.2) either: (i) assume that  $g(u) - \lambda u$  is monotone (see [B-N], [R], [W]), or (ii) assume enough symmetry on  $g, q$ , and  $r$  so that the kernel of  $u_{tt} - u_{xx}$ , (1.2) reduces to  $\{0\}$  (see [Co]), or (iii) use dichotomy on whether the Palais-Smale condition holds proving existence for values of  $cq+r$  which cannot be given explicitly (see [H], [W]). Our main result (Theorem A below) does not fall in any of the above three classes.

Let  $\Omega = [0, \pi] \times [0, 2\pi]$ . Let  $H^1, L^2$ , and  $L^\infty$  denote the Sobolev spaces  $H^1(\Omega), L^2(\Omega)$ , and  $L^\infty(\Omega)$  respectively. We let  $\|\cdot\|_1, \|\cdot\|$ , and  $\|\cdot\|_\infty$  denote the norms in  $H^1, L^2$ , and  $L^\infty$  respectively. Let

$$N = \left\{ u \in L^2: u = \sum_{k=1}^{\infty} (a_k \sin(kx) \sin(kt) + b_k \sin(kx) \cos(kt)) \right\}.$$

Let  $N^\perp \subseteq L^2$  denote the orthogonal complement to  $N$  in  $L^2$ . Let  $P$  denote the orthogonal projection onto  $N$  and  $Q$  the orthogonal projection

onto  $N^\perp$ . Let  $\varphi$  be a solution to

$$(1.4) \quad \begin{aligned} \varphi_{tt} - \varphi_{xx} + \lambda\varphi &= q(x, t), & (x, t) \in [0, \pi] \times [0, 2\pi], \\ \varphi(0, t) = \varphi(\pi, t) &= 0, & \varphi(x, t) = \varphi(x, t + 2\pi). \end{aligned}$$

Throughout the rest of this paper we will assume that

$$(1.5) \quad m\{(x, t) \in \gamma; |\varphi(x, t)| < \delta\} \rightarrow 0 \quad \text{as } \delta \rightarrow 0$$

uniformly in  $\gamma$ , where  $\gamma$  is any characteristic of the operator  $\partial_{tt} - \partial_{xx}$ . Our main result is

**THEOREM A.** *If  $(P(r))_t \in L^2$ ,  $\varphi_t \in L^\infty$ , and (1.5) holds, then there exists  $c_0$  such that for  $|c| > c_0$  the problem (1.1)–(1.2) has a weak solution  $u \in H^1 \cap L^\infty$ .*

Our proof is based on an iteration argument that resembles the proof of the inverse function theorem. We give (see §3) estimates in  $H^1 \cap L^\infty$  which show the convergence of the scheme. Even though our arguments and main result suggest applicability of the standard inverse function theorem, our hypotheses are not enough to guarantee it.

**2. Preliminaries and notations.** A direct calculation shows that if  $u \in N$ , then  $u_{tt} - u_{xx} = 0$  in the sense of distributions. Let  $Au = u_{tt} - u_{xx}$ . We recall (see [B-N]) that  $A$  subject to condition (1.2) is selfadjoint, the range of  $A$  is closed in  $L^2$ , and  $R(A) = N^\perp$ . The eigenvalues of  $A$  subject to (1.2) form the set  $\{k^2 - j^2: k = 1, 2, 3, \dots, j = 0, 1, 2, \dots\}$ . The corresponding eigenfunctions are  $\sin(kx) \sin(jt)$  and  $\sin(kx) \cos(jt)$ . The operator  $A^{-1}$  is compact from  $N^\perp$  into  $N^\perp$ , and there exists a real number  $c$  such that

$$\begin{aligned} \|A^{-1}f\|_\infty &\leq c\|f\| & \text{for all } f \in R(A), \\ \|A^{-1}f\|_1 &\leq c\|f\| & \text{for all } f \in R(A). \end{aligned}$$

Using Fourier series it is easy to show that if  $u \in H^1, v \in N \cap H^1$  then

$$(2.1) \quad \iint (P(u))_t v_t = \iint u_t v_t,$$

$$(2.2) \quad \int_0^{2\pi} (v)^2(x, s) ds \leq \int \int_\Omega (v)^2 dx dt \quad \text{for all } x \in [0, \pi].$$

Let  $A_1$  denote the operator defined by

$$A_1 \left( \sum_{\substack{k=1 \\ j=0 \\ k \neq j}}^{\infty} (a_{kj} \sin(kx) \sin(jt) + b_{kj} \sin(kx) \cos(jt)) \right) \\ = \sum_{\substack{k=1 \\ j=0 \\ k \neq j}}^{\infty} \left[ \frac{a_{kj}}{k^2 - j^2 + \lambda} \sin(kx) \sin(jt) + \frac{b_{kj}}{k^2 - j^2 + \lambda} \sin(kx) \cos(jt) \right].$$

It is easy to see that if  $f \in L^2$ , then  $w = A_1(f)$  is a weak solution to  $w_{tt} - w_{xx} + \lambda w = f$ . An elementary Fourier series argument shows that if  $w = A_1(Q(f))$ , then

$$(2.3) \quad \int_0^{2\pi} ((w_x)^2(x, t) + (w_t)^2(x, t)) dt \leq b_0^2 \|Q(f)\|^2$$

for all  $x \in [0, \pi]$ .

where  $b_0 = \max\{(2/\pi)(k^2 + j^2)/(k^2 - j^2 + \lambda)^2 : k \neq j, k = 1, 2, 3, \dots, j = 1, 2, 3, \dots\}$ . In particular

$$(2.4) \quad \|A_1(Q(f))\|_1 \leq b_0 \pi^{1/2} \|Q(f)\|.$$

Also it is easy to show that

$$(2.5) \quad \|A(Q(f))\|_{\infty} \leq b_1 \|Q(f)\|$$

where

$$b_1 = \left( \sum_{\substack{k=1 \\ j=0 \\ k \neq j}}^{\infty} \frac{2}{(k^2 - j^2 + \lambda)^2} \right)^{1/2}$$

We can rewrite (1.1) as the following:

$$(2.6) \quad u_{tt} - u_{xx} + \lambda u = (c/R)Rq(x, t) + r(x, t) + g(u)$$

where

$$R = \max\{2(2)^{1/2}dK\lambda, 2(2)^{1/2}, 16d^4 \|g'\|_{\infty}^3 \lambda, 16 \|g'\|_{\infty}^3 / \lambda\} / (\|\varphi_t\|_{\infty} + 2^{1/2}),$$

$K = b_0 \pi^{1/2} + b_1$  and  $d > 0$  is a constant such that  $\|u\|_{L^4} \leq d \|u\|_1$  and  $\|u\| \leq d \|u\|_1$  for all  $u \in H^1$  (the existence of  $d$  follows the fact that

the embeddings  $H^1 \rightarrow L^4$  and  $H^1 \rightarrow L^2$  are continuous (see [A]). Let  $w = Ru/c$  and  $\beta = R/c$ . Then (2.6) becomes

$$(2.7) \quad w_{tt} - w_{xx} + \lambda w = Rq(x, t) + \beta(r(x, t) + g(w/\beta)).$$

Inductively we define  $w_0 = 0$ ,  $w_1$  as the solution to

$$(2.8) \quad \begin{aligned} (w_1)_{tt} - (w_1)_{xx} + \lambda w_1 &= \beta(r(x, t) + g(R\varphi/\beta)), \\ w_1(0, t) = w_1(\pi, t) &= 0, \quad w_1(x, t) = w_1(x, t + 2\pi), \end{aligned}$$

and  $w_{n+1}$  as the solution to

$$(2.9) \quad \begin{aligned} (w_{n+1})_{tt} - (w_{n+1})_{xx} + \lambda w_{n+1} &= \beta(r(x, t) + g((R\varphi + w_n)/\beta)), \\ w_{n+1}(0, t) = w_{n+1}(\pi, t) &= 0, \quad w_{n+1}(x, t) = w_{n+1}(x, t + 2n). \end{aligned}$$

**3. Estimates.** For the sake of simplicity we will assume throughout the rest of this paper that  $\lambda > 0$ , and  $c > 0$ . The case  $\lambda < 0$ , or  $c < 0$  requires only obvious modifications.

**LEMMA 3.1.** *Let  $\{w_n\}_n$  be defined by (2.8) and (2.9). Under the assumption of Theorem A, there exists  $\beta_1 > 0$  such that if  $\beta \in (0, \beta_1)$  then for all  $n = 1, 2, 3, \dots$  we have*

$$(3.1) \quad \|w_n\|_1 + \|w_n\|_\infty \leq 1.$$

*Proof.* Since for each characteristic  $\gamma$ ,  $m\{(x, t) \in \gamma : |\varphi(x, t)| < \delta\} \rightarrow 0$  as  $\delta \rightarrow 0$  uniformly in  $\gamma$  then there exists  $\delta_0 > 0$  such that if  $0 < \delta < \delta_0$  then

$$(3.2) \quad \begin{aligned} m\{(x, t) \in \gamma : |\varphi(x, t)| < \delta\} \\ < (\lambda/(32\pi R(\|g'\|_\infty + 1)(\|\varphi_t\|_\infty + 2^{1/2})))^2 \end{aligned}$$

for any characteristic  $\gamma$ . Because of (1.3) there exists  $L \geq 0$  such that for all  $u \in \mathbf{R}$

$$(3.3) \quad |g(u)| \leq (|u|/64\pi KR(\|\varphi\|_\infty + 1)) + L,$$

and there exists  $M > 0$  such that if  $|u| > M$  then

$$(3.4) \quad |g'(u)| < \lambda/(64\pi^2 R(\|\varphi_t\|_\infty + 2^{1/2})).$$

Now we define  $\beta_1$  by

$$(3.5) \quad \beta_1 = \min\{1/(32(M + (L + 1)\pi K + K\|r\|)), \\ \lambda/(32\pi(\|r_t\| + 1)), R\delta_0/M\}.$$

Next we prove (3.1) by induction. First we observe that

$$\begin{aligned}
 (3.6) \quad \|g(R\varphi/\beta)\|^2 &= \iint_{\Omega} (g(R\varphi/\beta))^2 \\
 &\leq 2 \iint_{\Omega} L^2 + 2 \left( \iint_{\Omega} (R\varphi/\beta)^2 \right) / (64\pi KR(\|\varphi\|_{\infty} + 1))^2 \\
 &\leq 4\pi^2 L^2 + (1/32K\beta)^2.
 \end{aligned}$$

We write  $w_1 = v_1 + z_1$  with  $v_1 \in N$  and  $z_1 \in N^{\perp}$ . From (2.4), (2.5), and (2.7) it follows that if  $\beta \in (0, \beta_1)$  then

$$(3.7) \quad \|z_1\|_1 + \|z_1\|_{\infty} \leq 1/4.$$

Projecting (2.8) into  $N$ , differentiating with respect to  $t$ , multiplying by  $(v_1)_t$ , and integrating over  $\Omega$  we have

$$\begin{aligned}
 \lambda \iint_{\Omega} ((v_1)_t)^2 dx dt &= \beta \iint_{\Omega} (P(r + g(R\varphi/\beta)))_t (v_1)_t dx dt \\
 &= \beta \iint_{\Omega} (r + g(R\varphi/\beta))_t (v_1)_t dx dt \\
 &= \beta \iint_{\Omega} r_t (v_1)_t + \iint_{\Omega} g'(R\varphi/\beta)(R\varphi_t)(v_1)_t \\
 &= \beta \|r_t\| \cdot \|(v_1)_t\| + R \|\varphi_t\|_{\infty} \|(v_1)_t\| \left( \iint_{\Omega} (g'(R\varphi/\beta))^2 \right)^{1/2}.
 \end{aligned}$$

Therefore

$$\|(v_1)_t\| \leq \left( \beta \|r_t\| + R \|\varphi_t\|_{\infty} \left( \iint_{\Omega} (g'(R\varphi/\beta))^2 \right)^{1/2} \right) / \lambda.$$

In order to estimate  $\iint_{\Omega} (g'(R\varphi/\beta))^2$  we define  $s_{\beta} = \{(x, t) : |R\varphi(x, t)| \leq M\beta\}$  and  $c_{\beta} = \Omega - s_{\beta}$ . Since  $M\beta/R < \delta_0$  we have

$$m(s_{\beta}) < (\lambda / (32\pi R(\|g'\|_{\infty} + 1)(\|\varphi_t\|_{\infty} + 2^{1/2})))^2$$

(see (3.2) and (3.5)); then

$$\begin{aligned}
 \left( \iint_{\Omega} (g'(R\varphi/\beta))^2 \right)^{1/2} &= \left[ \iint_{s_{\beta}} (g'(R\varphi/\beta))^2 + \iint_{c_{\beta}} (g'(R\varphi/\beta))^2 \right]^{1/2} \\
 &\leq [ \|g'\|_{\infty}^2 m(s_{\beta}) + 2\pi^2 \lambda^2 / (64\pi^2 R(\|\varphi_t\|_{\infty} + 2^{1/2}))^2 ]^{1/2} \\
 &\leq [ 2\lambda^2 / (32R(\|\varphi_t\|_{\infty} + 2^{1/2}))^2 ]^{1/2} \\
 &\leq \lambda / (16R(\|\varphi_t\|_{\infty} + 2^{1/2})).
 \end{aligned}$$

Hence if  $\beta \in (0, \beta_1)$  then  $\|(v_1)_t\| \leq 1/8$ . Since  $\|v_t\| = \|v_x\|$  for all  $v \in N \cap H^1$ , we have  $\|v_1\|_1 \leq 1/4$ . Because of (2.2) we have

$$\begin{aligned} |v_1(x, t)| &= \left| \int_0^t (v_1)_t(x, s) ds \right| \leq \left( \int_0^t ((v_1)_t)^2(x, s) ds \right)^{1/2} \left( \int_0^t ds \right)^{1/2} \\ &\leq (2\pi)^{1/2} \|(v_1)_t\| \leq (2\pi)^{1/2}/8 \leq 1/3. \end{aligned}$$

Therefore

$$(3.8) \quad \|v_1\|_1 + \|v_1\|_\infty \leq 7/12.$$

Combining (3.7) and (3.8) we have

$$\|w_1\|_1 + \|w_1\|_\infty \leq 1.$$

Suppose now that  $\|w_n\|_1 + \|w_n\|_\infty \leq 1$ . We write  $w_{n+1} = v_{n+1} + z_{n+1}$  with  $v_{n+1} \in N$  and  $z_{n+1} \in N^\perp$ . Again from (2.4), (2.5) and (2.7) we have

$$\|z_{n+1}\|_1 + \|z_{n+1}\|_\infty \leq \beta K(\|r\| + \|g((R\varphi + w_n)/\beta)\|).$$

In order to estimate  $\|g((R\varphi + w_n)/\beta)\|$  we observe that

$$\begin{aligned} &\iint_{\Omega} (g((R\varphi + w_n)/\beta))^2 \\ &\leq 2 \iint_{\Omega} L^2 + 2 \left( \iint_{\Omega} ((R\varphi + w_n)/\beta)^2 \right) / (64\pi RK(\|\varphi\|_\infty + 1))^2 \\ &\leq 4\pi^2 L^2 + 8\pi^2 [(R\|\varphi\|_\infty + 1) / (64\pi RK(\|\varphi\|_\infty + 1))]^2 / \beta^2. \end{aligned}$$

Therefore if  $\beta \in (0, \beta_1)$  then

$$(3.9) \quad \|z_{n+1}\|_1 + \|z_{n+1}\|_\infty \leq 1/4.$$

Now projecting (2.9) into  $N$ , differentiating with respect to  $t$ , multiplying by  $(v_{n+1})_t$  and integrating over  $\Omega$  we have

$$\begin{aligned} (3.10) \quad &\lambda \iint_{\Omega} ((v_{n+1})_t)^2 \\ &= \beta \left( \iint_{\Omega} (r_t)(v_{n+1})_t \right. \\ &\quad \left. + \iint_{\Omega} g'((R\varphi + w_n)/\beta)((R\varphi + w_n)/\beta)_t (v_{n+1})_t \right) \\ &\leq \beta \|r_t\| \cdot \|(v_{n+1})_t\| \\ &\quad + (2\pi R\|\varphi_t\|_\infty + 2^{1/2}) \left[ \iint_{\Omega} (g'((R\varphi + w_n)/\beta))^2 ((v_{n+1})_t)^2 \right]^{1/2}. \end{aligned}$$



Now we consider

$$(3.11) \quad I = \iint_{\Omega} (g'((R\varphi + w_n)/\beta))^2 ((v_{n+1})_t)^2 \\ = \int_0^{2\pi} \int_0^\pi (g'((R\varphi + w_n)/\beta))^2 ((v_{n+1})_t)^2 dx dt.$$

Without loss of generality we can assume that  $v_{n+1} = h(x - t)$  or  $(v_{n+1})_t = -h'(x - t)$ . Because the integrand in (3.11) is  $2\pi$  periodic in  $t$  we have

$$I = \int_0^\pi \int_x^{2\pi+x} (g'((R\varphi + w_n)(x, t)/\beta))^2 (h'(x - t))^2 dt dx.$$

By defining  $\eta = x$ ,  $\zeta = -x + t$ ,  $\gamma_\zeta = \{(s, s + \zeta) : s \in [0, \pi]\}$  and  $A_\beta = \{(x, t) \in \Omega : |R\varphi(x, t)| \leq M\beta + 1\}$  we have

$$(3.12) \quad I = \int_0^{2\pi} \int_0^\pi (g'((R\varphi(\eta, \eta + \zeta) + w_n(\eta, \eta + \zeta))/\beta))^2 (h'(-\zeta))^2 d\eta d\zeta \\ = \int_0^{2\pi} (h'(-\zeta))^2 \left( \int_0^\pi (g'((R\varphi(\eta, \eta + \zeta) + w_n(\eta, \eta + \zeta))/\beta))^2 d\eta \right) d\zeta \\ = \int_0^{2\pi} (h'(-\zeta))^2 \left[ \int_{\gamma_\zeta \cap A_\beta} (g'((R\varphi(\eta, \eta + \zeta) + w_n(\eta, \eta + \zeta))/\beta))^2 d\eta \right. \\ \left. + \int_{\gamma_\zeta - (\gamma_\zeta \cap A_\beta)} (g'((R\varphi(\eta, \eta + \zeta) + w_n(\eta, \eta + \zeta))/\beta))^2 d\eta \right] d\zeta \\ \leq \|(v_{n+1})_t\|^2 [(\|g'\|_\infty \lambda / (32\pi(\|g'\|_\infty + 1)))^2 \\ + (\lambda\pi^{1/2}/64\pi^2)^2] / (2R(\|\varphi_t\|_\infty + 2^{1/2}))^2.$$

Hence if  $\beta \in (0, \beta_1)$  then

$$\|(v_{n+1})_t\| \leq 1/8.$$

Imitating the argument in (3.8) we have

$$(3.13) \quad \|v_{n+1}\|_1 + \|v_{n+1}\|_\infty \leq 7/12.$$

Combining (3.9) and (3.13) we have

$$\|w_{n+1}\|_1 + \|w_{n+1}\|_\infty \leq 1,$$

which proves the lemma.

**LEMMA 3.2.** *If  $\{a_n\}$  is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq \tau(a_n + a_{n-1}), \quad n = 2, 3, 4, \dots,$$

then

$$a_{n+1} \leq c_{n+1}k\tau^{[n/2]},$$

where  $k = \max\{a_1, a_2\}$ , and  $c_n$  is the  $n$ th Fibonacci number of the sequence defined by  $c_{n+1} = c_n + c_{n-1}$ ,  $c_2 = 2$  and  $c_1 = 0$ , and  $[x]$  denotes the largest integer less than or equal to  $x$ . In particular the series  $\sum a_n$  converges if  $\tau$  is small enough.

*Proof.* We prove the lemma by induction. For  $n = 2$  we have

$$a_3 \leq \tau(a_2 + a_1) \leq 2\tau k = (c_2 + c_1)\tau k = c_3k\tau^{[2/2]}.$$

Suppose that

$$a_n \leq c_nk\tau^{[(n-1)/2]}.$$

If  $n$  is even we have

$$\begin{aligned} a_{n+1} &\leq \tau(a_n + a_{n-1}) \leq \tau(c_nk\tau^{[(n-1)/2]} + c_{n-1}k\tau^{[(n-2)/2]}) \\ &\leq \tau k(c_n\tau^{(n-2)/2} + c_{n-1}\tau^{(n-2)/2}) \\ &\leq (c_n + c_{n-1})k\tau^{n/2} \leq c_{n+1}k\tau^{[n/2]}. \end{aligned}$$

Similarly if  $n$  is odd we have

$$\begin{aligned} a_{n+1} &\leq \tau(c_nk\tau^{(n-1)/2} + c_{n-1}k\tau^{(n-3)/2}) \\ &= c_nk\tau\tau^{(n-1)/2} + c_{n-1}k\tau^{(n-1)/2} \leq (c_n + c_{n-1})k\tau^{[n/2]}, \end{aligned}$$

which proves Lemma 3.2.

**4. Proof of Theorem A.** For  $n = 1, 2, 3, \dots$  we write  $w_n = v_n + z_n$  where  $v_n \in N$  and  $z_n \in N^\perp$ . Let  $n > 2$ . From the Sobolev imbedding theorem and (2.4) and (2.5) we have

$$\begin{aligned} (4.1) \quad \|z_{n+1} - z_n\|^2 &\leq d^2 \|z_{n+1} - z_n\|_1^2 \\ &\leq (dK\beta)^2 \iint_{\Omega} (g((R\varphi + w_n)/\beta) - g((R\varphi + w_{n-1})/\beta))^2 \\ &= (dK)^2 \iint_{\Omega} (g'((R\varphi + \zeta)/\beta))^2 (w_{n-1} - w_n)^2 \\ &\leq 2(dK)^2 \left[ \iint_{\Omega} (g'((R\varphi + \zeta)/\beta))^2 (v_n - v_{n-1})^2 \right. \\ &\qquad \qquad \qquad \left. + \iint_{\Omega} (g'((R\varphi + \zeta)/\beta))^2 (z_n - z_{n-1})^2 \right] \end{aligned}$$

where  $\zeta \in [w_n(x, t), w_{n-1}(x, t)] \cup [w_{n-1}(x, t), w_n(x, t)]$ . Since  $v_n$  and  $v_{n-1} \in N$ , imitating the argument in (3.11)–(3.12) we see that

$$\begin{aligned} &\iint_{A_\beta} (g'((R\varphi + \zeta)/\beta))^2 (v_n - v_{n-1})^2 \\ &\leq (\lambda \|v_n - v_{n-1}\| / 32\pi R (\|\varphi_t\|_\infty + 2^{1/2}))^2. \end{aligned}$$

Thus

$$\begin{aligned}
 (4.2) \quad & 2(dK)^2 \iint_{\Omega} (g'((R\varphi + \zeta)/\beta))^2 (v_n - v_{n-1})^2 \\
 & \leq 2(dK)^2 \left[ \iint_{A_\beta} (g'((R\varphi + \zeta)/\beta))^2 (v_n - v_{n-1})^2 \right. \\
 & \quad \left. + \iint_{\Omega - A_\beta} (g'((R\varphi + \zeta)/\beta))^2 (v_n - v_{n-1})^2 \right] \\
 & \leq 2(dK)^2 [(\lambda/32\pi R(\|\varphi_t\|_\infty + 2^{1/2}))^2 \\
 & \quad + (\lambda/64\pi R(\|\varphi_t\|_\infty + 2^{1/2}))^2] \|v_n - v_{n-1}\|^2 \\
 & \leq 2(dK)^2 (\lambda/16R(\|\varphi_t\|_\infty + 2^{1/2}))^2 \|v_n - v_{n-1}\|^2.
 \end{aligned}$$

where we also have used that  $|(R\varphi + \zeta)/\beta| > M$  for  $(x, t) \in \Omega - A_\beta$  (see (3.4)). On the other hand we have

$$\begin{aligned}
 (4.3) \quad & 2(dK)^2 \iint_{\Omega} (g'((R\varphi + \zeta)/\beta))^2 (z_n - z_{n-1})^2 \\
 & \leq 2(dK)^2 \left[ \iint_{A_\beta} (g'((R\varphi + \zeta)/\beta))^2 (z_n - z_{n-1})^2 \right. \\
 & \quad \left. + \iint_{\Omega - A_\beta} (g'((R\varphi + \zeta)/\beta))^2 (z_n - z_{n-1})^2 \right] \\
 & \leq 2(dK)^2 \left[ \|g'\|_\infty^2 \iint_{A_\beta} (z_n - z_{n-1})^2 \right. \\
 & \quad \left. + (\lambda/64\pi^2 R(\|\varphi_t\|_\infty + 2^{1/2}))^2 \|z_n - z_{n-1}\|^2 \right] \\
 & \leq 2(dK)^2 \left[ \|g'\|_\infty^2 \iint_{\Omega} (\chi_{A_\beta} (z_n - z_{n-1}))^2 \right. \\
 & \quad \left. + (\lambda/64\pi^2 R(\|\varphi_t\|_\infty + 2^{1/2}))^2 \|z_n - z_{n-1}\|^2 \right] \\
 & \leq 2(dK)^2 \left[ \|g'\|_\infty^2 \left( \iint_{\Omega} (\chi_{A_\beta})^2 \right)^{1/2} \left( \iint_{\Omega} (z_n - z_{n-1})^4 \right)^{1/2} \right. \\
 & \quad \left. + (\lambda/64\pi^2 R(\|\varphi_t\|_\infty + 2^{1/2}))^2 \|z_n - z_{n-1}\|^2 \right] \\
 & \leq 2(dK)^2 [\|g'\|_\infty^2 \lambda d^2 \|z_n - z_{n-1}\|^2 / 32\pi R(\|g'\|_\infty + 1)(\|\varphi_t\|_\infty + 2^{1/2}) \\
 & \quad + (\lambda/64\pi^2 R(\|\varphi_t\|_\infty + 2^{1/2}))^2 \|z_n - z_{n-1}\|^2],
 \end{aligned}$$

where we have used that by the Sobolev imbedding theorem (see [A])  $\|z_n - z_{n-1}\|_{L^4} \leq d\|z_n - z_{n-1}\|_1$ . Also since  $g'$  is bounded,

$$\|\beta(g(\varphi + w_n)/\beta) - g((\varphi + w_{n-1})/\beta)\| \leq \|g'\|_\infty \|w_{n-1} - w_{n-2}\|_1.$$

This, (4.2) and (4.3) give

$$\begin{aligned} (4.4) \quad & 2(dK)^2 \iint_{\Omega} (g'((R\varphi + \zeta)/\beta))^2 (z_n - z_{n-1})^2 \\ & \leq 2(dK)^2 \frac{\|g'\|_\infty^4 \lambda(dK)^2 \|w_{n-1} - w_{n-2}\|^2}{32\pi R(\|g'\|_\infty + 1)(\|\varphi_t\|_\infty + 2^{1/2})} \\ & \leq 2(dK)^2 [\|g'\|_\infty^3 \lambda(dK)^2 \|w_{n-1} - w_{n-2}\|^2 / 32R(\|\varphi_t\|_\infty + 2^{1/2}) \\ & \quad + (\lambda/64\pi(\|\varphi_t\|_\infty + 2^{1/2}))^2 \|z_n - z_{n-1}\|^2]. \end{aligned}$$

Combining (4.2) and (4.4) we have

$$\begin{aligned} (4.5) \quad \|z_{n+1} - z_n\| & \leq 2^{1/2} dK \lambda \|w_n - w_{n-1}\| \\ & \quad + 2^{1/2} (dK)^2 \|g'\|_\infty^{3/2} \|w_{n-1} - w_{n-2}\| \\ & \quad \times (\lambda/32R(\|\varphi_t\|_\infty + 2^{1/2}))^{1/2}. \end{aligned}$$

Also

$$\begin{aligned} \lambda^2 \|v_{n+1} - v_n\|^2 & \leq \iint_{\Omega} (g'((\varphi + \zeta)/\beta))^2 (w_n - w_{n-1})^2 \\ & \leq 2 \iint_{\Omega} (g'((\varphi + \zeta)/\beta))^2 [(v_n - v_{n-1})^2 + (z_n - z_{n-1})^2]. \end{aligned}$$

Using now (4.2) and (4.4) we have

$$\begin{aligned} \lambda^2 \|v_{n+1} - v_n\|^2 & \leq 2(\lambda/16R(\|\varphi_t\|_\infty + 2^{1/2}))^2 \|v_n - v_{n-1}\|^2 \\ & \quad + 2[\|g'\|_\infty^3 \lambda(dK)^2 \|w_{n-1} - w_{n-2}\|^2 / 32R(\|\varphi_t\|_\infty + 2^{1/2}) \\ & \quad + (\lambda\|z_n - z_{n-1}\|/64R(\|\varphi_t\|_\infty + 2^{1/2}))^2]. \end{aligned}$$

Hence

$$\begin{aligned} (4.6) \quad \|v_{n+1} - v_n\| & \leq 2^{1/2} \|w_n - w_{n-1}\| / 8R(\|\varphi_t\|_\infty + 2^{1/2}) \\ & \quad + 2^{1/2} \|g'\|_\infty^{3/2} \|w_{n-1} - w_{n-2}\| / (32\lambda R(\|\varphi_t\|_\infty + 2^{1/2}))^{1/2}. \end{aligned}$$

Combining (4.5) and (4.6) and using the definition of  $R$  we have

$$\|w_{n+1} - w_n\| \leq (\|w_n - w_{n-1}\| + \|w_{n-1} - w_{n-2}\|) / 8.$$

Hence by Lemma 3.2 we have

$$\|w_{n+1} - w_n\| \leq k2^n (1/8)^{[n/2]}, \quad n = 2, 3, 4, \dots$$

Hence  $\sum_{n=3}^{\infty} \|w_n - w_{n-1}\|$  converges. Thus the sequence  $\{w_n = (w_n - w_{n-1}) + (w_{n-1} - w_{n-2}) + \cdots + (w_2 - w_1) + (w_1 - w_0)\}$  converges in  $L^2$  to some  $w \in L^2$ . Since  $\{w_n\}$  is bounded in  $H^1 \cap L^\infty$  we see that  $w$  also belongs to  $H^1 \cap L^\infty$ . Hence by (2.11) we see that  $w + R\phi$  is a solution to (1.2), (2.7). Hence  $u = c(w + R\phi)$  is a solution to (1.1)–(1.2) which proves the Theorem.

**REMARK.** Double checking the proofs it is easily seen that Theorem A also holds when the limits in (1.3) are allowed to be in some interval of the form  $(-s, s)$ , with  $s$  depending on the distance from  $\lambda$  to  $\{k^2 - j^2: k = 1, 2, 3, \dots, j = 0, 1, 2, 3, \dots\}$ .

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