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Synopsis

This article provides an overview of the limitations of checking out a few cases to prove conjectures in mathematics. To that end, I present a purposeful collection of number-theoretic conjectures where extensive checking of cases has found counterexamples, with emphasis on the historical backgrounds. Historical examples of long-term attempts to prove or disprove such conjectures could help individuals to realize more deeply that a limited number of observations does not guarantee the correctness of a conjecture, even though there may be many examples in its favor.

Keywords: *conjectures, counterexamples, disproving conjectures, rejected conjectures*

1. Introduction

A natural starting point to come up with mathematical rules is to look at a few of cases. Checking out a few cases may lead individuals to make some conjectures. Sometimes, these conjectures are turned into formulas or rules after the determination of their truths through mathematical proof, and sometimes, further examination disproves them. There are also cases where several mathematicians have attempted to solve a conjecture for many years, even though no proof or counterexample has been found yet.

Hanna and Barbeu [4] argue that conjectures play a significant role in mathematical development, and, like the experts in other fields, mathematicians use them. De Villiers [15] presents the function of conjecturing as "looking for an inductive pattern, generalization, analogy, and so on" (page 398).

Hanna and Barbeu [4] also express the idea that conjectures help us guide mathematical generalizations towards valid results. Moreover, a brief look at the history of mathematics shows that by testing conjectures, mathematicians have achieved important results. From an educational viewpoint, Warren and Cooper [16] note that examining the truth of a conjecture is a mathematical activity. This activity has some interesting outcomes arising from long-term attempts to prove or disprove conjectures.

This article provides an overview of the limitations of checking out a few cases to prove conjectures in mathematics. To this end, I present a purposeful collection of number-theoretic conjectures where extensive checking of cases has found counterexamples. I also include some historical background on some of these conjectures.

2. Attempts to prove or disprove conjectures

As Knuth [7] says, "Certainly not all conjectures and theorems lend themselves to constructing explanatory proofs or to generating explanatory counterexamples" (page 489). Among good examples, one can mention the four-color theorem and Fermat's last theorem, both proved in the twentieth century, after decades (or centuries) of effort. On the other hand, however, are other conjectures that have not been proved yet; many mathematicians are still working on them. These include the twin prime conjecture (the conjecture that there are an infinite number of twin primes) and the Goldbach conjecture the conjecture that every positive integer can be expressed as the sum of two prime numbers).

While making conjectures and trying to determine their validity are useful mathematical activities, proving or disproving conjectures, even as a simple school practice, is not often an easy task; and such exercises may prove to be challenging not only for the students but also for the teacher. Part of the difficulty comes from the fact that "[b]efore constructing a proof for a true statement or generating a counterexample for a false one, students and teachers need to be able to accurately decide the truth or falsity of a given proposition" [8, page 68].

Studies also emphasize that undergraduate students and mathematics teachers have difficulty verifying the truth and falsehood of a given statement.

There is a good deal of evidence here which shows that among common invalid methods used to examine the validity of a statement are working with examples to obtain conclusions [5, 6], and relying on empirical findings [1, 7] as valid criteria of proof. Rips and Asmuth [11] use the expression *proof by multiple examples* or *example-based proof strategy* to denote these types of "proofs".

The examination of multiple examples, putting significant weight on observation, helps to reinforce the correctness of a statement, so these criteria are usually considered as valid methods by students, but our students are not alone; there is much historical precedent for people believing in the truth of a statement due to a preponderance of validating instances.

3. The Weight of Observations and Finding Counterexamples

A glance at the history of mathematics shows that there is evidence for generalization of patterns based on a limited number of observations; in other words, observation of a number of examples has often been used by past mathematicians as a justification for the truth of a general assertion. According to Hanna and Barbeu [4], "[f]or the early Egyptians, Babylonians and Chinese, in fact, the weight of observational evidence was enough to justify mathematical statements as well" (page1). Shahriari [12] also asserts that once it was thought that if a result is correct for twenty consecutive steps, it will be true for the rest.[1]

What might be unpleasant to encounter for some while testing cases to verify a conjecture is a *counterexample*. During the process of the examination of a conjecture, if we encounter a counterexample at any point, we reject the conjecture, because in mathematics, "[o]ne counterexample is enough to say that the statement is not true, even though there will be many examples in its favor" [13, page 17]. This might be confusing for some students, but it is after all a fact of (mathematical) life.

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[1] This might be familiar to our students. Stylianides and Stylianides [14] quote a pupil saying "Checking 5 cases is not enough to trust a pattern in a problem. Next time I work with a pattern problem, I'll check 20 cases to be sure" (page 7).

Sometimes, a counterexample is found in the early stages. For example, in the examination of the conjecture "every positive integer is equal to the sum of two integer squares", in the case $n = 3$, a counterexample is found. Consider the conjecture "if n is prime, then $2^n - 1$ is prime". When this statement is true, then $2^n - 1$ is known as a Mersenne prime. The first value of n that gives a counter-example to the conjecture is 11.

$$2^{11} - 1 = 2047 = (23)(89).$$

Consider the proposition "there is no natural number n such that the number $\sqrt{89n^2 + 1}$ is a natural number". If one is to give a counterexample through the examination of the natural numbers, they will not be successful soon. But if one is not discouraged and keeps examining, the first counterexample will be found at value $n = 530000$ [12]:

$$89(53000)^2 + 1 = 250001000001 = 500001^2.$$

For $\sqrt{61n^2 + 1}$, one needs to be much more patient and continue until the 226153980th step:

$$61(226153980)^2 + 1 = 1766319049^2.$$

Similarly, the conjecture that the number $\sqrt{1141n^2 + 1}$ is not an integer for all $n \geq 1$ is false via the following counterexample (see [10, page 60]):

$$n = 30\,693\,385\,322\,765\,657\,197\,397\,208.$$

Stylianides and Stylianides [14] use the term *Monstrous Counterexample* to describe this type of counterexample.

According to [12], the Polish mathematician Sierpinski succeeded in finding the smallest number n with 29 digits, for which the number $991n^2 + 1$ is a perfect square, as follows:

$$n = 12\,055\,735\,790\,331\,359\,447\,442\,538\,737.$$

From this it follows that the number $\sqrt{991n^2 + 1}$ is not always an irrational number.

In the next section we look at several historical examples of conjectures introduced by mathematical celebrities, and later rejected either by others or simply through the passing of time.

4. Rejected conjectures

Most conjectures are considered stronger the longer they survive, but for some predictions, the opposite may happen. In 1532, Stifel predicted that the world would end on October 19, 1533. In 1953, Napier tried to prove that the end of the world would take place in the years between 1688 and 1700, and his book ran through twenty-one editions [2]. However, the passage of time showed that their predictions were wrong.

In mathematics, a conjecture becomes stronger over time, requiring a valid proof to make one accept it, or a counterexample to reject it. However, there are some examples of conjectures rejected after a long time. For example, in 1769 Euler conjectured that for every natural number greater than 2, at least n integers raised to the nth power are required to provide a sum that is itself an nth power. After many years, in 1966, Lander and Parkin rejected the conjecture through the following counterexample [2]:

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

Another example is Euler's conjecture that had not been proved for many years. This conjecture asserted that there were no integers, x, y, z, and w such that $x^4 + y^4 + z^4 = w^4$. After almost 200 years, Noam Elkies presented the following counterexample to disprove the conjecture [15]:

$$2,682,440^4 + 15,365,639^4 + 18,796,760^4 = 20,615,673^4.$$

As another example, we look at Fermat, who conjectured that all the Fermat numbers $F_n = 2^{2^n} + 1$ are prime. This conjecture was rejected by Euler in 1732 through the following counterexample:

$$F_5 = 2^{32} + 1 = 4294967297 = 641 \cdot 6700417.$$

According to [12], based on long tests, the Russian mathematician Grave guessed that if p was a prime number, then the number $2^{p-1} - 1$ would not be divisible by p^2. It was later shown that this proposition fails for $p = 1093$ since the number $2^{1092} - 1$ is divisible by 1093^2. However, as we have observed so far, finding a counterexample is not easy, while, as mentioned in [1], from the perspective of logic, it should perhaps be trivial because one counterexample is enough to reject a conjecture.

Sometimes the effort to find counterexamples continues even after a proof is found by others. This can be characterized as distrust in proofs. We explore this concept in the next section.

5. Distrust in Proofs

There are many problems in the history of mathematics that remained open as conjectures for a long time. We already have proofs for some of these today, and yet people try to prove or disprove them. Examples of this distrust can also be found in the history of mathematics. The problem of the *Quadrature of the Circle* is a famous example.

Here the statement is that it is not possible to construct a square with the same area as a given circle using only a finite number of operations of a straightedge and compass. A counterexample to the statement would be an algorithm for compass and straightedge that accomplished the task.

In 1775, the French Academy of Sciences was deluged with proposed counterexamples, all of which turned out to be incorrect. The Academy mathematicians eventually took the profound step of deciding that they would no longer examine any solution of this problem [2]. Note that this is over a hundred years before the logical proof that squaring the circle was impossible! All numbers constructable with compass and straightedge are algebraic, so if squaring the circle were possible, then $\sqrt{\pi}$ would need to be algebraic. Lindemann showed in 1882 that π is transcendental and therefore so is $\sqrt{\pi}$.

Today problems of *doubling the cube* and *trisecting the angle* are almost in the same condition.

Here is a still-open problem involving π. A real number is called *simply normal* whenever in its decimal expansion all digits appear with limiting frequency $1/10$. It is *normal* when the limiting frequency of any group of k numbers is $1/10^k$. It is not resolved even now if π is normal, simply normal, or neither. Consider trying to calculate π to many digits in order to test!

No finite number of calculations can resolve this question either affirmatively or negatively. That being said, all experimental evidence points to π being normal.

6. Concluding Remarks

Here, the significance of proof in mathematics becomes visible. Even long consecutive observations cannot make one sure of the validity of a conjecture and, as noted, at any time somebody may provide a counterexample and thus reject an old conjecture. As Hanna and Barbeu [4] state, "...it seemed to be accepted in the nineteenth century that all continuous functions defined on an interval had a derivative at virtually all the points in the interval. Karl Weierstrass was able to give an example of a function that was continuous but did not have a derivative anywhere" (page 6).

Our historical citations and examples here indicate that no weight of observations should suffice to conclude the truth of a conjecture. Historical discussions of concepts can throw light on knowledge construction and creation of better understanding through the path traveled. After reviewing conjectures that historically had remained unresolved for a long time, we should probably know that, at any moment, the validity of a conjecture may be rejected by a counterexample, although it may not be so soon, and "there's no endpoint at which you can stop and be sure that no counterexamples will be found" [11, page 22].

After great efforts, Guy [3] presents his best formula of The Strong Law of Small Numbers, as follows:

> There aren't enough
> Small numbers to meet the
> Many demands made of them. (page 707)

Moreover, as mentioned, counterexamples and proofs have an essential role in the development of mathematics and, as Ko and Knuth [9] state, they are important tools in deepening understanding of mathematics.

From a humanistic perspective, we know that attempts to prove or disprove conjectures have often led to new insights in mathematics. From a pedagogical viewpoint, historical examples of the longterm attempts to prove or disprove the conjectures could help students to realize more deeply that a limited number of observations does not guarantee the correctness of a conjecture, even though there may be many examples in its favor.

The various examples in the paper also point towards a better understanding of how human efforts have led to the development of mathematics through new results. Here we explored a range of conjectures; some were eventually proven to be correct and others were eventually refuted. The correct conjectures illustrate once again the central role of proof in mathematics, while the refuted ones highlight the importance of counterexamples.

References

[1] O. Buchbinder and O. Zaslavsky, "How to decide? Students' ways of determining the validity of mathematical statements", *CERME*, Volume **5** (2007), pages 561–570.

[2] H. Eves, *An introduction to the history of mathematics (6th edition)*, Saunders College Publishing, 1990.

[3] R. K. Guy, "The strong law of small numbers", *American Mathematical Monthly*, Volume **95** (1988), pages 697–712.

[4] G. Hanna and E. Barbeau, "Proof in mathematics", 2009. Available at http://www.math.toronto.edu/barbeau/hannajoint.pdf, last accessed on January 29, 2019.

[5] G. Harel and L. Sowder, "Students' proof schemes", *Research in collegiate mathematics education*, Volume **3** (1998), pages 234–282.

[6] M. Inglis and L. Alcock, "Expert and novice approaches to reading mathematical proofs", *Journal for Research in Mathematics Education*, Volume **43** Number 4 (2012), pages 358–390.

[7] E. Knuth, "Proof as a tool for learning mathematics", *Mathematics Teacher*, Volume **95** Number 7 (2002), pages 486–490.

[8] Y. Ko and E. J. Knuth, "Undergraduate mathematics majors' writing performance producing proofs and counterexamples about continuous functions", *Journal of Mathematical Behavior*, Volume **28** (2009), pages 68–77.

[9] Y. Ko and E. J. Knuth, "Validating proofs and counterexamples across content domains: Practices of importance for mathematics majors", *Journal of Mathematical Behavior*, Volume **32** (2013), pages 20–35.

[10] M. T. Michaelson, "A literature review of pedagogical research on mathematical induction", *Australian Senior Mathematics Journal*, Volume **22** Number 2 (2008), pages 57–62.

[11] L. J. Rips and J. Asmuth, "Mathematical induction and induction in mathematics", in *Induction*, edited by A. Feeney and E. Heit (Cambridge, Cambridge University Press, 2007), pages 248–268.

[12] P. Shahriari, *Mathematical induction (3rd ed)*, Ministry of education, Research and Educational Planning, Madreseh Publication, 2001.

[13] A. Stefanowicz, *Proofs and mathematical reasoning*, University of Birmingham, 2014.

[14] A. J. Stylianides and G. J. Stylianides, "'Cognitive conflict' as a mechanism for supporting developmental progressions in students' knowledge about proof", 2008. Article available at the website of the 11th International Congress on Mathematical Education, under Topic Study Group 18 (http://tsg.icme11.org/tsg/show/19). Monterrey, Mexico, 2008.

[15] M. De Villiers, "The role and function of quasi-empirical methods in mathematics", *Canadian Journal of Science, Mathematics and Technology Education*, Volume **4** Number 3 (July 2004), pages 397–418.

[16] E. Warren and T. Cooper, "Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking", *Educational Studies in Mathematics*, Volume **67** (2008), pages 171–185.