11-1-1980

Existence and Uniqueness for a Variational Hyperbolic System without Resonance

Peter W. Bates
Texas A & M University - College Station

Alfonso Castro
Harvey Mudd College

Recommended Citation

Bates, Peter W. and Alfonso Castro. "Existence and uniqueness for a variational hyperbolic system without resonance" Nonlinear Analysis TMA, Vol. 4, No. 6(1980), pp. 1151-1156.

Nonlinear Analysis, Theory, Methods & Applications, Vol. 4, No. 6, pp. 1151–1156.

0362-546X/80/1101-1151 $02.00/0

EXISTENCE AND UNIQUENESS FOR A VARIATIONAL HYPERBOLIC SYSTEM WITHOUT RESONANCE

PETER W. BATES* and ALFONSO CASTRO†

* Texas A&M University, College Station, Texas 77843, U.S.A.; and † Centro de Investigacion del IPN, Apartado Postal 14-740, Mexico D.F.

(*Received* 21 *May* 1979; *revised* 15 *October* 1979)

Key Words and Phrases: Nonlinear wave equation, system, minimax, critical point, Galerkin approximation, weak solution, boundary value problem.

1. INTRODUCTION

IN THIS PAPER we study the existence of weak solutions of the problem

$$\begin{cases} \Box u + \nabla G(u) = f(t, x) & (t, x) \in \Omega \equiv (0, \pi) \times (0, \pi) \\ u(t, x) = 0 & (t, x) \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Box$ is the wave operator $\partial^2/\partial t^2 - \partial^2/\partial x^2$, $G: \mathbf{R}^n \to \mathbf{R}$ is a function of class C^2 such that $\nabla G(0) = 0$ and $f: \bar{\Omega} \to \mathbf{R}^n$ is a continuous function having first derivative with respect to t in $(L_2(\Omega))^n$ and satisfying

$$f(0, x) = f(\pi, x) = 0 \tag{1.2}$$

for all $x \in [0, \pi]$.

We assume that there exist two $n \times n$ real symmetric matrices $A \leqslant B$ with eigenvalues $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_n$ and $\beta_1 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_n$ respectively, such that

$$\left(\bigcup_{i=1}^{n} [\alpha_i, \beta_i]\right) \cap \{k^2 - j^2; k, j \in \mathbf{N}\} = \varnothing \tag{1.3}$$

and

$$A \leqslant (\partial^2 G(u)/\partial u_i \partial u_j) \leqslant B \qquad \text{for all} \quad u \in \mathbf{R}^n. \tag{1.4}$$

Our main result is

THEOREM A. If (1.3) and (1.4) hold then (1.1) has a unique weak solution. In addition, such a weak solution belongs to $(\mathring{H}^1(\Omega))^n$.

A result analogous to Theorem A was proved by Ahmad [1] (existence) and Lazer [2] (uniqueness) for a second order system of ODEs.

Our interest in proving Theorem A came from noticing that a simple extension of the results in [4] to systems, shows that (1.1) has a unique weak solution if there exist two real numbers p

and q such that $[p, q] \cap \{k^2 - j^2 : k, j \in \mathbf{N}\} = \varnothing$ and

$$pI \leqslant (\partial^2 G(u)/\partial u_i \partial u_j) \leqslant qI \qquad \text{for all} \quad u \in \mathbf{R}^n. \tag{1.4'}$$

Unfortunately, the methods of [4] do not seem to extend to cover the case when we assume (1.4) rather than (1.4′).

Let us denote by $R(\square)$ and $\mathrm{Ker}(\square)$ the range and kernal respectively of the operator $\square : D(\square) \subset (L^2(\Omega))^n \to (L_2(\Omega))^n$ with Dirichlet boundary condition. We prove Theorem A using a Galerkin approximation procedure. At each finite dimensional step we prove the existence of an approximate solution by applying a minimax theorem due to Lazer–Landesman–Meyers [3]. Condition (1.4) allows us to give an *a priori* estimate in $(L_2(\Omega))^n$ for the approximate solutions. The fact that the operator $\square$ with Dirichlet boundary condition has a compact inverse on $R(\square)$ gives us the existence of $u \in R(\square)$ and $v \in \mathrm{Ker}(\square)$ so that $u + v$ satisfies (1.1) in a weak sense.

The methods used here apply to (1.1) with other boundary conditions (Neumann, periodic, mixed) with very little modification.

Finally we remark that if condition (1.4) is replaced by: there exists $r > 0$ such that

$$A \leqslant (\partial^2 G(u)/\partial u_i \partial u_j) \leqslant B \qquad \text{for} \quad \|u\| \geqslant r, \tag{1.4''}$$

then it can be proved that (1.1) has a solution. This solution is not necessarily unique and in $(\mathring{H}^1(\Omega))^n$. Assuming (1.4) rather than (1.4″) gives us the advantage of obtaining a much simpler variational characterization of the approximate solutions which has numerical analytic implications (see [3, Section 7]).

2. NOTATIONS AND PRELIMINARY LEMMAS

We let $\{a_i; i = 1, \ldots, n\}$ and $\{b_i; i = 1, \ldots, n\}$ be orthonormal bases of $\mathbf{R}^n$ such that

$$Aa_i = \alpha_i a_i \qquad Bb_i = \beta_i b_i \qquad \text{for} \quad i = 1, \ldots, n. \tag{2.1}$$

We denote by $\phi_{kl} : \bar{\Omega} \to \mathbf{R}$ the function defined by $\phi_{kl}(t, x) = (2/\pi)\sin(kt)\sin(lx)$. Clearly $\{\phi_{kl}; k, l \in \mathbf{N}\}$ is a complete orthonormal set in $L_2(\Omega)$. Moreover, $\{\phi_{kk}; k = 1, 2, \ldots\}$ is a complete orthonormal set in $\mathrm{Ker}(\square)$.

For each positive integer N we define

$$X_N = \left\{ \sum_{i,k,l} \mu_{ikl}\phi_{kl}b_i; 1 \leqslant i \leqslant n, k^2 - l^2 > \beta_i, |k^2 - l^2| \leqslant N, k^2 \leqslant N, \mu_{ikl} \in \mathbf{R} \right\},$$

$$Y_N = \left\{ \sum_{i,k,l} \mu_{ikl}\phi_{kl}b_i; 1 \leqslant i \leqslant n, k^2 - l^2 < \beta_i, |k^2 - l^2| \leqslant N, k^2 \leqslant N, \mu_{ikl} \in \mathbf{R} \right\},$$

$$Z_N = \left\{ \sum_{i,k,l} \mu_{ikl}\phi_{kl}a_i: 1 \leqslant i \leqslant n, k^2 - l^2 < \alpha_i, |k^2 - l^2| \leqslant N, k^2 \leqslant N, \mu_{ikl} \in \mathbf{R} \right\},$$

and

$$E_N = \left\{ \sum_{k,l} \phi_{kl}c_{kl}; |k^2 - l^2| \leqslant N, k^2 \leqslant N, c_{kl} \in \mathbf{R}^n \right\}. \tag{2.2}$$

Clearly $X_N \oplus Y_N = E_N$ and $\bigcup_{N=1}^{\infty} E_N$ is dense in $(L_2(\Omega))^n$. We let $\langle , \rangle_0$ and $\| \|_0$ denote the usual inner product and norm in $(L_2(\Omega))^n$, respectively, and let $\| \|_1$ denote the norm in $(\mathring{H}^1(\Omega))^n$ given by $\|v\|_1^2 = \int_\Omega (|\partial v/\partial t|^2 + |\partial v/\partial x|^2)$.

We will need the main result of [3] restated here as:

LEMMA 2.1. Let j be a C^2 functional on a Hilbert space H. Suppose X and Z are closed subspaces of H (not necessarily orthogonal) so that X is finite dimensional and $H = X \oplus Z$. If there exist constants $m_1, m_2 > 0$ such that

$$\langle D^2 j(u)w, w\rangle \geqslant m_1 \|w\|^2 \qquad \text{and}$$
$$\langle D^2 j(u)v, v\rangle \leqslant -m_2 \|v\|^2 \qquad \text{for all} \quad u \in H, w \in Z, v \in X,$$

then there exists a unique $u_0 \in H$ such that $\nabla j(u_0) = 0$ and $j(u_0) = \max_{x \in X} \min_{y \in Z} j(x + y)$.

Define the functional $J: (\mathring{H}^1(\Omega))^n \to \mathbf{R}$ by

$$J(u) = \int_\Omega [(\partial u/\partial x, \partial u/\partial x) - (\partial u/\partial t, \partial u/\partial t)]/2 + G(u) - (f, u).$$

where (,) denotes the usual inner product in $\mathbf{R}^n$. Since we are assuming that G is of class C^2 and that $(\partial^2 G(u)/\partial u_i \partial u_j)$ is uniformly bounded it follows that J is of class C^2. We observe that if J_N denotes the restriction of J to E_N then

$$\langle \nabla J_N(u), v\rangle = \int_\Omega (\partial u/\partial x, \partial v/\partial x) - (\partial u/\partial t, \partial v/\partial t) + (\nabla G(u), v) - (f, v) \tag{2.3}$$

for all $u, v \in E_N$, where $\langle \ , \ \rangle$ denotes the duality pairing. We denote by $D^2 J_N(u)$ the Hessian of J_N at u.

LEMMA 2.2: For each positive integer N there exists a unique $u_N \in E_N$ such that $\nabla J_N(u_N) = 0$. Moreover,

$$J_N(u_N) = \max_{x \in X_N} \min_{z \in Z_N} J(x + z).$$

Proof. Fix $u \in E_N$; then for each $w = \sum_{i,k,l} \mu_{ikl} \phi_{kl} a_i \in Z_N$

$$\begin{aligned} \langle D^2 J_N(u)w, w\rangle &= \int_\Omega |\partial w/\partial x|^2 - |\partial w/\partial t|^2 + ((\partial^2 G(u)/\partial u_i \partial u_j)w, w) \qquad (2.4) \\ &\geqslant \int_\Omega |\partial w/\partial x|^2 - |\partial w/\partial t|^2 + (Aw, w) \\ &= \sum_{i,k,l} (l^2 - k^2 + \alpha_i)\mu_{ikl}^2 \geqslant m_1 \|w\|_0^2 \end{aligned}$$

where $m_1 = \min\{l^2 - k^2 + \alpha_i : 1 \leqslant i \leqslant n, l, k \in \mathbf{N}, l^2 - k^2 + \alpha_i > 0\}$. Similarly, for $v \in X_N$,

$$\langle D^2 J_N(u)v, v\rangle \leqslant \sum_{i,k,l} (l^2 - k^2 + \beta_i)\, \mu_{ikl}^2 \leqslant -m_2 \|v\|_0^2 \tag{2.5}$$

where $m_2 = \min\{k^2 - l^2 - \beta_i : 1 \leqslant i \leqslant n,\ l, k \in \mathbb{N},\ l^2 - k^2 + \beta_i < 0\}$. From this we conclude that $X_N \cap Z_N = \{0\}$. It is easy to see that $\dim Z_N = \dim Y_N$ and so $E_N = X_N \oplus Z_N$. In addition, (2.4) and (2.5) show that the hypotheses of Lemma 2.1 are satisfied, which completes the proof.

We now write $u_N = u_N^1 + u_N^2$ with $u_N^1 \in X_N$ and $u_N^2 \in Z_N$. By Lemma 2.2 we have

$$\begin{aligned}
0 &= \langle \nabla J_N(u_N), u_N^1 - u_N^2 \rangle \\
&= \int_\Omega (\partial u_N^1/\partial x, \partial u_N^1/\partial x) - (\partial u_N^1/\partial t, \partial u_N^1/\partial t) - (\partial u_N^2/\partial x, \partial u_N^2/\partial x) + (\partial u_N^2/\partial t, \partial u_N^2/\partial t) \\
&+ \left(\int_0^1 (\partial^2 G(su_N)/\partial u_i \partial u_j) u_N \, ds, (u_N^1 - u_N^2)\right) - (f, u_N^1 - u_N^2) \\
&\leqslant \int_\Omega (\partial u_N^1/\partial x, \partial u_N^1/\partial x) - (\partial u_N^1/\partial t, \partial u_N^1/\partial t) - (\partial u_N^2/\partial x, \partial u_N^2/\partial x) + (\partial u_N^2/\partial t, \partial u_N^2/\partial t) + (Bu_N^1, u_N^1) \\
&- (Au_N^2, u_N^2) + \|f\|_0 \cdot (\|u_N^1\|_0 + \|u_N^2\|_0) \\
&\leqslant -m_1\|u_N^1\|_0^2 - m_2\|u_N^2\|_0^2 + \|f\|_0(\|u_N^1\|_0 + \|u_N^2\|_0),
\end{aligned} \tag{2.6}$$

where m_1 and m_2 are as in Lemma 2.2. Since m_1 and m_2 are positive and independent of N inequality (2.6) proves that $\{u_N\}_N$ is bounded in $(L_2(\Omega))^n$.

It is well documented that for each $h \in R(\square)$ there exists a unique w in the orthogonal complement of Ker($\square$) which is a weak solution of $\square u = h$ in Ω, $u \equiv 0$ on $\partial\Omega$. Moreover, $w \in (\mathring{H}^1(\Omega))^n$ and there exists a constant $c > 0$ such that

$$\|w\|_1 \leqslant c\|h\|_0. \tag{2.7}$$

Now we write $u_N = v_N + w_N$ with $v_N \in R(\square)$ and $w_N \in \text{Ker}(\square)$. We let Q_N denote the orthogonal projection onto $E_N \cap R(\square)$. Because of Lemma 2.2 we have $\square v_N = Q_N(f - \nabla G(u_N))$ in Ω, $v_N \equiv 0$ on $\partial\Omega$. Hence by (2.7), we have

$$\begin{aligned}
\|v_N\|_1 &\leqslant c\left(\|f\|_0 + \left\|\int_0^1 (\partial^2 G(su_N)/\partial u_i \partial u_j) u_N \, ds\right\|_0\right) \\
&\leqslant c(\|f\|_0 + \sup\{|\alpha_1| \, \|u_N\|_0, |\beta_n| \, \|u_N\|_0; N = 1, 2, \ldots\}) \\
&\equiv K.
\end{aligned} \tag{2.8}$$

3. PROOF OF THEOREM A

First we claim that $\{u_N\}_N$ is bounded in $(\mathring{H}^1(\Omega))^n$. To show this, by (2.8) it is sufficient to prove that $\{w_N\}_N$ is bounded in $(\mathring{H}^1(\Omega))^n$. To do so we write $w_N = w_N^+ + w_N^-$, where $w_N^- \in X_N$ and $w_N^+ \in Z_N$. Since $v = \partial^2 w_N^+/\partial t^2 - \partial^2 w_N^-/\partial t^2 \in E_N \cap \text{Ker}(\square)$, by Lemma 2.2 we have

$$0 = \int_\Omega (\nabla G(u_N), v) - (f, v). \tag{3.1}$$

Therefore, integrating by parts, using (1.2) and $\nabla G(0) = 0$ and noting that $\|\partial w_N/\partial t\|_0 = \|\partial w_N/\partial x\|_0$

we have

$$\|\partial f/\partial t\|_0(\|w_N^+\|_1 + \|w_N^-\|_1) \geqslant \int_\Omega ((\partial^2 G(u_N)/\partial u_i \partial u_j)(\partial u_N/\partial t), \partial w_N^+/\partial t - \partial w_N^-/\partial t)$$

$$\geqslant -C_1\|w_N\|_1 + \int_\Omega (A\partial w_N^+/\partial t, \partial w_N^+/\partial t) - \int_\Omega (B\partial w_N^-/\partial t, \partial w_N^-/\partial t)$$

$$\geqslant -C_1\|w_N\|_1 + C_2\|w_N^+\|_1^2 + C_3\|w_N^-\|_1^2, \tag{3.2}$$

where C_1 comes from the facts that $(\partial^2 G(u)/\partial u_i \partial u_j)$ and $\{\|v_N\|_1\}_N$ are uniformly bounded, $2C_2 = \min\{\alpha_i; \alpha_i > 0\}$ and $2C_3 = \min\{-\beta_i; \beta_i < 0\}$. From (3.2) it is clear that $\{\|u_N\|_1\}$ is bounded. Let $(v_0, w_0) \in (\mathring{H}^1(\Omega))^n \times (\mathring{H}^1(\Omega))^n$ be such that some subsequence $\{(v_{N_j}, w_{N_j})\}_j$ converges weakly in $(\mathring{H}^1(\Omega))^n \times (\mathring{H}^1(\Omega))^n$ to (v_0, w_0).

We claim that $u_0 = v_0 + w_0$ is a weak solution of (1.1). Let $\phi: \Omega \to \mathbf{R}^n$ be any C^∞ function with compact support in Ω. Let ϕ_j be the orthogonal projection of ϕ on E_{N_j}. Since $\bigcup_{j=1}^\infty E_{N_j}$ is dense in $(L_2(\Omega))^n$ we see that $\phi_j \to \phi$ in $(L_2(\Omega))^n$ and $\Box\phi_j \to \Box\phi$ in $(L_2(\Omega))^n$. Therefore, using Lemma 2.2 we have

$$\int_\Omega (v_0, \Box\phi) + (\nabla G(u_0), \phi) - (f, \phi)$$

$$= \int_\Omega (v_0, \Box(\phi - \phi_j)) + (\nabla G(u_0), \phi - \phi_j) - (f, \phi - \phi_j)$$

$$+ (\nabla G(u_0) - \nabla G(v_{N_j} + w_{N_j}), \phi_j) + (v_0 - v_{N_j}, \Box\phi_j). \tag{3.3}$$

It is clear that the right-hand side of (3.3) tends to zero as j tends to infinity. Hence u_0 is a weak solution of (1.1) which by construction is in $(\mathring{H}^1(\Omega))^n$ and this proves the existence part of Theorem A.

Finally, we prove that (1.1) has at most one weak solution. Suppose that u^1 and u^2 are two such solutions. For $i = 1, 2$, let $u_N^i = x_N^i + z_N^i$ be the projection of u^i onto E_N, where $x_N^i \in X_N$ and $z_N^i \in Z_N$. Let $v_N = x_N^1 - x_N^2$, $w_N = z_N^1 - z_N^2$. We have

$$0 = \int_\Omega \left\{(u^1 - u^2, \Box(v_N - w_N)) + \left(\int_0^1 (\partial^2 G(u^2 + s(u^1 - u^2))/\partial u_i \partial u_j)(u^1 - u^2)\,ds, v_N - w_N\right)\right\}$$

$$= \int_\Omega \left\{(v_N + w_N, \Box(v_N - w_N)) + \left(\int_0^1 (\partial^2 G(u^2 + s(u^1 - u^2))/\partial u_i \partial u_j)(v_N + w_N)\,ds, v_N - w_N\right)\right.$$

$$+ \left(\int_0^1 (\partial^2 G(u^2 + s(u^1 - u^2))/\partial u_i \partial u_j)\right.$$

$$\left.\left.(u^1 - u_N^1 + u_N^2 - u^2)\,ds, v_N - w_N\right)\right\}$$

$$\leqslant \int_\Omega \{(\Box v_N + Bv_N, v_N) - (\Box w_N + Aw_N, w_N)\} + C(\|u^1 - u_N^1\|_0 + \|u^2 - u_N^2\|_0)$$

$$\leqslant -m_2\|v_N\|_0^2 - m_1\|w_N\|_0^2 + C(\|u^1 - u_N^1\|_0 + \|u^2 - u_N^2\|_0).$$

Consequently, v_N and w_N tend to zero as $N \to \infty$, and so $u^1 = u^2$.

REFERENCES

1. AHMAD S., An existence theorem for periodically perturbed conservative systems, *Mich Math. J.* **20**, 385–392 (1973).
2. LAZER A. C., Application of a lemma on bilinear forms to a problem in nonlinear oscillations, *Proc. Am. math. Soc.* **33**, 311–327 (1970).
3. LAZER A. C., LANDESMAN E. M. & MEYERS D. R., On saddle point problems in the calculus of variations, the Ritz algorithm, and monotone convergence, *J. math. Analysis Applic.* **52**, 594–614 (1975).
4. MAWHIN J., *Solutions périodiques d'equations aux dérivées partielles hyperboliques nonlinéares*, pp. 301–315, Mélanges Th. Vogel, Presses Univ. de Bruxelles (1978).