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Multiple Solutions for a Dirichlet Problem with Jumping Nonlinearities, II

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1. INTRODUCTION

In this paper we consider the existence of solutions for the problem

$$-x''(t) = g(x(t)) - p(t) - c, \qquad t \in [0, 1] \tag{1.1}$$

$$x(0) = x(1) = 0, \tag{1.2}$$

where $p(t)$ is a continuous function and c is a constant. We assume that g is of class C^1, and that g is positive and strictly increasing on $(0, \infty)$. In addition, we assume that there exist real numbers M and $\rho > 0$ such that

$$\lim_{u \to -\infty} (g(u)/u) = M, \tag{1.3}$$

$$\lim_{u \to \infty} (g(u)/u^{1+\rho}) = \infty. \tag{1.4}$$

Our main result is:

THEOREM 1.1. *If p and g are as above, then there exists an increasing sequence $\{c(n) := c(n, p); n > 2(\max\{0, M\})^{1/2}/\pi\}$ tending to $+\infty$ such that*

(A) *If $n > 1 + 2(\max\{0, M\})^{1/2}/\pi$ and $c > c(n)$ then (1.1)–(1.2) have a solution with n interior zeroes and $x'(0) > 0$.*

(B) *If $n > 2(\max\{0, M\}^{1/2})/\pi$ and $c > c(n)$ then (1.1)–(1.2) have a solution with n interior zeroes and $x'(0) < 0$.*

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Further if $g(u)-\pi^2 u$ is bounded below then there exists c_* such that for $c<c_*$ Eqs. (1.1)–(1.2) *have no solution.*

Our proofs are based on the analysis of the phase-plane. Most of our arguments resemble those of [2], where under the hypothesis $\lim_{|u|\to\infty}(g(u)/u)=\infty$ the existence of infinitely many solutions for (1.1)–(1.2) was proved. We do not make use of the maximum principle. This is why, unlike in previous work (see [1, 5] and references therein), we are not restricted to $M<\lambda_1=\pi^2$.

The outline of the proof of Theorem 1.1 is as follows. In the first place show that there exist constants $\alpha\in(0,1)$ and c^* such that if $c>c^*$ and $|a|\geqslant 1$ then the solution to Eq. (1.1) satisfying $x(0)=0$, $x'(0)=ac^{\alpha}$ exists on $[0,2]$ (see Lemma 2.1) and that its zeroes on $[0,2]$ are non-degenerate (see Lemma 2.2). That is, the orbit starting at $(0, ac^{\alpha})$ does not go through the origin in the (x,x')-plane for $t\in[0,2]$. Thus for such (t,a,c) a continuous argument function $\theta(t,a,c)$ is well defined (see [4]). Next we show that if $|a|\in[1,3]$ then $\lim_{c\to\infty}\theta(1,a,c)=\infty$ (see Lemmas 2.3 and 2.4) and for $c>c^*$, $\lim_{|a|\to\infty}\theta(1,a,c)\leqslant\pi+2(\max\{0,M\})^{1/2}$ (see Lemmas 2.5 and 2.6). Using these two limits and the intermediate value theorem we conclude that given suitably large n (see Theorem 1.1) there exists $c(n)$ such that if $c>c(n)$ then for some a, $\theta(1,a,c)=(n+1)\pi$, which by the definition of θ implies that the corresponding orbit is a solution to (1.1)–(1.2) with n interior zeroes.

Theorem 1.1 extends the work of [3], where the case $p=0$ was studied using the so-called quadrature method. Our motivation to study problem (1.1)–(1.2) is due to the results of [5], where $M<\pi^2$ and $\lim_{u\to\infty}(g(u)/u)\in(N^2\pi^2,(N+1)^2\pi^2)$. In turn, A. C. Lazer and P. J. McKenna in [5] were motivated by developments that go back to the classical Ambrosetti–Prodi result [1].

2. Phase-Plane Analysis

In what follows we extend p to $[0,\infty)$ as $p(x)=p(1)$ for $x\geqslant 1$. Also, without loss of generality, we can assume that

$$\max\{|p(x)|; x\geqslant 0\}\leqslant 1 \qquad \text{and} \qquad g(0)=0. \tag{2.1}$$

Lemma 2.1. *If a, b, c are arbitrary numbers then the solution to Eq.* (1.1) *satisfying $x(0)=a$, $x'(0)=b$ exists for all t.*

Proof. Since g is of class C^1, by (1.3), (1.4), and (2.1), we see that there exists a real number $M_1\geqslant 0$ such that $g_1(u):=g(u)+M_1u$ satisfies

$$ug_1(u)\geqslant 2u^2 \qquad \text{for all } u\in R. \tag{2.2}$$

Let $G_1(u) = \int_0^u g_1(s)\, ds$ and

$$E_1(t) = (x'(t))^2/2 + G_1(x(t)),$$

where $x(t)$ is the solution to (1.1) satisfying $x(0) = a$, $x'(0) = b$. Hence

$$\begin{aligned} \frac{dE_1}{dt} &= (x'(t))[x''(t) + g_1(x(t))] \\ &= x'(t)[-g(x(t)) + p(t) + c + g(x(t)) + M_1 x(t)] \\ &\leqslant |c+1| \sqrt{2E_1} + M_1 \sqrt{2E_1}\, |x(t)|, \end{aligned} \tag{2.3}$$

where we have used that $G_1 \geqslant 0$ (see (2.2)). Also from (2.2) we have $G_1(u) \geqslant u^2$ for all $u \in R$. This, (2.3), and the fact that $E_1 \geqslant 0$ yield

$$\begin{aligned} \frac{dE_1}{dt} &\leqslant |c+1| \sqrt{2E_1} + M_1 \sqrt{2E_1} \cdot \sqrt{E_1} \\ &\leqslant \sqrt{2}\, |c+1| + (|c+1| + M_1 \sqrt{2})\, E_1. \end{aligned} \tag{2.4}$$

Hence if we let $k = -(|c+1| + \sqrt{2}\, M_1)$, then multiplying (2.4) by e^{kt} we have

$$\frac{d}{dt}(e^{kt} E_1) \leqslant e^{kt} \sqrt{2}\, |c+1|. \tag{2.5}$$

Thus we have

$$e^{kt} E_1(t) - E_1(0) \leqslant \int_0^u e^{ks} \sqrt{2}\, |c+1|\, ds,$$

i.e.,

$$e^{kt} E_1(t) \leqslant \frac{b^2}{2} + G_1(a) + \frac{\sqrt{2}\, |c+1|\, e^{kt}}{k} - \frac{\sqrt{2}\, |c+1|}{k},$$

i.e.,

$$E_1(t) \leqslant e^{-kt} \left\{ \frac{b^2}{2} + G_1(a) - \frac{\sqrt{2}\, |c+1|}{k} \right\} + \frac{\sqrt{2}\, |c+1|}{k}. \tag{2.6}$$

Therefore E_1 is bounded on bounded intervals. Hence, since $G_1(u) \geqslant u^2$ (see (2.2)), (x, x') is bounded on bounded intervals. This proves that (x, x') does not blow up in finite time, which proves that (x, x') is defined for all time and the lemma is proven.

Now let α be defined by

$$\alpha = (\rho + 3)/(2\rho + 3). \tag{2.7}$$

Let $x(t, \alpha)$ denote the solution to (1.1) satisfying $x(0)=0$, $x'(0)=ac^{\alpha}$ with $a \in R$. Let $r^2(t, a)=(x(t, a))^2+(x'(t, a))^2$. Now for $a \in [1, \infty)$ and for $T>0$ such that $r(t, a)>0$ on $[0, T]$, there exists the function $\theta(t, a)$ such that

$$x(t, a)=r(t, a)\sin(\theta(t, a)), \tag{2.8}$$

$$x'(t, a)=r(t, a)\cos(\theta(t, a)), \tag{2.9}$$

$$\theta(0, a)=0. \tag{2.10}$$

A simple computation shows that

$$\theta'(t, a)=\frac{(x'(t, a))^2-x(t, a)\,x''(t, a)}{r^2(t, a)} \tag{2.11}$$

$$=(\cos^2(\theta(t, a)))+\frac{g(r(t, a)\sin(\theta(t, a)))\sin(\theta(t, a))}{r(t, a)}$$

$$-(p(t)+c)\frac{\sin(\theta(t, a))}{r(t, a)}. \tag{2.12}$$

Since the dependence of the various functions on a is clear from the context we will eventually drop this variable.

LEMMA 2.2. *There exists c^* such that if $c>c^*$ and $|a| \geqslant 1$ then $r(t, a)>0$ for all $t \in [0, 2]$.*

Proof. Let M_1, G_1, and E_1 be as before. Suppose $E_1(t', a)=0$ for some $t' \in (0, 2]$ and $E_1>0$ for $t \in [0, t')$. By the continuity of E_1 we can assume that there exists $t''<t'$ such that $E_1(t'', a)=c^{2\alpha}/2$ and $0<E_1(t, a)<c^{2\alpha}/2$ for all $t \in (t'', t')$. Then, since $G_1 \geqslant 0$, we have $|x'|<c^{\alpha}$ on (t'', t'). Also from (1.4) we obtain

$$x(t'')<Kc^{2\alpha/(2+\rho)}, \tag{2.13}$$

with $K>0$ a constant independent of **c** and **a**. Now since $dE_1/dt=(c+p(t)+M_1x(t))\,x'(t)$, integrating in (t'', t') we obtain

$$0=E_1(t')=E_1(t'')+\int_{t''}^{t'}(c+p(s)+M_1x(s))\,x'(s)\,ds$$

$$\geqslant\frac{c^{2\alpha}}{2}-cx(t'')-2c^{\alpha}-\frac{M_1}{2}x^2(t'')$$

$$>\frac{c^{2\alpha}}{2}-Kc^{1+(2\alpha/(2+\rho))}-2c^{\alpha}-\frac{M_1K^2}{2}c^{4\alpha/(2+\rho)}. \tag{2.14}$$

But by the definition of α (see (2.7)), $2\alpha > \max\{1 + 2\alpha/(2+\rho), 4\alpha/(2+\rho)\}$ and hence (2.14) is not possible for c large. That is, there exists c^* such that for $c > c^*$, $E_1(t, a) > 0$ for all $t \in [0, 2]$. Hence $r(t, a) > 0$ if $|a| \geqslant 1$, $t \in [0, 2]$, and $c > c^*$, which proves Lemma 2.2.

LEMMA 2.3. *Let $a_1 \in [1, 3]$, $T \in [0, 1]$, and K be a non-negative integer. Given any positive integer n there exists $d = d(n)$ such that if $c > d$, $\theta(T, a_1) = 2K\pi$, and $x'(T, a_1) = ac^\alpha$ with $a \in [1, 3]$, then there exists $T_1 \in [0, 1/n]$ such that $\theta(T + T_1, a_1) = (2K+1)\pi$. Moreover*

$$|x'(T + T_1, a_1) + ac^\alpha| < c^\alpha/(2n^2). \tag{2.15}$$

Proof. For the sake of clarity we divide this proof in three steps. Throughout this proof ε denotes a fixed real number satisfying

$$0 < \varepsilon < (1-\alpha)/2. \tag{2.16}$$

Step 1. There exists d_1 such that if $c > d_1$ then there exists $t_1 \leqslant T + (c^{-\varepsilon}/2)$ such that $x(t_1, a_1) = g^{-1}(c - 1 + c^{\alpha+\varepsilon})$ and $x'(t_1) > 0$.

First note that by (2.16), $\alpha + \varepsilon = (\alpha+1)/2 < 1$ since $\alpha < 1$. Further by (2.7) and (2.16) we have $1/(\rho+1) + \varepsilon - \alpha < 0$. These and (1.4) imply

$$\lim_{c \to \infty} \{c^\varepsilon [g^{-1}(c - 1 + c^{\alpha+\varepsilon})]/c^\alpha\} = 0. \tag{2.17}$$

Further, by (2.7) we have $1 + 1/(\rho+1) - 2\alpha < 0$. Hence

$$\lim_{c \to \infty} \{(c+1)[g^{-1}(c - 1 + c^{\alpha+\varepsilon})]/c^{2\alpha}\} = 0. \tag{2.18}$$

From (2.17) it follows that there exists $\delta_1 > 0$ such that if $c > \delta_1$ then

$$\cos^2(\arctan[g^{-1}(c - 1 + c^{\alpha+\varepsilon})/c^\alpha]) > 0.95. \tag{2.19}$$

Since the derivative with respect to z of $\sin(\arctan(z))$ at $z = 0$ is 1, by (2.18) we see that there exists $\delta_2 > 0$ such that if $c > \delta_2$ then

$$(c+1)\sin[\arctan(2g^{-1}(c - 1 + c^{\alpha+\varepsilon})/c^\alpha)]/(c^\alpha/2) < 0.05. \tag{2.20}$$

Again by (2.17) we see that there exists $\delta_3 > 0$ such that if $c > \delta_3$ then

$$g^{-1}(c - 1 + c^{\alpha+\varepsilon})/(ac^\alpha) < 0.1c^{-\varepsilon} \tag{2.21}$$

and

$$\arctan[2g^{-1}(c - 1 + c^{\alpha+\varepsilon})/(ac^\alpha)] \leqslant 3g^{-1}(c - 1 + c^{\alpha+\varepsilon})/(ac^\alpha). \tag{2.22}$$

Let d_1 be defined by

$$d_1 = \max\{\delta_1, \delta_2, \delta_3, c^*, 2\}, \tag{2.23}$$

where c^* is as in Lemma 2.2.

Let $c > d_1$. Since by hypothesis $\theta(T, a_1) = 2K\pi$, by (2.8) we have $x(T, a_1) = 0$. This and the assumption $x'(T, a_1) = ac^\alpha > 0$ imply that there exists $t > T$ such that $x(s) \in (0, g^{-1}(c - 1 + c^{\alpha+\varepsilon}))$ for all $s \in (T, t)$. Thus the set $S = \{t > T;\ x(s) \in (0, g^{-1}(c - 1 + c^{\alpha+\varepsilon}))\}$ is non-empty. Therefore we have the following alternatives. Either

(A) The interval $[T, T + (c^{-\varepsilon}/2)] \subset S$, *or*

(B) There exists $t \in S$, $t \leqslant T + (c^{-\varepsilon}/2)$ such that $x(t) = g^{-1}(c - 1 + c^{\alpha+\varepsilon})$, *or*

(C) There exists $t \in S$, $t \leqslant T + (c^{-\varepsilon}/2)$ such that $x(t) = 0$.

Now we will show that neither (A) nor (C) is possible. Let $t \in S$ with $t \leqslant T + (c^{-\varepsilon}/2)$. Integrating (1.1) on (T, t) we have

$$\begin{aligned} x'(t) &= ac^\alpha + \int_T^t (p(\tau) + c - g(x(\tau)))\, d\tau \\ &\geqslant ac^\alpha + (c-1)(t-T) - (c - 1 + c^{\alpha+\varepsilon})(t-T) \end{aligned}$$

since $x(\tau) \leqslant g^{-1}(c - 1 + c^{\alpha+\varepsilon})$ and $g(x)$ is increasing in $(0, \infty)$. But $t - T \leqslant c^{-\varepsilon}/2$. Hence we have

$$x'(t) \geqslant ac^\alpha - (c^\alpha/2) \geqslant ac^\alpha/2. \tag{2.24}$$

Thus by the definition of $r(t, a)$ we have $r(s, a_1) \geqslant ac^\alpha/2$ for all $s \in [T, t)$. Now since $\theta(s, a_1) = \arctan(x/x')$ we get

$$\theta(s, a_1) \leqslant \arctan[2g^{-1}(c - 1 + c^{\alpha+\varepsilon})/(ac^\alpha)] \tag{2.25}$$

for all $s \in [T, t)$. Replacing this in (2.20) we have

$$(c+1)\sin(\theta(s, a_1))/r(s, a_1) \leqslant 0.05 \tag{2.26}$$

for $c > d_1$ and $s \in [T, t)$. Hence from (2.12), (2.19), and (2.26) we have

$$\theta'(s, a_1) \geqslant 0.9 \tag{2.27}$$

for all $s \in [T, t)$. Thus if (A) holds then

$$\theta(T + (c^{-\varepsilon}/2), a_1) \geqslant 0.45c^{-\varepsilon}. \tag{2.28}$$

This, (2.25), (2.22), and (2.21) give

$$\begin{aligned} 0.45c^{-\varepsilon} &\leqslant \arctan[2g^{-1}(c-1+c^{\alpha+\varepsilon})/(ac^{\alpha})] \\ &\leqslant 3g^{-1}(c-1+c^{\alpha+\varepsilon})/(ac^{\alpha}) \\ &\leqslant 0.3c^{-\varepsilon}, \end{aligned}$$

which is clearly false. This contradiction shows that (A) cannot occur. On the other hand, if (C) occurs then by the definition of the set S we have $x(s)>0$ for all $s \in (T, t)$. Therefore

$$x'(t) \leqslant 0. \tag{2.29}$$

Since (2.29) contradicts (2.24), we see that (C) cannot occur either. Thus only (B) holds, and by (2.24), $x'(t_1) \geqslant ac^{\alpha}/2 > 0$. Hence Step 1 holds.

Step 2. Let t_1 and d_1 be as in Step 1. We claim that there exists $d_2 \geqslant d_1$ such that if $c > d_2$ then for some $t_2 \in (t_1, t_1 + [2ac^{\alpha}/(c^{\alpha+\varepsilon}-2)])$ we have $x(t_2) = g^{-1}(c-1+c^{\alpha+\varepsilon})$, $x'(t_2)<0$, and $x(s) > g^{-1}(c-1+c^{\alpha+\varepsilon})$ for $s \in (t_1, t_2)$.

By Step 1 (in particular see (2.24))

$$x'(t_1) \geqslant (ac^{\alpha}/2) > 0. \tag{2.30}$$

Therefore there exists $t > t_1$ such that $x(s) > g^{-1}(c-1+c^{\alpha+\varepsilon})$ for $s \in (t_1, t)$. Integrating (1.1) on $[t_1, t]$ we have

$$\begin{aligned} x'(t) &= x'(t_1) + \int_{t_1}^{t} (c + p(\tau) - g(x(\tau)))\, d\tau \\ &\leqslant x'(t_1) + [c+1-(c-1+c^{\alpha+\varepsilon})](t-t_1) \\ &\leqslant x'(t_1) + (2-c^{\alpha+\varepsilon})(t-t_1). \end{aligned} \tag{2.31}$$

Since, by hypothesis, g is an increasing function on $[0, \infty)$ and x is also an increasing function on $[T, t_1)$ (see (2.24)), we have $c+p(s)-g(x(s)) \leqslant 0$ whenever $x(s) \geqslant g^{-1}(c+1)$ and $s \in [T, t_1)$. Therefore by (1.1) we have

$$x'(t_1) \leqslant x'(s_1), \tag{2.32}$$

where $s_1 \in [T, t_1)$ is such that

$$x(s_1) = g^{-1}(c+1). \tag{2.33}$$

The existence of such an s_1 follows for c larger than $2^{1/\alpha}$ and by the continuity of x.

Again using (2.24) we have

$$g^{-1}(c+1) = 0 + \int_T^{s_1} x'(s)\,ds$$
$$\geqslant (s_1 - T)(ac^\alpha/2). \tag{2.34}$$

Next we observe that there is a $\delta_4 \geqslant d_1$ such that if $c \geqslant \delta_4$ then

$$g^{-1}(c+1) \leqslant c^{1/(\rho+1)}, \tag{2.35}$$

$$2c^{-\varepsilon} < c^\alpha, \tag{2.36}$$

and

$$2(c+1)\, c^{(1/(\rho+1))-\alpha} < c^\alpha. \tag{2.37}$$

Inequality (2.35) follows immediately from (1.4), whereas (2.36) follows from the hypotheses $\varepsilon > 0$ and $\alpha > 0$. By (2.7) we have $1 + (1/(\rho+1)) < 2\alpha$, which implies (2.37). Thus we have established the existence of δ_4 satisfying (2.35)–(2.37).

Let $c > d_2 = \text{Max}\{\delta_4, 2^{1/\alpha}\}$. Then replacing (2.35) into (2.34) we have

$$(s_1 - T) \leqslant (2/a)\, c^{(1/(\rho+1))-\alpha} \tag{2.38}$$

for all $c \geqslant d_2$. Hence integrating (1.1) on $[T, s_1]$ we obtain

$$x'(s_1) = x'(T) + \int_T^{s_1} (c + p(\tau) - g(x(\tau)))\,d\tau$$
$$\leqslant ac^\alpha + (c+1)(s_1 - T)$$
$$\leqslant ac^\alpha + (c+1)(2/a)\, c^{(1/(\rho+1))-\alpha}. \tag{2.39}$$

Now replacing (2.37) in the last term of (2.39) and using (2.32) we arrive at

$$x'(t_1) \leqslant ac^\alpha + c^\alpha/a \leqslant 2ac^\alpha \tag{2.40}$$

since $a \geqslant 1$.

Suppose now that $t > t_1$ and $x(s) > g^{-1}(c - 1 + c^{\alpha+\varepsilon})$ for all $s \in (t_1, t)$. From (2.31) and (2.40) we have

$$x(t) - x(t_1) \leqslant 2ac^\alpha(t - t_1) + (2 - c^{\alpha+\varepsilon})(t - t_1)^2. \tag{2.41}$$

Hence if $t > t_1 + [2ac^\alpha/(c^{\alpha+\varepsilon} - 2)]$ we infer

$$x(t) - x(t_1) < 0. \tag{2.42}$$

This and the continuity of x imply that there exists $t_2 \in (t_1, t_1 + [2ac^\alpha/(c^{\alpha+\varepsilon}-2)])$ such that $x(t_2) = x(t_1) = g^{-1}(c-1+c^{\alpha+\varepsilon})$ and $x(s) > g^{-1}(c-1+c^{\alpha+\varepsilon})$ for all $s \in (t_1, t_2)$. Clearly $x'(t_2) \leqslant 0$. We will now show that, in fact, $x'(t_2) < 0$. Since $x(s) > g^{-1}(c-1+c^{\alpha+\varepsilon})$ for $s \in (t_1, t_2)$, by (1.1) we have

$$\begin{aligned} x''(s) &< (c+1) - (c-1+c^{\alpha+\varepsilon}) \\ &= 2 - c^{\alpha+\varepsilon} < 0 \end{aligned} \tag{2.43}$$

for all $s \in (t_1, t_2)$, since $c > d_2 \geqslant 2^{1/\alpha}$. This and the fact that $x(t_1) = x(t_2) = g^{-1}(c-1+c^{\alpha+\varepsilon})$ imply that there exists a unique $s_2 \in (t_1, t_2)$ with $x(t_2) = \mathrm{Max}\{x(s);\ t_1 \leqslant s \leqslant t_2\}$ and $x'(s_2) = 0$.

Thus, using again that $x''(s) < 0$ on (t_1, t_2), we have

$$x'(t_2) < x'(s_2) = 0. \tag{2.44}$$

Since also $x(t_2) = g^{-1}(c-1+c^{\alpha+\varepsilon})$, and $x(s) > g^{-1}(c-1+c^{\alpha+\varepsilon})$, for all $s \in (t_1, t_2)$ Step 2 is proven.

Step 3. Let t_1, t_2, d_1, d_2 be as above. Then there exists $d_3 \geqslant d_2$, $t_3 \in (t_2, t_2 + c^{\beta(\alpha-1)}]$, where

$$1 < \beta < 1 + 1/(\rho+1) \tag{2.45}$$

such that if $c > d_3$ then

$$x(t_3) = 0 \qquad \text{and} \qquad x(s) > 0 \qquad \text{for all } s \in (t_2, t_3). \tag{2.46}$$

We will first estimate $x'(t_2)$. Let $c > d_2$. Suppose $x'(t_2) < -3ac^\alpha$. Then by (2.40) and (2.43), $\mathrm{Max}\{|x'(s)|;\ s \in [t_1, t_2]\} = -x'(t_2)$.

Now multiplying (1.1) by $x'(s)$ and integrating on $[t_1, t_2]$ we have

$$\begin{aligned} [x'(t_2)]^2 &= [x'(t_1)]^2 - 2G(x(t_2)) + 2G(x(t_1)) \\ &\quad + 2c\int_{t_1}^{t_2} x'(s)\,ds + 2\int_{t_1}^{t_2} p(s)\,x'(s)\,ds, \end{aligned} \tag{2.47}$$

where G is the primitive of g. But $x(t_1) = x(t_2)$ and so $G(x(t_1)) = G(x(t_2))$, $\int_{t_1}^{t_2} x'(s)\,ds = 0$. Hence by (2.47) and (2.40) we have

$$[x'(t_2)]^2 \leqslant 4a^2c^{2\alpha} - 2x'(t_2)(t_2 - t_1),$$

that is,

$$[x'(t_2)]^2 + 2x'(t_2)(t_2 - t_1) - 4a^2c^{2\alpha} \leqslant 0$$

and so

$$x'(t_2) \geqslant -(t_2 - t_1) - \sqrt{(t_2 - t_1)^2 + 4a^2c^{2\alpha}}. \tag{2.48}$$

Now by Step 2, there exists $\delta_5 \geqslant d_2$ such that if $c > \delta_5$ then

$$t_2 - t_1 < 3ac^{-\varepsilon}. \tag{2.49}$$

Hence for such $c > \delta_5$ by (2.48) we get

$$-3ac^{\alpha} > -3ac^{-\varepsilon} - \sqrt{9a^2c^{-2\varepsilon} + 4a^2c^{2\alpha}},$$

which is clearly false for large enough c, since $\varepsilon > 0$. Hence there exists $\delta_6 \geqslant \delta_5$ such that for $c > \delta_6$

$$x'(t_2) \geqslant -3ac^{\alpha}. \tag{2.50}$$

In the following let $c > \delta_6$. Then by (2.40), (2.43), and (2.50) we have

$$|x'(s)| \leqslant 3ac^{\alpha} \qquad \text{for all } s \in [t_1, t_2]. \tag{2.51}$$

Hence using (2.47), (2.30), and (2.49) we get

$$\begin{aligned}[x'(t_2)]^2 &\geqslant (ac^{\alpha}/2)^2 - 6ac^{\alpha}(t_2 - t_1)\\ &\geqslant (ac^{\alpha}/2)^2 - (6ac^{\alpha})(3ac^{-\varepsilon})\\ &\geqslant (ac^{\alpha}/3)^2\end{aligned}$$

provided c is large enough. Hence there exists δ_7 such that if $c > \delta_7$ then

$$x'(t_2) \leqslant -(ac^{\alpha}/3). \tag{2.52}$$

Let $c > \delta_7$. Suppose now that for all $s \in [t_2, t_2 + c^{\beta(\alpha-1)}]$, $x(t) > 0$. This, (1.1), and (2.52) imply that for all $s \in [t_2, t_2 + c^{\beta(\alpha-1)}]$,

$$\begin{aligned}x'(s) &\leqslant -(ac^{\alpha}/3) + \int_{t_2}^{s} [c + 1 - g(x(\tau))]\, d\tau\\ &\leqslant -(ac^{\alpha}/3) + (c+1)(s - t_2).\end{aligned} \tag{2.53}$$

Thus

$$\begin{aligned}0 < x(t_2 + c^{\beta(\alpha-1)}) &\leqslant x(t_2) - (ac^{\alpha}/3)\, c^{\beta(\alpha-1)} + (c+1)[c^{2\beta(\alpha-1)}/2]\\ &= g^{-1}(c - 1 + c^{\alpha+\varepsilon}) + \frac{c^{1+2\beta(\alpha-1)}}{2} + \frac{c^{2\beta(\alpha-1)}}{2} - \frac{a}{3}c^{\alpha+\beta(\alpha-1)}.\end{aligned}$$

That is,

$$0 < g^{-1}(c-1+c^{\alpha+\varepsilon})\, c^{-\alpha-\beta(\alpha-1)} + \frac{c^{(\beta-1)(\alpha-1)}}{2} + \frac{c^{\beta(\alpha-1)-\alpha}}{2} - \frac{a}{3}. \tag{2.54}$$

Now clearly $(\beta-1)(\alpha-1)<0$, $\beta(\alpha-1)-\alpha<0$ and

$$\begin{aligned}\frac{1}{\rho+1}-\alpha-\beta(\alpha-1) &= \frac{1}{\rho+1}-\alpha+\beta(1-\alpha)\\ &< \frac{1}{\rho+1}-\alpha+\left(1+\frac{1}{\rho+1}\right)(1-\alpha)\\ &= \left(\frac{\rho+3}{\rho+1}\right)-\alpha\left(\frac{2\rho+3}{\rho+1}\right)=0,\end{aligned}$$

since $\alpha=(\rho+3)/(2\rho+3)$. Then by (1.4), we can find $d_3 \geqslant \delta_7$ such that if $c>d_3$ then (2.54) is false. That is, there must exist a $t_3 \in [t_2, t_2+c^{\beta(\alpha-1)}]$ such that $x(t_3)=0$ and $x(s)>0$ for all $s\in[t_2, t_3)$, which proves Step 3.

We will now prove Lemma 2.3. Let $c>d_3$. From Steps 1–3

$$t_3 \leqslant T+(c^{-\varepsilon}/2)+[2ac^{\alpha}/(c^{\alpha+\varepsilon}-2)]+c^{\beta(\alpha-1)}. \tag{2.55}$$

Now since $\varepsilon<(1-\alpha)/2$ and $\beta>1$, $-\varepsilon>\beta(\alpha-1)$. Hence there exists $d_4 \geqslant d_3$ such that if $c>d_4$ then (2.55) gives

$$t_3 \leqslant T+5ac^{-\varepsilon}. \tag{2.56}$$

Next we estimate $|x'(s)|$ for $s\in[T, t_3]$. Now integrating (1.1) on $[T, s]$ for $s\leqslant t_1$ we have

$$\begin{aligned}x'(s) &\leqslant ac^{\alpha}+\int_T^s (c+1)\, d\tau\\ &\leqslant ac^{\alpha}+(c+1)(c^{-\varepsilon}/2).\end{aligned}$$

But $x'(s)>0$ (see (2.24)) and hence

$$|x'(s)| \leqslant ac^{\alpha}+(c+1)(c^{-\varepsilon}/2) \tag{2.57}$$

in $[T, s]$, where $s\leqslant t_1$. From (2.51) we already have $|x'(s)|\leqslant 3ac^{\alpha}$ for all $s\in[t_1, t_2]$. Now for $s\in[t_2, t_3]$ integrating (1.1) on $[t_2, s]$, where $s\leqslant t_3$, we have

$$x'(s)=x'(t_2)+\int_{t_2}^s [c+p(\tau)-g(x(\tau))]\, d\tau$$

and so

$$x'(t_2)+[c-1-g(x(t_2))](s-t_2)\leqslant x'(s)\leqslant x'(t_2)+(c+1)(s-t_2),$$

that is, using (2.50) and (2.52) we get

$$-3ac^{\alpha}-c^{\alpha+\varepsilon}c^{\beta(\alpha-1)}\leqslant x'(s)\leqslant -ac^{\alpha}/3+(c+1)\,c^{\beta(\alpha-1)}.$$

Hence

$$-3a-c^{\varepsilon+\beta(\alpha-1)}\leqslant\frac{x'(s)}{c^{\alpha}}\leqslant -\frac{a}{3}+(c+1)\,c^{-\alpha+\beta(\alpha-1)}. \tag{2.58}$$

But $\varepsilon+\beta(\alpha-1)<(1-\alpha)/2+\beta(\alpha-1)<0$ since $\beta>1$ and $1-\alpha+\beta(\alpha-1)=(\alpha-1)(\beta-1)<0$ since $\alpha<1$, $\beta>1$. Then there exists $d_5\geqslant d_4$ such that if $c>d_5$ (see (2.58)) we have

$$-4ac^{\alpha}\leqslant x'(s)\leqslant -(a/4)\,c^{\alpha}, \tag{2.59}$$

for $s\in[t_2,t_3]$. Hence combining (2.59), (2.51), and (2.57), for $c>d_5$ we have

$$|x'(s)|\leqslant \text{Max}\{ac^{\alpha}+(c+1)\,c^{-\varepsilon}/2,\,4ac^{\alpha}\}\qquad\text{for}\quad s\in[T,t_3]. \tag{2.60}$$

But $1-\varepsilon>\alpha$. Hence there exists $d_6\geqslant d_5$ such that if $c>d_6$ then for $s\in[T,t_3]$ we have

$$|x'(s)|\leqslant c^{1-\varepsilon}. \tag{2.61}$$

Now multiplying (1.1) by $x'(s)$ and integrating on $[T,t_3]$, using the fact that $x(t_3)=x(T)=0$, we get

$$x'(t_3)^2-x'(T)^2=2\int_T^{t_3}p(s)\,x'(s)\,ds. \tag{2.62}$$

Hence by using (2.61), (2.56), and the fact that $x'(T)=ac^{\alpha}$ we have

$$|x'(t_3)^2-a^2c^{2\alpha}|\leqslant 10ac^{1-2\varepsilon}. \tag{2.63}$$

Now (2.63) gives

$$|x'(t_3)+ac^{\alpha}|\leqslant\frac{10ac^{1-2\varepsilon}}{|x'(t_3)-ac^{\alpha}|}$$

and since $x'(t_3)<0$ (see (2.59)) we get

$$|x'(t_3)+ac^{\alpha}|\leqslant 10c^{1-\alpha-2\varepsilon}\leqslant c^{\alpha}[10c^{1-2\alpha-2\varepsilon}]. \tag{2.64}$$

But $1-2\alpha-2\varepsilon<0$ since $2\alpha=(2\rho+6)/(2\rho+3)>1$ and $\varepsilon>0$. Hence there exists $d(n)\geqslant d_6$ such that if $c>d=d(n)$

$$10c^{1-2\alpha-2\varepsilon}<1/(2n^2), \tag{2.65}$$

and

$$5ac^{-\varepsilon}<1/n. \tag{2.66}$$

Now for $c>d$, combining (2.56) and (2.66) we get

$$t_3\leqslant T+1/n, \tag{2.67}$$

and combining (2.65) and (2.64) we get

$$|x'(t_3)+ac^{\alpha}|<c^{\alpha}/(2n^2). \tag{2.68}$$

Hence clearly $T_1=t_3-T$ satisfies the requirements of Lemma 2.3. Thus Lemma 2.3 is proven.

LEMMA 2.4. *Let $x(t)$ be a solution to (1.1) satisfying $x(T)=0$, $x'(T)=-ac^{\alpha}$ with $(T,a)\in[0,1]\times[1,3]$. Given any positive integer n there exists $D:=D(N)$ such that if $c>D$ then there exists $T_1\in(0,1/n]$ with $x(T+T_1)=0$ and $x<0$ on $(T,T+T_1)$. Moreover*

$$|x'(T+T_1)-ac^{\alpha}|<c^{\alpha}/(2n^2). \tag{2.69}$$

Proof. Let $0<\tau<1$ be such that $M-\tau\neq 0$. By (1.3) there exists a real number η such that

$$(M-\tau)\,x-g(x)\geqslant\eta \qquad \text{for all } x\leqslant 0. \tag{2.70}$$

Now we define

$$f(u):=\begin{cases}(\sinh((\tau-M)^{1/2}(u-T)))/(\tau-M)^{1/2} & \text{if } \tau>M\\ (\sin((M-\tau)^{1/2}(u-T)))/(M-\tau)^{1/2} & \text{if } \tau<M.\end{cases} \tag{2.71}$$

Then rewriting (1.1) as $x''+(M-\tau)\,x=c+p(t)+(M-\tau)\,x-g(x(t))$ we have

$$x(t)=-ac^{\alpha}f(t)+\int_T^t f(T+t-s)(p(s)+c+(M-\tau)\,x(s)-g(x(s)))\,ds. \tag{2.72}$$

Now consider the interval $[T,T+c^{(\alpha-1)/4}]$. Since $0<\alpha<1$, there exists $D_1(n)$ such that if $c>D_1$

$$c^{(\alpha-1)/4}\leqslant\pi/(2\,|\tau-M|^{1/2}). \tag{2.73}$$

Let $c > D_1$. Then in $[T, T+c^{(\alpha-1)/4}]$, $f \geqslant 0$ and hence from (2.70)–(2.72) we have

$$x(t) \geqslant -ac^{\alpha} f(t) + (c+\eta-1) \int_T^t f(T+t-s)\, ds. \tag{2.74}$$

But in $[T, T+c^{(\alpha-1)/4}]$ we also have

$$2(t-T)/\pi \leqslant f(t) \leqslant (e^{\pi/2})/(2\,|\tau - M|^{1/2}) = K_1 \qquad \text{(say)}. \tag{2.75}$$

Now let $c > D_2$, where $D_2 \geqslant D_1$ is such that $c+\eta-1 \geqslant c/2$. Then from (2.74) and (2.75) we get

$$x(t) \geqslant -ac^{\alpha} K_1 + (c/2)(t-T)^2/\pi \tag{2.76}$$

for all $t \in [T, T+c^{(\alpha-1)/4}]$. In particular,

$$\begin{aligned} x(T+c^{(\alpha-1)/4}) &\geqslant -ac^{\alpha} K_1 + [cc^{(\alpha-1)/2}]/(2\pi) \\ &= -ac^{\alpha} K_1 + c^{(\alpha+1)/2}/(2\pi). \end{aligned} \tag{2.77}$$

But $(\alpha+1)/2 > \alpha$. So there exists $D_3 \geqslant D_2$ such that if $c > D_3$ then $x(T+c^{(\alpha-1)/4}) > 0$. Hence choosing $c \geqslant D_4 = D_4(n)$, where $D_4 \geqslant D_3$ is such that

$$c^{(\alpha-1)/4} < 1/n, \tag{2.78}$$

there exists $T_1 \in [0, 1/n]$ such that $x(T+T_1) = 0$ and $x(s) < 0$ on $(T, T+T_1)$.

We conclude the proof of Lemma 2.4 by establishing (2.69). From (2.76) we have that $x(t) \geqslant -K_1 ac^{\alpha}$ for all $t \in [T, T+T_1]$. Then using (1.3), there exists $K_2 > 0$, D_5, where $D_5 \geqslant D_4$ such that for $c > D_5$

$$|x''(t)| \leqslant K_2 c \tag{2.79}$$

for all $t \in [T, T+T_1]$ and so

$$|x'(t) - x'(T)| \leqslant K_2 c(t-T) \tag{2.80}$$

for all $t \in [T, T+T_1]$. But $T_1 < c^{(\alpha-1)/4}$ since $x(T+c^{(\alpha-1)/4}) > 0$. Hence

$$|x'(t) - x'(T)| \leqslant K_2 c^{1-(1-\alpha)/4},$$

that is,

$$-ac^{\alpha} - K_2 c^{1-\varepsilon_1} \leqslant x'(t) \leqslant -ac^{\alpha} + K_2 c^{1-\varepsilon_1} \tag{2.81}$$

for all $t \in [T, T+T_1]$, where

$$\varepsilon_1 = (1-\alpha)/4. \tag{2.82}$$

Now clearly $1-\varepsilon_1 > \alpha$. Hence there exists $K_3 > 0$, D_6, where $D_6 \geqslant D_5$ such that for $c > D_6$

$$|x'(t)| \leqslant K_3 c^{1-\varepsilon_1} \tag{2.83}$$

for all $t \in [T, T+T_1]$. The rest of the proof of (2.69) follows the same pattern of proving (2.15). For this reason we refer the reader to (2.62)–(2.68). This concludes the proof of Lemma 2.4.

LEMMA 2.5. *Let $m > 0$, $\delta > 0$ be such that $M+m > 0$. Let $x(t)$ be a solution to* (1.1) *and let c^* be as in Lemma 2.2. For each $c > c^*$ there exists $B := B(c, m, \delta) > 0$ such that if $b > B$, if $x(t_1) = 0$, and if $x'(t_1) = -b$ for some $t_1 \in [0, 1]$, then $x(t) < 0$ for all $t \in (t_1, t_1 + \pi(M+m+\delta)^{-1/2})$.*

Proof. By (1.3)–(1.4) we see that there exists a constant $M^* < 0$ such that

$$g(u) - (M+m)\,u > M^* \tag{2.84}$$

for all $u \in R$. Let $\lambda = (M+m)^{1/2}$. Rewriting (1.1) as $x''(s) + \lambda^2 x(s) = c + p(s) - g(x(s)) + \lambda^2 x(s)$ we have for $s \in (0, \pi/\lambda)$

$$\begin{aligned} x(t_1+s) &= \lambda^{-1}\left\{x'(t_1)\sin(\lambda s) + \int_0^s [\sin(\lambda(s-u))(c+p(t_1+u)\right.\\ &\qquad \left. - g(x(t_1+u)) + \lambda^2(x(t_1+u)))]\,du\right\}\\ &\leqslant \lambda^{-1}\sin(\lambda s)\left\{-b + \frac{(c+1-M^*)[1-\cos(\lambda s)]}{\lambda\sin(\lambda s)}\right\}\\ &= \lambda^{-1}\sin(\lambda s)\left\{-b + \frac{(c+1-M^*)}{\lambda}\tan(\lambda s/2)\right\}, \end{aligned} \tag{2.85}$$

where we have used (2.84). From (2.85) it follows immediately that if

$$\begin{aligned} b &> [(c+1-M^*)/(M+m)^{1/2}]\tan([M+m]^{1/2}\,\pi/[2(M+m+\delta)^{1/2}])\\ &:= B(c, m, \delta) \end{aligned} \tag{2.86}$$

then $x(t) < 0$ for all $t \in (t_1, t_1 + \pi(M+m+\delta)^{-1/2})$, and Lemma 2.5 is proven.

LEMMA 2.6. *Let $c > c^*$ be given and $B(c, m, \delta)$ be as above. We claim that there exists $F := F(c, m, \delta) > 0$ such that if $x(t)$ satisfies* (1.1), $x(0) = 0$, $x'(0) = b$, *where* $|b| > F$, *then* $|x'(T)| > B(c, m, \delta)$ *whenever* $x(T) = 0$, $0 \leqslant T \leqslant 1$.

Proof. Let E_1 be as in Lemma 2.1. By a similar argument as in (2.3)–(2.4) we obtain

$$(dE_1/dt) \geqslant -\sqrt{2}\,|c+1| - (|c+1| + M_1\sqrt{2})\,E_1. \tag{2.87}$$

Multiplying (2.87) by e^{kt}, where $k = |c+1| + M_1\sqrt{2}$, we have

$$\frac{d}{dt}(e^{kt}E_1) \geqslant -\sqrt{2}\,|c+1|\,e^{kt},$$

and so integrating on $[0, T]$ we have

$$e^{kT}E_1(T) - E_1(0) \geqslant -(\sqrt{2}/k)\,|c+1|\,\{e^{kT} - 1\},$$

and since $E_1(0) = b^2/2$ we obtain

$$\begin{aligned} E_1(T) &\geqslant e^{-kT}\left\{(b^2/2) + \frac{\sqrt{2}\,|c+1|}{k}\right\} - \frac{\sqrt{2}\,|c+1|}{k} \\ &\geqslant e^{-k}(b^2/2) - (\sqrt{2}\,|c+1|/k). \end{aligned} \tag{2.88}$$

But $E_1(T) = [x'(T)]^2/2$. Hence from (2.88) we have $|x'(T)| = (2E_1(T))^{1/2} > B(c, m, \delta)$ whenever

$$\begin{aligned} |x'(0)| = |b| &> \{[B^2(c, m, \delta) + 2\sqrt{2}\,|c+1|\,k]\,e^k\}^{1/2} \\ &:= F(c, m, \delta), \end{aligned} \tag{2.89}$$

which proves Lemma 2.6.

3. PROOF OF THEOREM 1.1

Let $n > [2(\text{Max}\{0, M\}^{1/2})/\pi]$ be an integer. Let $c > \max\{d(n+1), D(n+1), c^*\}$ (see Lemmas 2.2–2.6). From (2.12) it follows immediately that if $\theta(t_1, a) = \pi K$ with K a non-negative integer, then

$$\theta(t, a) > \pi K \qquad \text{for all } t > t_1. \tag{3.1}$$

First we will let $n > [2(\text{Max}\{0, M\}^{1/2})/\pi] + 1$ be an integer and establish a solution $x(t)$ to (1.1)–(1.2) with n interior zeroes and satisfying $x'(0) > 0$.

Since $\theta(0, 2) = 0$, by Lemma 2.3 there exists $T_1 \in (0, 1/(n+1)]$ such that $\theta(T_1, 2) = \pi$ and by (2.15) we see that

$$x'(T_1, 2) \in [-(2 + 1/(2(n+1)^2))\, c^\alpha, -(2 - 1/(2(n+1)^2))\, c^\alpha]. \quad (3.2)$$

Hence by Lemma 2.4 we see that there exists $T_2 \in (T_1, 2/(n+1)]$ such that $\theta(T_2, 2) = 2\pi$ and

$$|x'(T_2, 2) + x'(T_1, 2)| \leqslant c^\alpha/(2(n+1)^2). \quad (3.3)$$

Then (3.1) yields

$$\theta(2/(n+1), 2) \geqslant 2\pi, \quad (3.4)$$

and from (3.2), (3.3) we have

$$\begin{aligned} |x'(T_2, 2) - 2c^\alpha| &\leqslant |x'(T_2, 2) + x'(T_1, 2)| \\ &\quad + |x'(T_1, 2) + 2c^\alpha| \\ &\leqslant c^\alpha/(2(n+1)^2) + c^\alpha/(2(n+1)^2) = c^\alpha/(n+1)^2. \quad (3.5) \end{aligned}$$

Iterating this argument (i.e., applying Lemmas 2.3 and 2.4 consecutively $(n+1)/2$ times) we obtain

$$\theta(1, 2) > (n+1)\pi. \quad (3.6)$$

Let now $m > 0$ and $\delta > 0$ be such that $M + m > 0$ and $n > [2(M + m + \delta)^{1/2}/\pi] + 1$. Let $b > \max\{2c^\alpha, F(c, m, \delta)\}$, where $F(c, m, \delta)$ is as in Lemma 2.6 and consider $x(t)$ satisfying (1.1), $x(0) = 0$, $x'(0) = b$. Since $c > c^*$, $b > c^\alpha$ by Lemma 2.2 there exists a non-negative integer I such that $I\pi \leqslant \theta(1, bc^{-\alpha}) < (I+1)\pi$. Further by (3.1) there exists an increasing sequence $0 = t_0 < t_1 < t_2 < \cdots < t_I \leqslant 1$ such that

$$\theta(t_i, bc^{-\alpha}) = \pi i, \qquad i = 0, 1, 2, ..., I. \quad (3.7)$$

Now since $b > F(c, m, \delta)$, by Lemmas 2.5 and 2.6 we obtain that

$$|x'(t_i)| \geqslant B(c, m, \delta), \qquad i = 1, 2, ..., I \quad (3.8)$$

and

$$t_{2j} - t_{2j-1} \geqslant \pi(M + m + \delta)^{-1/2}. \quad (3.9)$$

Since $t_I \leqslant 1$, from (3.9) we infer

$$\begin{aligned} 1 \geqslant t_I = \sum_{i=1}^{I} t_i - t_{i-1} &\geqslant \sum_{j=1}^{[[I/2]]} t_{2j} - t_{2j-1} \\ &\geqslant [[I/2]]\, \pi(M + m + \delta)^{-1/2}. \quad (3.10) \end{aligned}$$

Thus

$$\theta(1, bc^{-\alpha}) \leqslant (I+1)\pi \leqslant (2[[I/2]]+2)\pi$$
$$\leqslant 2(M+m+\delta)^{1/2} + 2\pi < (n+1)\pi. \tag{3.11}$$

From (3.6), (3.11), the continuous dependence of $\theta(1, a)$ on **a**, and the intermediate value theorem we see that there exists $a_1 \in [2, bc^{-\alpha}]$ such that

$$\theta(1, a_1) = (n+1)\pi. \tag{3.12}$$

By the definition of θ it follows then that $x(t, a_1)$ is a solution of (1.1)–(1.2). In addition by (3.1) it follows that $x(t, a_1)$ has exactly n interior zeroes, and the (A) part is proven.

Next we let $n > [2(\text{Max}\{0, M\}^{1/2})/\pi]$ be an integer and obtain a second solution to (1.1)–(1.2), $x_1(t)$, having n interior zeroes. Unlike our first solution, $x_1(t)$ satisfies

$$x_1'(0) < 0. \tag{3.13}$$

Let $x(t, a)$ denote the solution to (1.1) satisfying $x(0) = 0$, $x'(0) = ac^\alpha$ with $a \in (-\infty, -1]$. Then we can define a new angle function $\psi(t, a)$ by

$$x(t, a) = -r(t, a) \sin(\psi(t, a)) \tag{3.14}$$

$$x'(t, a) = -r(t, a) \cos(\psi(t, a)), \tag{3.15}$$

as long as $r(t, a) > 0$. Thus by Lemma 2.2, ψ is well defined on $[0, 2] \times (-\infty, -1]$ if $c > c^*$. Since we are assuming $c > D(n+1)$, by Lemma 2.4 there exists $\tau_1 \in (0, 1/(n+1)]$ such that

$$\psi(\tau_1, -2) = \pi, \qquad |x'(\tau_1, -2) - 2c^\alpha| < c^\alpha/(2(n+1)^2). \tag{3.16}$$

Applying now Lemma 2.3 we see that there exists $\tau_2 \in (\tau_1, 2/(n+1)]$ such that

$$\psi(\tau_2, -2) = 2\pi, \qquad |x'(\tau_2, -2) + x'(\tau_1, -2)| < c^\alpha/(2(n+1)^2), \tag{3.17}$$

and so

$$|x'(\tau_2, -2) + 2c^\alpha| \leqslant |x'(\tau_2, -2) + x'(\tau_1, -2)| + |2c^\alpha - x'(\tau_1, -2)|$$
$$\leqslant c^\alpha/(2(n+1)^2) + c^\alpha/(2(n+1)^2)$$
$$= c^\alpha/(n+1)^2. \tag{3.18}$$

Iterating this process and using (3.1) we have

$$\psi(1, -2) \geqslant (n+1)\pi. \tag{3.19}$$

Let now $m>0$ and $\delta>0$ be such that $M+m>0$ and $n>2(M+m+\delta)^{1/2}/\pi$. Let $b<0$ be such that $|b|>\max\{2c^{\alpha}, F(c, m, \delta)\}$, where $F(c, m, \delta)$ is as in Lemma 2.6, and consider $x(t)$ satisfying (1.1), $x(0)=0$, $x'(0)=b$. Since $c>c^*$, $|b|>c^{\alpha}$, by Lemma 2.2 there exists a non-negative integer K such that $K\pi \leqslant \psi(1, bc^{-\alpha})<(K+1)\pi$.

Further by (3.1) there exists an increasing sequence $0=\tau_0<\tau_1<\tau_2<\cdots<\tau_K\leqslant 1$ such that

$$\psi(\tau_i, bc^{-\alpha})=\pi i, \qquad i=0, 1, 2, ..., K. \tag{3.20}$$

Now since $|b|>F(c, m, \delta)$, by Lemmas 2.5 and 2.6 we obtain that

$$|x'(\tau_i)|\geqslant B(c, m, \delta), \qquad i=0, 1, 2, ..., K \tag{3.21}$$

and

$$\tau_{2j-1}-\tau_{2j-2}\geqslant \pi(M+m+\delta)^{-1/2}. \tag{3.22}$$

Since $\tau_K\leqslant 1$, from (3.22) we infer

$$\begin{aligned}1\geqslant \tau_K=\sum_{i=1}^{K} \tau_i-\tau_{i-1}\geqslant &\sum_{j=1}^{[[(K+1)/2]]} (\tau_{2j-1}-\tau_{2j-2})\\ &\geqslant [[(K+1)/2]]\,\pi(M+m+\delta)^{-1/2}.\end{aligned} \tag{3.23}$$

Thus

$$\begin{aligned}\psi(1, bc^{-\alpha})\leqslant (K+1)\pi &\leqslant (2[[(K+1)/2]]+1)\pi\\ &\leqslant 2(M+m+\delta)^{1/2}+\pi<(n+1)\pi.\end{aligned} \tag{3.24}$$

From (3.19), (3.24), the continuous dependence of $\psi(1, a)$ on **a**, and the intermediate value theorem we see that there exists $b_1\in[bc^{-\alpha}, -2]$ such that

$$\psi(1, b_1)=(n+1)\pi. \tag{3.25}$$

By the definition of ψ it follows then that $x_1(t)=x(t, b_1)$ is a solution of (1.1)–(1.2), which has exactly n interior zeroes and $x_1'(0)<0$. Hence the (B) part is proven.

In order to conclude the proof of Theorem 1.1 we need to show the existence of c_*. Since $g(u)-\pi^2u$ is bounded below, there exists a real number J such that

$$g(u)-\pi^2u>J \tag{3.26}$$

for all $u \in R$. Thus, if $x(t)$ is a soluton to (1.1)–(1.2) then multiplying (1.1) by $\sin(\pi t)$ and integrating by parts we have

$$c \int_0^1 \sin(\pi t)\, dt = \int_0^1 [g(x(t)) - \pi^2 x(t) - p(t)] \sin(\pi t)\, dt$$

$$\geqslant \int_0^1 [J - p(t)] \sin(\pi t)\, dt. \tag{3.27}$$

This shows that (1.1)–(1.2) has no solution if

$$c < c_* = (\pi/2) \left(\int_0^1 [J - p(t)] \sin(\pi t)\, dt \right), \tag{3.28}$$

which concludes the proof of Theorem 1.1.

Remark. While studying the case $p(t) \equiv 0$ in [3], we established two theorems related to our work here, one if $M < \sqrt{2}\,\pi^2$ and the other if $M \geqslant \sqrt{2}\,\pi^2$. However, we recently noticed (our thanks to Mr. Terry McCabe) that the above inequalities in [3] should be corrected to $M < \pi^2$ and $M \geqslant \pi^2$, respectively, for the two theorems to hold. This can be easily seen, since for $M \geqslant 0$, $\lim_{q \to -\infty} [-J(q)]$ is equal to $\pi/\sqrt{2M}$ and not π^2/M as stated in Eq. (3.2) in [3].

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