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A Collection of Ideas on Systems and Their Extensions

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Systems beget prejudice. Closed systems beget malicious prejudice. What's inside is familiar, comfortable. What's outside is not--it's considered unimportant, irrelevant, sometimes even nonexistent, and invariably it's given a **derogatory** name. But if perchance the system gets extended, and the extended system somehow or other, not only continues to work, but even works better, then a new closed system is born--a new, improved, superior system that is then given a *laudatory* name.

NUMBER SYSTEMS

The counting numbers 1, 2, 3, ..., also formally called the natural numbers, are the subject of Kronecker's famous quote, "God made the natural numbers, all the rest are the invention of man." They are familiar, they represent real things in the world. They are indeed natural. The numbers -1, -2, -3,..., in some sense a mirror image of the natural numbers, are very unnatural. They don't represent things which can be pointed at, and they are spoken of disparagingly as **negative** numbers. One way to justify calling them numbers is the following. An equation with an unknown can be considered as a question. "$x + 1 = 2$?" asks "What number upon adding 1 to it results in 2?." The answer is the number 1. "$x + 2 = 1$?" however, has no answer in the natural number system. By going outside the system one can posit -1 as the answer. In fact, by combining the naturals and the negatives with the number 0 as connecting link, namely ..., -3, -2, -1, 0,1, 2, 3, ..., this new system is powerful enough to answer the question "$x + a = b$?" for any numbers a and b in the system. This was impressive enough that it came to be called the *integer* number system. Integers have integrity. They are all honest-to-goodness numbers.

There are questions, however, like "$2x = 1$?" which have no answer in the integer number system. The answer 1/2 is a ratio of two integers, and, because it is not an integer, that is, a whole number, it was spoken of disparagingly as a **fraction**, namely a broken, or fractured, number. The fractions are not like a mirror image of the integers but are thought of as sitting in the spaces between the integers. By extending the integers to include the fractions, the resulting system became powerful enough to answer the question "$ax = b$?" for b any number in the system and a any number other than 0. It all seemed so reasonable to consider fractions as bona fide numbers, that the extended system came to be the *rational* number system.

The rational number system, in its turn, failed to answer many questions, some as simple as "$x^2 = 2$?" The answer in this case can be represented by the diagonal of a square whose sides have length 1, and this answer, $\sqrt{2}$, is of course spoken of disparagingly as an **irrational**. The irrationals are thought of as being distributed among the rationals, namely along the same number line, which at first is hard to conceive of, since the rationals by themselves are "dense" in the sense that between any two rationals, no matter how close, there is another rational. Even worse, it turns out the irrationals are "infinitely denser" than the rationals. By extending the rationals to include the irrationals, not only is the question "$x^a = b$?" answered for many combinations of a and b, but also many other kind of questions (e.g. "$dx = c$?" where d is the diameter and c the circumference of a circle). The irrationals turned out to be very useful, and because it seemed at the time that no more numbers could be squeezed into the number line, the combination of rationals and irrationals came to be called the *real* number system. Irrationality had bestowed upon it the mantle of reality.

These intimations of reality however did not provide omniscience, for there were still questions with no answers, such as "$x^2 = -1$?" The answer $\sqrt{-1}$, not being real, was spoken of jeeringly, one might say, as being **imaginary**, and for that reason was given the special symbol i. This imaginary number did not fit anywhere on the real number line. Further, multiplying i by a real number y suggested an imaginary number line iy which intersected the real number line at 0, and adding a real number x to iy, i.e. $x + iy$, produced a number plane, which turned out to be so incredibly useful in solving a myriad of complex problems (from those in formal mathematics to those in theoretical physics) that the numbers $x + iy$ came to be known as the *complex* number system. This extension transcends the real number system in a way quite different from the previous extensions, for in this case a new dimension has been added to the number system. The complex plane is not the only way to extend the real number line. The answer to "$|x| = -1$?" is given a symbol h and is spoken of as being **hallucinatory**, with the numbers $x + hy$ making up the *perplex* number system [1]. Also, the answer given to "$x^2 = 0$ and x not equal to 0?" might be given the symbol ℓ and be spoken of as being **ludicrous**, with the numbers $x + \ell y$ making up the *ethereal* number system [2]. These two number systems, while not as useful as the complex number system, nevertheless do have their uses, and lend credence to the idea that having more than one extension to a system doesn't mean they should compete with each other as to which is the correct one. As inventions of man, all are, so to speak, on the same footing.

PEOPLE SYSTEMS

People systems are closed by borders, which might be of various kinds, such as geographical, class, gender, and theological. Extensions of people systems can thus occur in various ways.

The Athenian city-state was a system in which the citizens of the state lived comfortably with the Athenian culture providing gracious living in an environment of theater, music, art, and philosophical discussion. Due to the lack of these things, the neighboring city-state was, in a sense, a mirror image of the Athenian one, and the word **Spartan** was used disparagingly by Athenians. There were questions, however, which had no answer when Athenians restricted themselves to their own state, such as "What military unit can serve as a rapid reaction force, with the ability to cover fifty miles in a day and survive for two weeks with no backup support?" By joining Athens and Sparta (and other city-states) together, the resulting combination became much more powerful, questions like the foregoing could now be answered, and all citizens of the new system were proudly known as *Hellenes*, or *Greeks* as we would now say.

If one were to ask "Who does the cooking, carrying and cleaning?", the answer would not be a Greek citizen, but rather a **slave**, who was not a citizen and who was spoken of disparagingly. One might further add that slaves were distributed in the spaces between citizens. It took a long time for the Greeks, and for the countries of the world in general, to extend full citizenship (implying freedom) to those who do the menial tasks of society, but there is a general consensus that doing so increases the well-being, strength, and productivity of that society, and this seems especially so when citizenship means the right to vote as in a genuine *democratic* system.

With the abolition of slavery, there still remained a large class of people which society depended greatly on, and yet were usually spoken of in a disparaging manner, namely **women**. They were distributed roughly equally in the spaces between men, but were not considered their equals in that they were not granted the same rights and privileges. Given the increase in power and effectiveness with previous extensions, it is an expectation of many that the same will be a consequence of extending equality to all people in a society, but achieving an *egalitarian* system is presently in all societies an ongoing struggle.

The first society considered consisted of humans, and the subsequent extensions all confined themselves to what might be called the human plane. There are questions asked, however, for which no human is an answer, such as "Who makes the thunder, the earthquakes, the floods and the droughts?" To pose an answer to such questions means transcending the human plane by adding a new dimension, enabling gods of various and diverse sorts to appear. Initially, these gods are more like **adversaries**, who humans could just as well do without, but often a relationship develops and a god becomes a protector and enabler for a particular group of humans, resulting in a *theocentric* system. This has a powerful effect on that group, granting it cohesion, resolve and purpose, which ends up affecting the people's daily habits of working, eating, playing, and worshipping, as well as their hopes and fears, their loves and hates, their courage and fortitude and peace of mind in the midst of life's troubles. The fact that different pantheons of gods are posited by different groups shows that there is more than one

continued on page 50

individuals share common terms and assumptions from which they discover new truths or facts, and within which they give value to certain observations. Since those assumptions, truths, and facts are not universally known, the truth of in-discipline reasoning is not self-evident to society as a whole. Assumptions may be passed on with the authority of the discipline, and thus may not be questioned by the masses, but that is due to the enforceability of authority, not to their reasonability. Aristotle's presentation of logic in the *Organon* and the *Rhetoric* assumes that Plato's model for reasoning is the basis for the development of a discipline, a science. If one assumes, as scientists and mathematicians too frequently do, that the principles and assumptions of scientific discourse are self-evident in the public arena, one tends to lose debates.

In his discussion of Logic in the *Organon* (Topics, Categories, Prior and Posterior Analytics), Aristotle retains this distinction, setting up syllogism as a method of reasoning from demonstrated premises to demonstrable conclusions, and setting up dialectic (the basis of logos in the *Rhetoric*) as discussion from common assumptions and opinions that are simply accepted without needing to be demonstrated. Thus, in doing rhetoric, you argue from your audience's opinions. In science, you argue from demonstrated truths. However, one must note that just as some audience opinions may be wrong (and the ethical character of the speaker may be sacrificed in the long run if audiences perceive him/her to be relying on audience beliefs that he/she knows are wrong), demonstrated premises may in the future be discarded by a scientific community -alchemy, for example. Aristotle points out that rhetorical argument aims to persuade audiences, not to do science. Rhetoric (and dialectic) is about public speaking (and informal discussion), not about understanding the nature of the mind, insects, the weather, or morality.

However, Aristotle does point out that rhetoric and dialectic can play a role in discovering truth. These subjects are useful in education (= propaedeutic), and they can be useful in assessing first principles or premises. The problem with first principles is that they have not been demonstrated to be true. Rhetoric and dialectic cannot demonstrate their truth—nothing can. But rhetoric and dialectic can assist in comparing the meaning and effect of statements of first principles, and there are advantages in being able to do that. It is easy to explain the particular balance between dialectic and syllogism in the medieval age given its pre-scientific situation and the dominance of religious perspectives in education and social understanding. The classical model for educated discourse provided in Aristotle is a complex weaving of social practice and theoretical understanding that values both. Since Descartes and Bacon, the balance in our mode of discussion has been shifting toward emphasizing and valuing scientific rather than rhetorical reasoning. As a result, educated discourse has become more arcane and alienated from the common discourse. The classical model of the Greeks provides a guide to righting this balance with the assumption that any educated person needs to be able to operate in both public and within-discipline modes. Not being able to do so constitutes a cultural handicap which we must define our educational principles and educational principles and methods to correct.

A Collection of Ideas on Systems and their Extensions

continued from page 46

way to extend the human plane in a new dimension.

This leads, almost invariably it seems, to competing claims as to which extension is the correct one. This is rather hard to avoid when various of these posited gods each reveal to a chosen messenger on earth that it is the one true god and that all others are the invention of man.

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