10-1-2012

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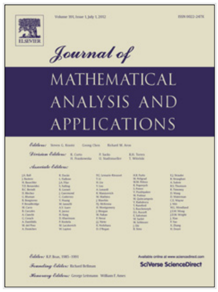
Accepted Manuscript



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PII: S0022-247X(12)00270-3
DOI: 10.1016/j.jmaa.2012.04.005
Reference: YJMAA 16656

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 9 December 2011

Please cite this article as: A. Castro, L. Sankar, R. Shivaji, Uniqueness of nonnegative solutions for semipositone problems on exterior domains, *J. Math. Anal. Appl.* (2012), doi:10.1016/j.jmaa.2012.04.005

Uniqueness of Nonnegative Solutions for Semipositone Problems on Exterior Domains

Alfonso Castro[*], Lakshmi Sankar[†], R. Shivaji [‡]

Abstract

We consider the problem

$$\begin{cases} -\Delta u = \lambda K(|x|)f(u), & x \in \Omega \\ u = 0 & \text{if } |x| = r_0 \\ u \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where λ is a positive parameter, $\Delta u = \mathrm{div}(\nabla u)$ is the Laplacian of u, $\Omega = \{x \in \mathbb{R}^n; n > 2, |x| > r_0\}$, $K \in C^1([r_0,\infty),(0,\infty))$ is such that $\lim_{r\to\infty} K(r) = 0$ and $f \in C^1([0,\infty),\mathbb{R})$ is a concave function which is sublinear at ∞ and $f(0) < 0$. We establish the uniqueness of nonnegative radial solutions when λ is large.

1 Introduction

Consider the boundary value problem:

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where λ is a positive parameter, $\Delta u = \mathrm{div}(\nabla u)$ is the Laplacian of u, Ω is a bounded domain and $f : [0,\infty) \to \mathbb{R}$ is a C^1 function. The case when $f(0) < 0$ is referred in the literature as semipositone problems. When Ω is a bounded domain, existence and uniqueness of nonnegative solutions of semipositone problems have been studied over the years, see [1]-[10]. Recently in [11] the existence of a positive solution for λ large was established when Ω is an exterior domain. In this paper, we extend this to establish the uniqueness of such solutions.

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In particular we consider:

$$\begin{cases} -\Delta u = \lambda K(|x|)f(u), & x \in \Omega \\ \quad u = 0 & \text{if } |x| = r_0 \\ \quad u \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1}$$

where λ is a positive parameter, $\Delta u = \operatorname{div}\big(\nabla u\big)$ is the Laplacian of u, $\Omega = \{x \in \mathbb{R}^n, n > 2|\,|x| > r_0\}$ is an exterior domain and f satisfies:

(H_1) f is increasing, $f(0) < 0$ and $\lim_{s\to\infty} f(s) = \infty$

(H_2) $\lim_{s\to\infty} \frac{f(s)}{s} = 0$.

Using the transformations $r = |x|$, $s = (\frac{r}{r_0})^{2-n}$ we can reduce (see appendix of [11]) equation (1) to the boundary value problem

$$\begin{cases} -u''(s) = \lambda h(s) f(u(s)), & 0 < s < 1 \\ \quad u(0) = u(1) = 0 \end{cases} \tag{2}$$

where $h(s) = \frac{r_0^2}{(2-n)^2} s^{\frac{-2(n-1)}{n-2}} K(r_0 s^{\frac{1}{2-n}})$. When the weight function K is such that $K \in C([r_0,\infty),(0,\infty))$ and satisfies:

(H_3) $K(r) \leq \dfrac{1}{r^{n+\rho}}$ for $r \gg 1$ and for some ρ such that $0 < \rho < n-2$,

the existence of positive radial solutions for (2) was established in [11] for λ large. Note that if K satisfies (H_3) then $h \in C((0,1],(0,\infty))$, is singular at 0, $\hat{h} = \inf_{t\in(0,1)} h(t) > 0$ and satisfies:
$(H_3)^*$ There exists $\epsilon_1 > 0$ and a constant $c > 0$ such that

$$h(t) \leq \frac{c}{t^\alpha} \quad \text{for all} \quad t \in (0,\epsilon_1) \quad \text{where} \quad \alpha = \frac{(n-2)-\rho}{n-2}.$$

To establish our uniqueness result we further assume :

(H_4) f is concave

(H_5) K is C^1 and $\frac{K(x^{-1})}{x^{2(n-1)}}$ is decreasing for $x > 0$.

We prove:

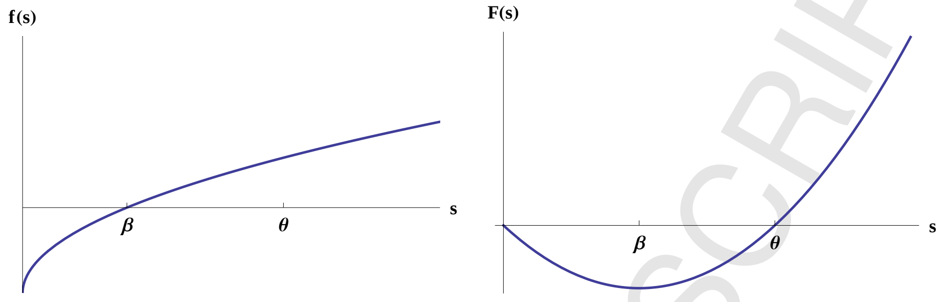
Theorem 1.1. *Assume $(H_1)-(H_5)$ hold. Then (1) has a unique nonnegative radial solution for $\lambda \gg 1$.*

Simple examples of the reaction term and the weight function satisfying our hypotheses are $f(s) = (s+1)^\gamma - 2$, where $\gamma \in (0,1)$ and $K(r) = \frac{1}{r^{n+\rho}}, \rho < n-2$. In Section 2 we establish some important a priori estimates and in Section 3 we prove Theorem 1.1. We note that once the crucial a priori estimates are established, the proof of Theorem 1.1 follows as in [6].

Remark 1: When $\rho \geq n-2$, h turns out to be nonsingular at 0 making the arguments less complicated. We restrict the focus of this paper to the more difficult case $\rho < n-2$.

2 A priori estimates

Let $F(s) = \int_0^s f(t)dt$. Note that there exist positive real numbers β, θ such that $f(\beta) = 0$ and $F(\theta) = 0$ and $\beta < \theta$. (See Figure 1).



(a) (b)

Figure 1: Graphs of $f(s)$ and $F(s)$

Lemma 2.1. *Let u be a nonnegative solution of (2). Then u has only one interior maximum, say at t_0, and $u(t_0) > \theta$.*

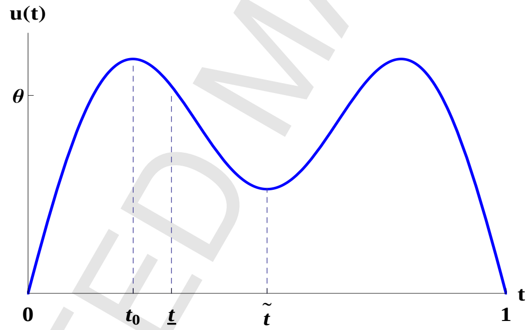


Figure 2: A solution with more than one maximum

Proof. Let $E(t) := \lambda F(u(t))h(t) + \frac{[u'(t)]^2}{2}, t \in (0,1)$. Hence $E'(t) = \lambda F(u(t))h'(t)$. Note that by (H_5), $h(s)$ decreases for $s > 0$. Thus $E(t)$ increases when $u(t) < \theta$ and decreases when $u(t) > \theta$. Let $t_0 \in (0,1)$ be the first point at which u has a local maximum, and assume $u(t) \leq \theta, \forall t \leq t_0$. Integrating (2) from t to $t_0, t < t_0$, and using $(H_3)^*$,

$$u'(t) \;=\; \lambda \int_t^{t_0} h(s)f(u(s))ds \leq \lambda \frac{df(\theta)}{1-\alpha}(t_0^{1-\alpha} - t^{1-\alpha}) \leq \lambda \frac{df(\theta)}{1-\alpha} \tag{3}$$

where $d \geq c$ is such that $h(t) \leq \frac{d}{t^\alpha}$ for all $t \in (0,1)$ and $\alpha \in (0,1)$. Integrating (3) again from 0 to $t, t < t_0$, $u(t) \leq \lambda M_0 t$ where $M_0 = \frac{df(\theta)}{1-\alpha}$. Since f is continuous, there exists $k_0 > 0$ such that $|F(u)| \leq k_0 u$ for $u \in [0, \theta]$. Hence

$$\lim_{t \to 0+} \lambda |F(u(t))| h(t) \leq \lim_{t \to 0+} k_0 \lambda M_0 d t^{1-\alpha} = 0,$$

which implies $\lim_{t\to 0+} E(t) \geq 0$. Since $E(t)$ increases on $[0, t_0]$, $E(t_0) = \lambda F(u(t_0))h(t_0) > 0$ which is a contradiction if $u(t_0) \leq \theta$. Hence $u(t_0) > \theta$.

Now suppose u has two interior maxima. Let $\tilde{t} \in (t_0, 1)$ be such that $u'(\tilde{t}) = 0$ and $u''(\tilde{t}) \geq 0$ (as in Figure 2). Since $u''(\tilde{t}) = -\lambda h(\tilde{t})f(u(\tilde{t})) \geq 0$ we see that $u(\tilde{t}) \leq \beta$ and thus $E(\tilde{t}) < 0$. Let $\underline{t} \in (t_0, \tilde{t})$ be such that $u(\underline{t}) = \theta$. Since $E(\underline{t}) \geq 0$ and E increases in $(\underline{t}, \tilde{t})$, $E(\tilde{t}) > 0$ which is contradiction. Hence u can have only one interior maximum and that maximum value is bigger than θ.

Lemma 2.2. *If $t_1, \hat{t_1}$ are such that $t_1 < \hat{t_1}$ and $u(t_1) = u(\hat{t_1}) = \beta$, then $t_1, 1 - \hat{t_1} \leq O(\lambda^{-\frac{1}{2}})$.*

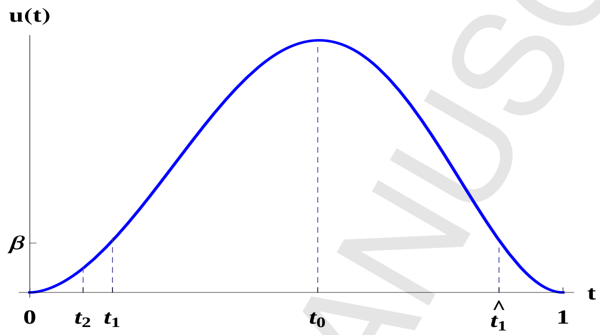


Figure 3: Graph of a solution

Proof. Let t_2 be the first point in (0, 1) such that $u(t_2) = \frac{\beta}{2}$. Integrating (2) from 0 to $t, t < t_2$,

$$\begin{aligned} u'(t) &= u'(0) - \lambda \int_0^t h(s)f(u(s))ds \\ &\geq \lambda \hat{h} t\big(- f(\frac{\beta}{2})\big). \end{aligned}$$

Integrating again from 0 to t_2, we obtain,

$$t_2 \leq \tilde{c}\lambda^{-\frac{1}{2}}, \text{ where } \tilde{c} = \Big(\frac{-\beta}{\hat{h}f(\frac{\beta}{2})}\Big)^{\frac{1}{2}} > 0. \tag{4}$$

By the Mean Value Theorem, there exists a $\bar{t} \in [0, t_2]$ such that $u(t_2) - u(0) = u'(\bar{t})(t_2)$ and by (4), $u'(\bar{t}) \geq \frac{\beta}{2\tilde{c}}\lambda^{\frac{1}{2}}$. Since u' increases in $[0, t_1]$, this implies

$$u'(t) \geq \frac{\beta}{2\tilde{c}}\lambda^{\frac{1}{2}}, \ \forall t \in [t_2, t_1]. \tag{5}$$

Integrating (5) from t_2 to t_1 we see that $(t_1 - t_2) \leq \tilde{c}\lambda^{-\frac{1}{2}}$. This and (4) implies $t_1 \leq O(\lambda^{-\frac{1}{2}})$. Similarly we can also prove $1 - \hat{t_1} \leq O(\lambda^{-\frac{1}{2}})$.

Lemma 2.3. *Given $M > 0$, there exists $\lambda(M)$ such that if $\lambda > \lambda(M)$ then $u(\hat{t}) \geq M$ for some $\hat{t} \in (t_1, \hat{t_1})$.*

Proof. Let $v := u - \beta$, then $v > 0$ in $(t_1, \hat{t_1})$ and satisfies:

$$\begin{cases} -v'' = \lambda h(t) \dfrac{f(u)}{u-\beta} v, \quad 0 < t < 1 \\ v(t_1) = v(\hat{t_1}) = 0. \end{cases} \tag{6}$$

Also,

$$-\Big(sin\Big(\frac{\pi(t-t_1)}{(\hat{t_1}-t_1)}\Big)\Big)'' = \frac{\pi^2}{(\hat{t_1}-t_1)^2} sin\Big(\frac{\pi(t-t_1)}{(\hat{t_1}-t_1)}\Big). \tag{7}$$

Multiplying (6) by $sin(\frac{\pi(t-t_1)}{(\hat{t_1}-t_1)})$ and integrating from t_1 to $\hat{t_1}$, we have

$$\int_{t_1}^{\hat{t_1}} cos\Big(\frac{\pi(s-t_1)}{(\hat{t_1}-t_1)}\Big)\frac{\pi}{(\hat{t_1}-t_1)} v' ds = \int_{t_1}^{\hat{t_1}} \lambda h(s) \frac{f(u)}{u-\beta} v sin\Big(\frac{\pi(s-t_1)}{(\hat{t_1}-t_1)}\Big) ds \tag{8}$$

and multiplying (7) by v and integrating from t_1 to $\hat{t_1}$, we have

$$\int_{t_1}^{\hat{t_1}} cos\Big(\frac{\pi(s-t_1)}{(\hat{t_1}-t_1)}\Big)\frac{\pi}{(\hat{t_1}-t_1)} v' ds = \int_{t_1}^{\hat{t_1}} \frac{\pi^2}{(\hat{t_1}-t_1)^2} v sin\Big(\frac{\pi(s-t_1)}{(\hat{t_1}-t_1)}\Big) ds. \tag{9}$$

Now subtracting (9) from (8) we see easily that,

$$\lambda \frac{f(u)}{u-\beta} h(t) = \frac{\pi^2}{(\hat{t_1}-t_1)^2} \text{ for some } t \in (t_1, \hat{t_1}). \tag{10}$$

Note that $\inf_{t\in(0,1)} h(t) > 0$ and from Lemma 2.2 without loss of generality we can assume $(\hat{t_1} - t_1) > \frac{1}{2}$. Thus for $\lambda \gg 1$, (10) is true only if $\frac{f(u)}{u-\beta} \to 0$. Since f satisfies (H_2) this implies $||u||_\infty \to \infty$ when $\lambda \to \infty$.

Lemma 2.4. *There exists $k > 0$ such that $u(t) > \lambda k$ for $t \in [\frac{1}{4}, \frac{3}{4}]$ if $\lambda \gg 1$.*

Proof. We first claim $u(t) > \frac{\beta+\theta}{2}$ for $t \in [\frac{1}{4}, \frac{3}{4}]$. Recall $t_0 \in (t_1, \hat{t_1})$ is the point at which u has it's maximum. By Lemma 2.3 given $M > 0, \exists\ \lambda(M)$ such that if $\lambda > \lambda(M)$ then $u(t_0) \geq M$. Since $u'' < 0$ in (t_1, t_0), for $t \in [t_1, t_0]$ we have

$$u(t) \geq \frac{(u(t_0)-\beta)}{t_0 - t_1}(t - t_1) + \beta. \tag{11}$$

Similarly for $t \in [t_0, \hat{t_1}]$, we can get

$$u(t) \geq \frac{(u(t_0)-\beta)}{\hat{t_1} - t_0}(\hat{t_1} - t) + \beta. \tag{12}$$

Now by Lemma 2.2, for $\lambda \gg 1$ we can assume $t_1 < 0.2$ and $\hat{t_1} > 0.8$. Hence from (11), (12) and Lemma 2.3, the claim $u(t) > \frac{\beta+\theta}{2}$ holds when λ is large. Now let $G(t,s)$ be the Green's function associated with problem (2). Then

$$\begin{aligned}u(t) &= \lambda \int_0^1 G(t,s)h(s)f(u(s))ds \\ &\geq \lambda\Big[\int_0^{t_1} G(t,s)h(s)f(u(s))ds + \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)h(s)f(u(s))ds \\ &\quad + \int_{\hat{t_1}}^{1} G(t,s)h(s)f(u(s))ds\Big].\end{aligned}$$

But by Lemma 2.2, $t_1 \to 0$ and $\hat{t_1} \to 1$ as $\lambda \to \infty$. Hence for $\lambda \gg 1$, $u(t) \geq \lambda k$ for $t \in [\frac{1}{4}, \frac{3}{4}]$, where $k = \frac{1}{2}\hat{h} f(\frac{\beta+\theta}{2}) \inf_{[0,1]} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)ds$, which proves the lemma.

Lemma 2.5. *There exists $\bar{\lambda}$ such that if $\lambda \geq \bar{\lambda}, u(t) \geq \lambda d(t, \partial\Omega)$, where $\Omega = (0,1)$.*

Proof. Let σ be the unique solution of

$$\begin{cases} -\sigma''(t) = \chi_{[\frac{1}{4},\frac{3}{4}]}h(t), \quad 0 < t < 1 \\ \sigma(0) = \sigma(1) = 0, \end{cases} \tag{13}$$

where χ is the characteristic function. By Hopf's maximum principle there exists $\bar{c} > 0$ such that $\sigma(t) \geq \bar{c}e(t)$ $\forall t \in [0,1]$, where e is the solution of $-e''(t) = h(t)$ in $(0,1)$ and $e(0) = e(1) = 0$. Let $\underline{M} > 0$ be such that $P = \bar{c}f(\underline{M}) + f(0) > 0$ and let u_1, u_2 satisfy $-u_1'' = \lambda f(\underline{M})\chi_{[\frac{1}{4},\frac{3}{4}]}h(t)$ in $(0,1), u_1(0) = u_1(1) = 0$ and $-u_2'' = -\lambda f(0)h(t)$ in $(0,1), u_2(0) = u_2(1) = 0$. Then by Lemma 2.4, if $\lambda > \frac{M}{k}$, we have

$$\begin{aligned} -u'' &= \lambda f(u)h(t) \\ &\geq \lambda f(\underline{M})\chi_{[\frac{1}{4},\frac{3}{4}]}h(t) + \lambda f(0)h(t) \end{aligned}$$

and thus, by the maximum principle, $u(t) \geq u_1(t) - u_2(t) = \lambda f(\underline{M})\sigma(t) + \lambda f(0)e(t)$. Hence

$$u(t) \;\geq\; \lambda f(\underline{M})\bar{c}e(t) + \lambda f(0)e(t) = \lambda P e(t), \quad \forall t \in (0,1).$$

Let $L > 0$ be such that $e(t) \geq Ld(t, \partial\Omega)$ for all $t \in [0,1]$. Hence $u(t) \geq \lambda\tilde{K}d(t,\partial\Omega)$ for all $t \in (0,1)$ where $\tilde{K} = PL$. Now let $D := [\epsilon, 1-\epsilon]$, for some $\epsilon > 0$. Then $u(t) \geq \lambda\tilde{K}\epsilon$ for all $t \in D$. Let u_3 be the unique solution to $-u_3''(t) = \chi_D h(t)$ in $(0,1), u_3(0) = u_3(1) = 0$. Since f satisfies (H_1), for $\lambda \gg 1$, $f(\lambda\tilde{K}\epsilon)u_3(t) + f(0)e(t) \geq d(t, \partial\Omega)$ in $[0,1]$. Hence for $\lambda \gg 1$, $-u'' = \lambda h(t)f(u(t)) \geq \lambda\Big(f(\lambda\tilde{K}\epsilon)\chi_D h(t) + f(0)h(t)\Big)$, and thus by the maximum principle $u(t) \geq \lambda\Big(f(\lambda\tilde{K}\epsilon)u_3(t) + f(0)e(t)\Big) \geq \lambda d(t, \partial\Omega)$ for all $t \in [0,1]$, if λ is large, which proves the lemma.

Lemma 2.6. *For each $\lambda > 0$, there exists $\bar{M}(\lambda)$ such that $||u||_\infty \leq \bar{M}$.*

Proof. Due to hypothesis (H_3), $\int_0^1 h(s)ds \equiv A < \infty$. By (H_2), there exists $\bar{K}$ such that $f(z) \leq \lambda^{-1}(A+1)^{-1}z + \bar{K}$, for all $z \geq 0$. Since $G(s,t) \leq 1/4$ for all $s, t, \in [0,1]$ we have

$$\begin{aligned} \|u\|_\infty &= u(t_0) \\ &= \lambda \int_0^1 G(s,t_0)h(s)f(u(s))ds \\ &\leq \lambda \int_0^1 G(s,t_0)h(s)(\lambda^{-1}(A+1)^{-1}u(t_0) + \bar{K})ds \\ &\leq \frac{1}{2}u(t_0) + \lambda\bar{K}A. \end{aligned} \tag{14}$$

Therefore $\|u\|_\infty \leq 2\lambda\bar{K}A$, which proves the lemma.

3 Proof of Theorem 1.1

We first claim that (2) has a maximal positive solution $\bar{u}$ for $\lambda \gg 1$. Given $\lambda > 0$, choose $J = J(\lambda) > \lambda f(\bar{M}(\lambda))$ where $\bar{M}(\lambda)$ is as in the previous section. Further choose $J \gg 1$ so that $J \geq \lambda f(J||e||_\infty)$ where e is as before (see Lemma 2.5). This is possible since f satisfies (H_2). Now if v is any solution of (2), then $-(Je-v)''(t) = Jh(t) - \lambda f(v)h(t) \geq h(t)(J - \lambda f(\bar{M}(\lambda))) > 0$ in $(0,1)$. By the maximum principle we obtain $Je \geq v$ in [0, 1]. Also, $-(Je)''(t) = Jh(t) \geq \lambda f(Je(t))h(t)$ in $(0,1)$. Hence Je is a supersolution of (2) larger than any solution of (2). However from [11] we know that (2) has a positive solution for $\lambda \gg 1$. Hence (2) must have a maximal positive solution $\bar{u}$ for $\lambda \gg 1$.

Now let u be any other positive solution of (2). To establish our theorem, we will now show that $\bar{u} \equiv u$ for $\lambda \gg 1$. Since $\bar{u}$ and u are solutions of (2) we obtain

$$\begin{aligned} -(\bar{u}-u)''(t) &= \lambda h(t)\Big(f(\bar{u}(t)) - f(u(t))\Big), \quad 0<t<1 \\ (\bar{u}-u)(0) &= (\bar{u}-u)(1) = 0. \end{aligned} \tag{15}$$

By the Mean Value Theorem there exists ξ such that $u \leq \xi \leq \bar{u}$ in $[0,1]$ and

$$\begin{aligned} -(\bar{u}-u)''(t) &= \lambda h(t)f'(\xi)(\bar{u}(t)-u(t)), \quad 0<t<1 \\ (\bar{u}-u)(0) &= (\bar{u}-u)(1) = 0. \end{aligned} \tag{16}$$

Multiplying (2) by $(\bar{u}-u)$, (16) by u, integrating and using the fact that f is concave we obtain

$$\lambda \int_0^1 \Big(f(u) - f'(u)u\Big)h(s)(\bar{u}-u)ds \leq 0. \tag{17}$$

Now by (H_2), there exists $a > 0, b > 0$ such that $f(z) - f'(z)z \geq b$ whenever $z \geq a$ and from Lemma 2.5, $u(t) \geq a$ if $d(t, \partial\Omega) \geq \frac{a}{\lambda}$ when $\lambda \gg 1$. Let $\Omega_+ = [\frac{a}{\lambda}, 1-\frac{a}{\lambda}]$ and $\Omega_- = (0, \frac{a}{\lambda}) \cup (1-\frac{a}{\lambda}, 1)$. Then from (17) we obtain

$$I = \int_{\Omega_+} b(\bar{u}-u)h(s)ds + \int_{\Omega_-} f(0)(\bar{u}-u)h(s)ds \leq 0. \tag{18}$$

Here we have used $f(z)-zf'(z) \geq f(0)\ \forall z \geq 0$, which follows from the fact that f is concave.

Next let m_1, m_2 satisfy $-m_1''(t) = \chi_{\Omega_+}h(t)$ in $(0,1), m_1(0) = m_1(1) = 0$ and $-m_2''(t) = \chi_{\Omega_-}h(t)$ in $(0,1), m_2(0) = m_2(1) = 0$ respectively. Multiplying (16) by $bm_1(t) + f(0)m_2(t)$ and integrating by parts we obtain,

$$\begin{aligned} I &= \int_{\Omega_+} b(\bar{u}-u)h(s)ds + \int_{\Omega_-} f(0)(\bar{u}-u)h(s)ds \\ &= \lambda \int_0^1 f'(\xi)(\bar{u}-u)h(s)[bm_1(s) + f(0)m_2(s)]ds. \end{aligned} \tag{19}$$

As λ tends to $+\infty$, m_1 tends to e and m_2 tends to 0 in $C^1[0,1]$. Hence for $\lambda \gg 1$ $bm_1(t) + f(0)m_2(t) > 0$ in $(0,1)$. Thus from (18) and (19) we see that $I = 0$ for $\lambda \gg 1$, and from (19), we see that this is possible only if $\bar{u} \equiv u$ in $[0,1]$, which proves Theorem 1.1.

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