Humanistic Mathematics Network Journal

Issue 20 Article 31

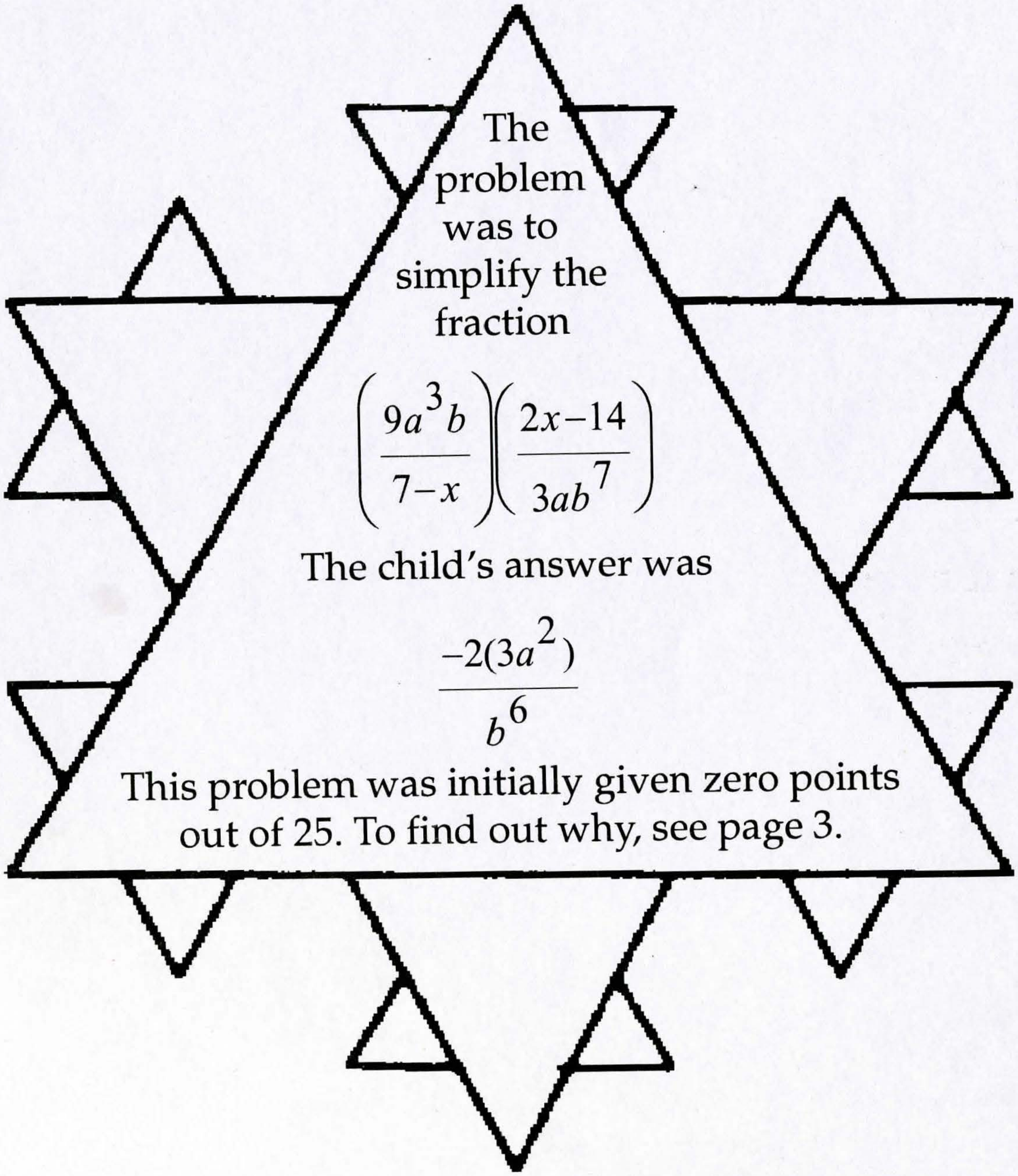
7-1-1999

Complete Issue 20, 1999

ISSN 1065-82

Humanistic Mathematics Network Journal

Issue #20 July 1999



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If possible, avoid footnotes; put references and bibliography at the end of the text, using a consistent style. Please put all figures on separate sheets of paper at the end of the text, with annotations as to where you would like them to fit within the text; these should be original photographs, or drawn in dark ink. These figures can later be returned to you if you so desire.

Two copies of your submission, double-spaced and preferably laser-printed, should be sent to:

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The Journal uses the programs Microsoft Word, PageMaker and Photoshop for the Macintosh. If you used any of these programs to prepare your paper or figures, it would be very helpful to us if you sent a copy of the files on diskette along with the printed copies of the paper.

NOTE TO LIBRARIANS

Humanistic Mathematics Network Journal issues #8–#20, ISSN#1065-8297, are the successors to the *Humanistic Mathematics Network Newsletter*, ISSN#1047-627X.

SUBSCRIPTIONS

The Humanistic Mathematics Network Journal is distributed free to all libraries and individuals who request copies and subscriptions. Please address subscription requests to Prof. Alvin White.

COVER

Educational blizzard: Inside the Koch Snowflake from Catherine A. Gorini's essay on the relation between art and mathematics is an example of poor math teaching given by Jerome Dancis. In this issue Dancis, Jack Lochhead and Frances Kurwahara Lang offer views on the state of K-12 mathematics education.

Publication of the Humanistic Mathematics Network Journal is supported by a grant from the
EXXON EDUCATION FOUNDATION.

Humanistic Mathematics Network Journal #20
July 1999

The poems featured in this issue are by sixth and seventh grade students taught by Margaret Schaffer at Hale Middle School in Los Angeles, CA.

From Newsletter #1

Dear Colleague,

This newsletter follows a three-day **Conference to Examine Mathematics as a Humanistic Discipline** in Claremont 1986 supported by the Exxon Education Foundation, and a special session at the AMS-MAA meeting in San Antonio January 1987. A common response of the thirty-six mathematicians at the conference was, "I was startled to see so many who shared my feelings."

Two related themes that emerged from the conference were 1) teaching mathematics humanistically, and 2) teaching humanistic mathematics. The first theme sought to place the student more centrally in the position of inquirer than is generally the case, while at the same time acknowledging the emotional climate of the activity of learning mathematics. What students could learn from each other and how they might come to better understand mathematics as a meaningful rather than arbitrary discipline were among the ideas of the first theme.

The second theme focused less upon the nature of the teaching and learning environment and more upon the need to reconstruct the curriculum and the discipline of mathematics itself. The reconstruction would relate mathematical discoveries to personal courage, discovery to verification, mathematics to science, truth to utility, and in general, mathematics to the culture within which it is embedded.

Humanistic dimensions of mathematics discussed at the conference included:

a) An appreciation of the role of intuition, not only in understanding, but in creating concepts that appear in their finished versions to be "merely technical."
b) An appreciation for the human dimensions that motivate discovery: competition, cooperation, the urge for holistic pictures.
c) An understanding of the value judgments implied in the growth of any discipline. Logic alone never completely accounts for what is investigated, how it is investigated, and why it is investigated.
d) A need for new teaching/learning formats that will help discourage our students from a view of knowledge as certain or to-be-received.
e) The opportunity for students to think like mathematicians, including chances to work on tasks of low definition, generating new problems and participating in controversy over mathematical issues.
f) Opportunities for faculty to do research on issues relating to teaching and be respected for that area of research.

This newsletter, also supported by Exxon, is part of an effort to fulfill the hopes of the participants. Others who have heard about the conferences have enthusiastically joined the effort. The newsletter will help create a network of mathematicians and others who are interested in sharing their ideas and experiences related to the conference themes. The network will be a community of support extending over many campuses that will end the isolation that individuals may feel. There are lots of good ideas, lots of experimentation, and lots of frustration because of isolation and lack of support. In addition to informally sharing bibliographic references, syllabi, accounts of successes and failures. . . the network might formally support writing, team-teaching, exchanges, conferences. . . .

Alvin White
August 3, 1987

From the Editor

Every week or so we are confronted with newspaper articles about new proposals to reform the teaching and learning of mathematics. Some of the proposals have merit.

The situations described in the two articles by Jerome Dancis, beginning on page 3, are not mentioned in editorials or proclamations from a governor's office. The victimization of those students is ignored by the politicians and editorial writers. I hope that the publication of the articles will contribute to an awareness of and the repair of the scandalous situation.

*****

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with the message: "subscribe HMNJ-L@hmc.edu."

After you have subscribed, to send mail to the list, write to

HMNJ-L@hmc.edu,

and your email will be forwarded to everyone on the list.

Gian-Carlo Rota: In Memoriam

Reprinted from the June 1999 issue of SIAM News.

Gian-Carlo Rota, a professor of applied mathematics and philosophy at the Massachusetts Institute of Technology, died of heart failure, apparently in his sleep, around April 18, 1999.

Rota was born in Vigevano, Italy, on April 27, 1932. Forced to flee Mussolini's death squads, his family left Italy in 1945, after which they lived for a time in Ecuador. His sister, Ester Rota Gasperoni, recounted the family's escape from Italy in two books, *Orage sur le Lac* and *L'arbre des Capulies*. Rota came to the United States in 1950; he received a BA from Princeton University and, in 1956, a PhD from Yale University under Jacob T. Schwartz.

He held postdoctoral positions at the Courant Institute and Harvard University before arriving at MIT in 1959. Except for a short hiatus at Rockefeller University (1965-67), he remained at MIT until his death, much to the good fortune of countless MIT undergraduates, graduate students, visitors, and faculty who were able to share his enthusiasm and joy for mathematics, philosophy, and life in general.

This brief chronicle of events, of course, in no way conveys what the man was really like. Being associated with Gianco meant far more than mathematical discussions and lectures. He became an important part of the personal lives of his friends and colleagues. He took a genuine interest in the well-being of all his associates and made many selfless sacrifices of time, money and intellectual effort on their behalf. It almost goes without saying that this altruism, together with his beautifully prepared and delivered lectures, made him one of the most popular and respected teachers at MIT.

Rota had far-ranging mathematical interests, but his first love (developed after he had received his doctorate in functional analysis) was combinatorics. He intuitively realized that combinatorics, which in the early 1960's was not considered a "serious" subject and was regarded with disdain by most leading mathematicians, had tremendous potential to develop into a mature and important area which would enrich many other seemingly unrelated parts of mathematics. Intuitive understanding of this type was characteristic of Rota's work—he was always looking for the "big picture" and trying to understand the true essence of any subject in which he was interested.

A seminal development for the future of combinatorics was the Foundations, a series of papers inaugurated by Rota with his now famous "On the Foundations of Combinatorial Theory I. Theory of Möbius Functions" (*Z. Wahrscheinlichkeitsthorie* 2 (1964) 340-368). This paper immediately captured the imagination of many young mathematicians (including myself) and planted the seeds for many subsequent developments within combinatorics, such as the theory of topological combinatorics and the tremendous expansion of matroid theory. Rota followed up the "Foundations I" paper with over 80 further papers in combinatorics (to say nothing of papers, essays, and reviews in many other areas of mathematics and in philosophy) that established him as the founding father and leading guru of the new subject of algebraic combinatorics.

An important watershed in the development of combinatorics was the NSF Advanced Science Seminar in Combinatorial Theory at Bowdoin College during the summer of 1971. Gian-Carlo presided over this meeting as a godfather of the "new combinatorics." He was involved in all aspects of the eight-week meeting, from the mathematical content to social activities. I recall one pedagogical innovation of his-the tandem lecture. He would choose about six people from the audience who had to leave the room and not talk to each other. He would then call them into the lecture hall one at a time to deliver a five-minute lecture. Each lecture had to be a continuation of the previous lecture, based on what the previous speaker had left on the blackboard.

There were many other facets of Rota's complex personality that I can only hint at here. He was deeply interested in phenomenology and wrote many papers and essays in this area. Although English was not his native language, he achieved a mastery of it that far surpassed the best efforts of most native speakers. He

wrote innumerable completely honest and lucid essays on mathematicians and the practice of mathematics, many of them collected into the books *Discrete Thoughts* and *Indiscrete Thoughts*. He was working on a book of provocative quotations entitled *Forbidden Thoughts* at the time of his death.

Rota developed *Advances in Mathematics* virtually single-handedly into one of the leading journals of research mathematics, and he was the editor-in-chief of the *Encyclopedia of Mathematics*, a book series that contains definitive expositions of a wide range of mathematical topics. In the course of his career, he held visiting positions at ten universities throughout the world, and, beginning in 1966, he was a consultant at Los Alamos National Laboratory. He received four honorary degrees (and was just about to receive another, from Nankai University); also among his honors are the Steele Prize of the American Mathematical Society (1988) and the Killian Faculty Achievement Award at MIT (1996). He was appointed the Norbert Wiener Professor of Mathematics at MIT for a five-year period beginning in 1998, and he was the Colloquium Lecturer of the American Mathematical Society in 1998.

Gianco had the extraordinary ability to touch deeply the lives of all with whom he associated, whatever their background and experience. Rarely, if ever, has the passing of a professional mathematician left such a large void.

Richard Stanley
Department of Mathematics, MIT

Math applies to everyday life
Whether you are measuring sugar or dealing with
strife.
Counting calories in a diet
Saving for a car, and it is time to buy it.
Planning ahead for your college years,
Calculating your interest in the bank with tears.
Taking a vacation in a car,
Planning for motels and the distance thus far.
Building a house and measuring the walls,
Driving the distance to the nearest malls.
Filling your car with a tank of gas,
Using a movie coupon as a pass.
Add, subtract, multiply or divide,
The choices are many, and you must decide.
Baking a pie at the right degree,
Means a lot to you and your family.
Geometry helps in the game of pool,
You need to know angles if you want to rule.
Music requires counting the beat,
Rhythm and rhyme and good timing meet.
Understanding math is important, you see,
It is passed through the years throughout history.
Math applies without a doubt to all that you do.
Understanding math can carry you through.

Geoffrey Smith

Algebra algebra,. You were so tough!
Of X's and Y's I'd had quite enough!
Expressions, equations, inequalities too,
As to finding solutions, I hadn't a clue!
Multiplication, addition, division, subtraction,
All your hard work drove me to distraction!
Your secretive variable was always unknown,
Unless you got lucky and found it alone!
But then came a lesson I learned from a friend,
And now I'm no longer at my wit's end.
When faced with a value that seems undefined,
The trick is to relax and open your mind,
So the knowledge that's in there can flow unarrested,
And answers can come easily in class when I'm tested.

Ryan Best

Math is a subject that is the best,
I like it better than all the rest.
It is so complex and so interesting,
I never know how much work it could be.
First with addition, then subtraction,
Then moving on to things like fractions.
Multiplication, division, and more,
Then on to geometry with shapes galore.
As I grew older with more math ahead,
It was so fun, some think I never could dread.
Then on to algebra and the metric system,
Having good attendance, I didn't want to miss 'em.

Alexandra Holliday

Middle School Math Teaching and How It Harms Our Children

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I wondered why American children learn so little arithmetic in middle school (grades 7 and 8). I also wondered why most American children need to be retaught arithmetic in Grade 9 and why many colleges even teach arithmetic (under the name of developmental mathematics). (In most countries, children complete their study of arithmetic in grade 7). As my children went through middle school, I watched as their self-confidence to do mathematics eroded. This forced me to monitor what was happening in their classes. I analyzed this from two perspectives: as a parent and as a college teacher of mathematics. I am writing this report to share my findings with other parents so that they may understand what is happening to their children.

THE CALLING-OUT-THE-ANSWERS METHOD OF INSTRUCTION

One professor of mathematics education has noted: "[Mathematics in middle schools is] delivered through an instructional model comprising call out the answers to the homework, work a few examples, assign seatwork. There is very little communication of ideas about mathematics among students or between teacher and students, either asked for or allowed in many classrooms. Kids leave thinking at the door of the classroom and go on automatic pilot." This teaching style is conducive to students collecting misconceptions about math as well as turning children into math haters. Also, when a child has a wrong answer he/she will not have a clue as to the errors or how to correct them (after merely hearing the answers called out).

COOKBOOK TYPE INSTRUCTION

It is standard for math textbooks and K-8th grade teachers to provide students with cookbook type directions of what to do in math. It is rare for students to be assigned problems that they have not been programmed to do. It is rare for textbooks and K-8th grade teachers to provide the students with understanding-based explanations which tell the whys and the wherefores of mathematics.

With skill-based instructional methods, the students spend large amounts of time mindlessly doing dull exercises in a rote manner. Doing calculations in a skill-based manner divorced from understanding results in students memorizing an excessive number of formulas which **are easily confused and easily forgotten over the summer**. Also, skill-based instruction leaves the students stymied when confronted with a problem that is only mildly different from the ones they have been programmed to do. This is one reason for low math SAT scores.

Providing students with understanding-based explanations of mathematics is not a common teaching technique. **"Explaining mathematics to students"** was **not** included in my local school system's Spring 1990 proposed list for "The [Math] teacher's role in discourse." The natural result is that while the students may develop some proficiency in math skills, they do not gain any understanding of the mathematics. This results in students collecting all sorts of misconceptions about mathematics and making a wide range of mistakes while doing calculations. This, in turn, results in less success in high school math classes. Remedying these misconceptions is difficult.

A major reason for the low achievement in mathematics in U.S. schools is that the mathematics curriculum is overly repetitive and fragmented.[1] In France addition of fractions is first taught in 8th grade; here it is taught in 4th, 5th, 6th, 7th and 8th grades. At the end of 8th grade, more French students know how to add fractions than American students.

INEFFICIENT USE OF CLASS TIME

Eighth grade math teachers use (on average) only 40% of class time for teaching, (that is for the teacher talking, class discussions and students working in groups). The other 60% of class time is used for tests

and seatwork. Increasing teaching time to 60% would enable students to progress through both the 7th grade and half of the 8th grade syllabi in a single year, thereby increasing the pupils' sense of accomplishment and reducing the level of boredom.[2]

SEATWORK IS USUALLY GLORIFIED BUSYWORK

Interesting and/or challenging seatwork is wonderful, and children need to do some exercises to learn anything, but unfortunately, in many classes most seatwork is just glorified busywork. Children learn very little while doing a large number of arithmetic exercises, and it even has a major counterproductive effect of turning misconceptions into "bad" habits. (This has been observed by Owen and Sweller (JRME Vol. 20) and by Dr. Kastner of U.MD. (at Towson) (retired) resp.)

TEACHING TO MASTERY

One of my children's teachers actually allocated more than half the class time to tests and quizzes. She would give a test on Monday, then a retest on Wednesday for those who did poorly on Monday's test, and a second retest on Friday for those who did poorly on Wednesday's test. The pupils who were not taking the retests were relegated to the boredom of busywork. This class took three quarters of the year to cover the first half of the syllabus. The excuses--lack of time and "teaching to mastery." The considerable amount of time squandered on retesting could have been allocated to teaching the untaught part of the syllabus. Parents, who insist that their children receive good grades, encourage and even coerce teachers to waste class time on retesting until the scores are high. This occurred at a middle school which received *national honors* from President Bush as a "School of Excellence." It is also a school of "choice" as it has a special "magnet" program. *All the parents* of children in this special program had specifically *chosen* it for their child even though it meant sacrificing their child's neighborhood friendships by bussing them to this non-neighborhood school.

That the overemphasis on testing sets students up to do poorly when they enter college was noted in discussions by faculty members from 10 departments of my university, together with high school teachers and college freshmen. The main conclusion was: **"The overemphasis on testing, skill development and fact content, etc. [in schools] seems to have inhibited [student] interest in learning, motivation, ability to work with and enjoy ideas, use creativity, and attain satisfaction from an educational experience."**

STUDENT WORK IS RARELY CORRECTED

Of course, all tests are graded, but teachers rarely specify just what the errors are or even if the errors are of a mathematical or a stylistic nature. Usually neither students nor teachers analyze patterns of errors. The natural consequence of not dealing with students' errors and misconceptions is that students do not learn from their mistakes, and they **repeat the same mistakes** on later tests. **"Correcting students' work"** was **not** on my local school system's Spring 1990 proposed list of "Instructional Interventions."

ALL-OR-NOTHING GRADING

The subject was factoring; the problems were to simplify several mathematical formulas. A child factored the numerator correctly, factored the denominator correctly, canceled like factors on top and bottom correctly, and then made the arithmetic error of $2x3 = 8$; score 0 out of 20 points because the final answer is wrong. On this test of factoring, the child factored *everything correctly*, clearly demonstrating full knowledge of factoring, and "earned" the flunking grade of 55%. (At my university, where we do not use all-or-nothing grading, the score on this *same* test paper would have been in the 90's.) On other tests, mathematically *correct* work received *no* credit when the form of the child's answer was different from the one in the teachers' manual. Sometimes this is the result of teacher rigidity; sometimes it is the result of the teacher's lack of mathematical knowledge, and of course, it takes several times as long to read test papers as it does to merely glance at the final answers.

Considering that teachers are already overworked, where could the extra time be found for correcting student work instead of merely doing all-or-nothing grading?

REDUCE TESTING

How to solve three problems with one command from the principal: Reduce testing time by half.

- This would free up one half of the time the teacher now spends on grading papers so that it could be used for correcting the work as well.
- This would free up much class time which could result in the students making significantly more

progress in a school year.
- This would result in the students becoming less test-oriented and suffering less test anxiety.

This reform of reducing testing time by half in middle school math classes could be instituted today at **no cost** to the taxpayers. I am aware that this reform is antithetical to middle school culture, and insisting on it is an infringement of the principal's prerogative of providing educational leadership as he/she wishes.

MISLEADING GRADING

All-or-nothing grading underevaluates a child's knowledge. Giving busywork the same importance as tests *often* produces grades that do *not* reflect learning. When a child receives a low grade and/or is told to repeat a course, the child, his/her parents and guidance counselors do not know whether the child has not learned the math or whether this is a consequence of all-or-nothing grading and/or the child not doing the busywork. In fact, these were the reasons my child's math grades were so low that the child was placed on the list to repeat the course even though, *according* to the teacher, the child had indeed learned all the mathematics! This occurred at the school of "excellence" and "choice" that my children had the priviledge of attending. On the other side, many students receive good grades without having learned the math. Last year, 10% of the freshmen at my university were ordered to take two giant steps backwards and retake Algebra 1 in spite of the fact that they received grades of A or B in Algebra 2 in high school.

THE MIDDLE SCHOOL *TEACHERS'* KNOWLEDGE OF MATHEMATICS MAY BE *CRUCIAL* TO YOUR CHILD'S SUCCESS IN HIGH SCHOOL MATHEMATICS

In California a survey by Ms. K. Culler revealed that among Grade 8 Algebra 1 students who were taught by a teacher who did **not** have a math major or minor in college, only 20% did well in Algebra 2 two years later. Students who were lucky enough to have a Grade 8 Algebra 1 teacher who *did* have a math major or minor in college did much better in Algebra 2. It is rare that a middle school principal checks whether or not a teacher is *fully* qualified to teach math before hiring or assigning a class.

THESE INSTRUCTIONAL STRATEGIES ARE BORING

At my children's school, it was not uncommon for students to pass notes via flying paper airplanes. I did my parental duty by asking my child why she was passing notes instead of listening to the teacher. The answer: "If I can learn everything by listening with one ear, why do I need to use both ears? In science class I don't pass notes because I need to listen with both ears." Most students will pay close attention to an interesting lesson, but boring lessons force students to "tune out." Passing notes is a harmless way for children to protect themselves from boredom; less polite students act out their boredom in disruptive manners. There was so much gossiping that the principal found it necessary to admonish the students to do less talking and socializing in class. When a student walks into a class he/she should sit down quietly and start working with no discussions in between. The principal did *not* say that there would be any reduction in the level of boredom at school. Basically, he was blaming the victims.

How common is middle school boredom? Clinical psychologist Dr. Robert Weigl and social worker Abby Sternberg's survey of causes of **stress** at Hayfield Secondary School in Alexandria, Virginia, (a suburb of Washington D.C.) found that 50% or more of the pupils in each grade from grade 7 to grade 12 were under **stress due to the boredom** of school work. Also, 40-50% of the pupils were under **stress due to too much school work**, especially too much *busy* work that took up too much time. School work being too hard was *not* a common source of stress. Of course, boredom results in students losing interest in school work.

Predictably, the instructional strategies described above do not work for middle school math even though the syllabus's level of sophistication is quite low, even by standards for 13-year-olds. This is why so many students need to be retaught arithmetic under a pseudonym like "General or Developmental Math" in grade 9. They will **not** prepare students for the MD State Dept. of Education's new grade 8 math test.

THE BOTTOM LINE

These methods of instruction do not result in much learning of mathematics, even by the best students. In Sept. 1990 the select students entering the special Science/Tech program at Eleanor Roosevelt High School (in a suburb of Washington D.C.) included 178 students who had all successfully completed Algebra

1 in middle school. Only 20% of these select students had learned enough algebra to score 70 or higher (out of 100) on the simple Algebra 1 Criterion Reference Test.

The types of instruction I have described are a *waste* of the taxpayers' money and of the teachers' and pupils' time. Worse yet, they are **counterproductive** since the main results are

- pupils with many misconceptions about arithmetic. The misconceptions become bad habits, difficult to remedy. This results in the pupils making many mistakes while studying algebra in high school.
- pupils with greatly reduced self-confidence
- pupils who hate math (many students leave elementary school liking math, but they quickly become math haters in middle school) and
- pupils (even bright pupils), their parents and guidance counselors who incorrectly believe that the pupils are poor learners of mathematics. The natural consequence of this is that the students choose non-challenging mathematics courses in high school, thereby limiting their career choices.

When I consulted other professors of mathematics and math education, I learned that the educational practices described here are very common in middle schools. Lowering class sizes, giving more tax money to schools, adding more days to the school year and school-based management (the current panaceas for improvement) will *not* address any of the problems mentioned in this report. In fact, school-based management may help to *perpetuate* these practices by reducing the pressure for improvement from a central administration.

Partially in reaction to the ineffectiveness of the types of instruction described above, the National Council of Teachers of Mathematics (NCTM) is recommending a new national math curriculum (outline) called the "Standards." These NCTM Standards call for a major deemphasis on skill-based learning and rote practice together with major emphasis on students gaining knowledge and understanding.

In 1987, in reaction to the ineffectiveness of math instruction, the MD State Dept. of Education was wise enough to issue the following:

SIDEBAR

The problem was to simplify the fraction

$$\left(\frac{9a^3b}{7-x}\right)\left(\frac{2x-14}{3ab^7}\right)$$

The child factored the numerator correctly and cancelled correctly. The child's answer was

$$\frac{-2(3a^2)}{b^6}$$

which is equal to

$$\frac{-6a^2}{b^6}$$

the answer demanded by the teacher. Initially, the teacher scored this question as zero points out of 25. On the remaining 3 questions full credit (75 points) was given. Thus, for this mathematically perfect test paper, the initial total score was 75% (C).

Of course the child had no idea as to why no credit was given for the work on problem #1. The following day, the child asked the teacher to point out the error. The teacher responded by giving partial credit of only 5 points out of 25 for this problem, raising the total score to 80%.

This occurred at a school of "**choice**" as it has a special "magnet" program. *The parents* (of the 200+ students in the "magnet" program) had *chosen* this program for their child even though this meant sacrificing their child's neighborhood friendships by having him/her bussed to this non-neighborhood school. When I informed the principal that this type of grading was occurring, as well as many other things listed in this report, he chose not to respond. Then I informed the assistant superintendent of schools, (the principal's boss) and also presented her with several test papers with mathematically correct answers marked wrong; her response was "Thank you for sharing this with us." The test papers were not regraded. The principal and the assistant superintendent had more important things to deal with; that year the school received *national honors* from President Bush as a "school of excellence," and the principal was promoted to be principal of a high school.

OFFICIAL GOALS FOR MATH INSTRUCTION[3]
(Emphasis added.)

Goal #1 is that students will "develop an appreciation of and a **positive attitude toward mathematics**."

Goal #2 is that students will develop an **understanding of mathematics**: concepts, properties and processes.

Goal #3 is that students will "acquire mathematical facts and skills."

Goal #4 is that students will "develop the ability to express and interpret mathematical ideas and relationships."

Goal #5 is that students will "develop the **mathematical reasoning ability** required in problem solving and decision-making situations."

Goal #6 is that students will develop the ability "to apply mathematics in personal, societal, technological, scientific and career settings."

Unfortunately, it is the rare middle school which has the expertise to make a serious effort to implement these goals.

NOTES

[1] This is one of the major conclusions of *The Underachieving Curriculum* by Curtis C. McKnight et al, a national report on the Second International Mathematics Study sponsored by the International Association for the Evaluation of Educational Achievement (1987). This report is the most thorough analysis of mathematics education in American schools. The name of the report comes from the report's conclusion that the mathematics curriculum in American schools is an **underachieving curriculum**.

[2] The data on how teachers use class time is listed in *The Underachieving Curriculum*.

[3] These goals are presented and discussed in the wonderful booklet: *Mathematics — A Maryland Curricular Framework*. This framework was developed by June M. Danaher, specialist in mathematics, Maryland State Dept. of Education. The superintendents of the county public school systems in Maryland have signed off on these goals.

Changing the Subject, Or, Would You Hire a Good Clarinet Teacher to Teach Your Child The Violin?

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Roughly, the first (milder) half of this article appeared in the Forum column of the Prince George's Extra Section of the Washington Post, April 21, 1999, Page 4. Parts enclosed in [ ]were not included.

Sarah McKinney-Ludd, a language arts teacher in Prince George's County, said her assignment to teach a middle school math class "robbed kids of a year of education. I stayed up every night for 180 days," she said. "I didn't sleep. I can't sleep because you have to stay ahead of the kids...It is emotionally bankrupting." (Linda Perlstein's front-page article, The Washington Post, Feb. 15, 1999).

Not assigning music teachers to teach math or vice versa is a no-brainer, except to too many school administrators and the school board.

Hospitals do not let a lung specialist fix broken bones on a slippery, icy day when there is an overload of patients with broken bones. Building contractors do not have plumbers and electricians filling in for the others' jobs. Parents never hire a good clarinet teacher to teach a child the violin.

That the head of a middle school math department be a certified math teacher is another no-brainer, except

to too many school administrators and the school board. In the Prince George's County public school system, the official mathematics leaders in middle and elementary schools are appointed at the full discretion/whim of the principals. No knowledge of mathematics required. After all, the school is the principal's fiefdom.

The lack of knowledge of mathematics by heads of middle and elementary school math departments had prompted the PGCPSS to write a grant proposal requesting federal funds to pay for mathematics instruction for these math departments' heads. The proposal stated that a reason for the deficiency in knowledge of math was that principals were under no obligation to choose teachers knowledgeable in math as the heads of their middle and elementary school math departments.

Some time ago, a friend from Mitchellville complained that a correct answer was marked wrong on her child's math test paper. The teacher's response was that she was a history teacher, and she was doing the best she could with the math class.

Occasionally, an art teacher will do a good job teaching math (there is some geometry in art), but this is the rare exception. [A teacher, teaching out of subject, will frequently have trouble explaining the material. They will often teach in a dull, follow-the-dull-textbook manner, even those who are exciting teachers of their own subjects. Following a half-decent textbook might be semi-tolerable for a good teaching-out-of-subject teacher. But textbooks commonly range from terrible to horrific, which traps the teacher trying to learn the subject from the students' book. It also traps students trying to compensate for the teacher by reading the book. Staying up every night for 180 days, as McKinney-Ludd did, is not a viable solution, and even that did not result in a successful class. Often, it is a case of sink and swim except here the students are sinking while the principal swims—he succeeded in assigning a teacher to each class, his butt is covered.]

The first year brings all sorts of new and difficult challenges to a new teacher. It is a no-brainer not to add to this daunting burden by adding a class preparation in a different subject. It is a double no-brainer not to add two class preparations in two different subjects. [Rookie biology teacher Mike Maerten is teaching two different math courses in addition to biology at Wilde Lake High School in Columbia.]

The PGCPSS has a crisis in recruitment and retention of teachers. Placing teachers in the loathed, high-stress, emotionally bankrupting situation of teaching subjects they do not know strongly encourages/pushes teachers to leave our schools. This is highly counterproductive, at a time when the PGCPSS should be doing everything to keep teachers.

The part below was not printed.

"Hold teachers accountable for the learning (or lack of learning) of their students" has become a popular slogan for the 1990's. But it is absurd to hold a language arts teacher accountable for the lack of learning of students in a math class.

It is not just that Ms. McKinney-Ludd's students were robbed of a year of math class, but the following year they walked into a trap; they "progressed" to the next math course in which they were expected to know the math of Ms. McKinney-Ludd's course. I call this "getting educationally beaten up;" it occurs when students are placed in a non-viable learning situation like having to learn Grade 8 math which builds on and heavily uses the Grade 7 math they did not learn. They stand little chance; they are trapped.

The middle school teachers' knowledge of mathematics may be crucial to your child's success in high school mathematics. In California a survey by Ms. K. Culler revealed that among Grade 8 Algebra I students who were taught by a teacher who did not have a math major or minor in college, only 20% did well in Algebra 2 two years later.

Algebra classes, taught by other than math teachers, are often a waste of the taxpayers' money and of the teachers' and pupils' time. Worse yet, they are counterproductive since the main results are:

- Pupils with many misconceptions about algebra. The misconceptions become bad habits, difficult to remedy. (A natural consequence of the teacher's difficulty in explaining algebra)
- Pupils entering the next math class with low level and insufficient knowledge of the math "taught" in the previous math class. This is a trap and a

non-viable learning situation

- This results in many students "getting educationally beaten up." Which in turn, results in pupils with greatly reduced self-confidence,
- Pupils who hate math.
- Pupils (even bright pupils), their parents and guidance counselors who incorrectly believe that the pupils are poor learners of mathematics.

The natural consequence of this is that the students are misguided to low content mathematics and science courses in high school, thereby limiting their career choices. Not desirable in this technological era.

In the 1990s, high school Algebra 1 has become my campus's biggest math course, even though all the students had passed Algebra 2 in high school, and even though UMCP has a selective admission policy. High school algebra classes not taught by teachers with state certification for math contribute to the many high school graduates who need to be retaught high school Algebra 1 in college.

Even with certified math teachers, the students in the PGCPSS are not learning much math. It is a no-brainer not to make a bad situation worse by assigning non-math teachers to teach math.

Principals argue that they need the flexibility to assign teachers where needed (whether qualified or not), otherwise many classes would have to be enlarged or canceled. This sounds like our children's education is being held hostage. I believe it is better to cancel a class than to have a counterproductive one where children's education gets mucked up. I would strongly prefer that my child be in a class of 40 students with a knowledgeable teacher than in a class of 25 students with a teacher half ignorant of the subject (teachers with oversize classes should receive extra pay).

Principals, concerned about filling classes and the school systems' teacher retention crisis, should be treating good teachers professionally, not shabbily. Last year, the principals' union complained that "[the CEAs are] kind of bullying [principals] around. It is disrespectful, unprofessional treatment" (PG Journal June 24, 1998). The CEAs are also principals of high schools. A CEA who is bullying principals is probably terrorizing teachers. A decade ago, there was an unofficial parents group in Greenbelt, whose self appointed task was to defend the good teachers from the principal (now retired) of ERHS. Considering the retention difficulties, the school board should order the removal of bullying principals (incompetent ones also).

Last year, the PGCPSS started a new open information policy of providing parents with more information about the schools. I challenge the PGCPSS to tell the parents what percent of classes at each school are taught by teachers certified in the subject; the advertisements for magnet programs should tell the parents what percent of classes at each magnet program are taught by appropriately certified teachers.

Time for the PGCPSS school board to follow Washington D.C. and Virginia by prohibiting this practice of assigning teachers to teach classes outside of their certification. The education of our children is too important to be left to the principals. Some common sense restrictions need to be imposed. Time to put children first, not principals.

Time for the PGCPSS school board to make state certification as a mathematics teacher a requirement for the chairmanship of a middle school mathematics department, same for all subjects,—even though this would tread on the principal's prerogative to hire and assign teachers in any manner that suits him/her. Time to put children first, not principals.

To the PG state delegates, who are looking for things to legislate to save our schools: legislate a requirement that heads of middle school math departments and teachers of math classes be certified in mathematics; same for all subjects. This will be good for the entire state. Also provide funds to pay for teachers to take additional college classes which lead to state certification.

Is Our Practice of Humanistic Mathematics Actually Humane?

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DeLiberate Thinking
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It would seem reasonable to assume that Humanistic Mathematics is good for students. Yet, I have recently been forced to question that assumption and to face the possibility that there may be dangerous unintended consequences to our efforts to make mathematics more meaningful for kids. I am in full agreement with the ideal; it is our current implementation that may have serious flaws.

My concern stems from a visit to a school system that is doing most things right. The hallways of the elementary school were decorated with an inspiring array of charts, graphs and reports, all showing various applications of mathematics to topics of interest to children. The teachers were skilled practitioners of student-centered learning and clearly knew how to keep their students mentally active, challenged and enthusiastic. So, I wondered, why were minority student test scores among the lowest in the state?

Later that day I visited the middle school. The building was spectacular. It was a spacious, well lit, inviting environment that had the feel of a place in which serious work would be expected. The teachers were dedicated and competent, though their instruction was a bit too teacher-centered for my taste. Yet when I got to see samples of student work it was pathetic. Placed side by side with no indication of the students' grade level, I would have judged the third grade work as seventh grade and the 7th grade work as 3rd grade. Students seemed to be sliding backwards at an alarming rate.

My visit to the elementary school had been fascinating; I remained interested and enthusiastic throughout the entire time. But, in the middle school my enthusiasm quickly waned, and, until the shock of seeing the abominable student work, I had been close to nodding off. What if students had the same reaction? I am certain my middle school years were no better than those offered in this school. But, when I went to middle school I knew school was supposed to be boring and that it was something I just had to put up with. How would I have reacted if I had thought school could be intellectually challenging and fun? Would I have had the stamina to make even half an effort?

A week later the 12th grade TIMSS scores were released. But what shocked me was not the scores but rather the reaction of the mathematical community to them. United States students are among the best in the world at 4th grade and among the worst by 12th grade. Clearly something is happening between the numbers 4 and 12 but the education community remains fascinated with the numbers 1 to 3. A bold, expensive initiative has been proposed to shrink class sizes in the early elementary years. Does anyone realize that the numbers 1, 2 and 3 come before 4, not after?

SOME REFLECTION

Of course most of this information is not really new. We have all known for a very long time that the middle school math years are usually a vast wasteland. What was new to me was the recognition that the wasteland might do more than put kids to sleep. Contrasted to a superior elementary education, it might permanently kill off any interest students would ever have in school learning. A conclusion that drastic demands to be questioned, so I asked myself about Japan. It is my understanding that in their education system a wonderfully student centered elementary program is followed by a highly structured, intensely pressured and thoroughly teacher-centered high school experience. Yet, few students seem to be damaged by that contrast. The difference seems to be in the culture and in the family. In Japan all students know that they must conform to the system and that to fail to do so is simply not an option. Back in the school district I visited the affluent students apparently get the same message. It is the students from less affluent families that seem to be giving up on school. There is no one back home to tell them they can't do that.

From all of the above I have derived three lemmas.

Lemma One
When a highly student-centered, intellectually demanding elementary school experience is followed by a less demanding, teacher-centered middle or high school program, it is natural for students to give up entirely on education. This will happen unless family and cultural constraints prevent it.

Lemma Two
As we improve our elementary programs, making them more student-centered and more intellectually challenging for students, we should expect to see degraded performance in the middle and high school years wherever teacher-centered instruction remains the norm.

The Die Lemma
If we continue to favor improvements in elementary math education without insuring that equivalent improvements are instituted at the secondary level, we will kill off many students' interest in mathematics.

SOME ACTION
Since reaching the above conclusions I have begun discussing them with other math educators. So far no one has taken serious issue with my logic. Yet, I have also found very little interest in taking on the challenge of changing secondary mathematics. Primary education, like Humanistic Mathematics, is warm and fuzzy; high schools are neither warm nor fuzzy and probably should not be made to be so. Thus, it seems that caring, concerned math educators don't care about high school and remain unconcerned about the hundreds of thousands of lives that are being destroyed by our failure to bring primary and secondary education in line with each other. In this country young underclass urban youth are dying at a rate comparable to the worst mass genocides of the Twentieth Century. A major contributing factor is the hopelessness of their educational options. It is for this reason that I have to ask whether Humanistic Mathematics (as it is currently practiced) is not leading directly to genocide and therefore a completely inhumane practice.

Let me be very clear about one thing. I am not against the changes we have made to the elementary mathematics curriculum. What I am against is the lack of coordination between what happens in the elementary and in the secondary mathematics classrooms. Also, I am not so idealistic as to believe we can change everything at once. I believe, based on extensive experience and observation, that efforts directed at changing the way university faculty teach are mostly futile. But, just as we agree it is inhuman to use 12 to 14-year-olds as soldiers, but somehow find it acceptable to use 18-year-olds, I am willing to put our college students at risk. I believe that college students can handle a sudden and arbitrary shift in the way they are taught math. And, if they can't handle it, I believe they can handle the faculty. I am not willing to ask the same from a 12 to 14-year-old, and I think it is inhumane to do so.

So, I wonder, is there anyone out there who shares my concern? Who out there has solved the problem of making the secondary math classroom as student centered and as exciting as the best primary classrooms? What chance is there of spreading such innovations widely, and what will it take to do that?

Editor's Note: Reader responses are welcome. Responses will be printed in a future issue and may also be posted on our listserve (email). For information on how to join the HMNJ Listserve, see "From the Editor."

"I am enough of an artist to draw freely upon my imagination. Imagination is more important than knowledge. Knowledge is limited. Imagination encircles the world."

--Albert Einstein

The Saturday Evening Post

Is Mathematics Education Taking a Step Backward?

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ABSTRACT
This article addresses the current political, socioeconomic, and educational state of mathematics education in California. The "back-to-basics" movement in mathematics mirrors the "back-to-phonics" movement in language arts. At a time when ethnic minorities have become the majority, the dominant culture has chosen to revert back to practices that are inequitable and empower the elite. Critical educators must carry on the dialogue necessary to empower the disenfranchised mathematically and undermine the social injustice and economic inequality that will result if this movement is embraced.

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It is time to step back and reflect on the multitude of political legislation that has taken place in recent years, how the change in state leadership will affect those actions, and the educational and economic implications they will have on innocent, powerless children. For example, classroom size reduction, affirmative action, bilingual education, vouchers, charter schools, teacher/principal accountability, no social promotion, elimination of "remedial" classes in the Cal State University system, and the shift back to phonics in reading and basic skills in mathematics are just a few that must be brought to the forefront. All of the aforementioned interact with one another to further undermine the success of the most disadvantaged urban youth and create structural conditions of social injustice and economic inequality.

While the hegemonic leadership claims to make decisions in the best interest of our children, it is easy to recognize that those decisions often work in concert with the political economy to maintain existing relations of domination and exploitation. Take the class size reduction initiative as a case in point. Despite its altruistic intentions, the rationale behind its implementation is less than effective. The most lucrative districts attract and hire the most qualified in a decreasing pool of candidates, and the least qualified are left to be hired by less attractive highly populated districts, namely high poverty inner city schools. All who are hired without certification are required to enroll in an accredited credential program and complete a minimum number of units a year which then permits them to renew their emergency status and continue teaching in the classroom. Some who are hired do not even hold the minimum GPA required for acceptance into a public program. These teachers who have GPA's below 2.5 must seek a program in a private institution that will accept them or leave their assignment after a year. What are the consequences of such a practice on children in grades K-2 who are in the critical stage of building their educational foundations of language, reading, and mathematics? This is just one example of inequitable educational opportunities that have long-term effects for children who are already disenfranchised economically, linguistically and politically.

This paper focuses on the new political policy that proposes to drive mathematics education forward, but in fact will result in a giant step backward for disenfranchised groups. A brief look at the history of mathematics education and its apparent recursive nature is critical to understanding the current political debates on what mathematics should be taught, what knowing mathematics means, how it should be taught, and who is capable of achieving in mathematics.

The American educational system is historically grounded in a philosophical framework that allowed those in powerful positions to mold and define the mathematics knowledge they deemed important to know, what it means to know, who would be privileged to know it, and in what pedagogical form (Martin, 1997). Elite white males were the ones privileged

to learn and profit from an education. Gradually women and people of color were allowed to attend school, but the content and pedagogy was still Anglo male driven. The pendulum has swung back and forth from a classical curriculum taught in a traditional behaviorist pedagogy to a reform contextualized curriculum taught in a constructivist pedagogy throughout history. Even with the shifts in philosophy, the predominant practiced pedagogy has been the "traditional" (Stigler & Hiebert, 1998). The closest we have come to reversing that practice has been the movement of the past 15 years. It has probably gained the most momentum because mathematics educators have redefined what mathematics is important to know and what it means to know it so that it makes sense to a much broader audience. By doing so, all children will have the opportunity to succeed in mathematics, not just an elite few. And just as the most recent reform movement was about to gain momentum and support from all constituencies, a shift back to the traditional is again alive.

Drafts of the new *Mathematics Framework for California Public Schools* and *Standards* documents have been adopted and are ready for printing. Previously, California looked to the national *Standards* document for direction. Unfortunately, while the national document espoused a commendable position, it was difficult for teachers at each grade level to delineate the specific expectations they were accountable for. The National Standards Committee, realizing this weakness, began work to clarify expectations. The new revised *Principles and Standards for School Mathematics* document has been embraced by all the states in the nation, and their state documents have been designed to support it, with California being the only exception. Instead, the state of California began its own work to create a state *Standards* document that claims to espouse a balance of conceptual understanding and skills, but in fact is clearly more skill-based. The document specifies by grade level what mathematics children should know. Unfortunately, much of what is expected is not developmentally appropriate and reads like a check-off list of skills.

> "
> ***The document specifies by grade level what mathematics children should know...much of what is expected is not developmentally appropriate and reads like a check-off list of skills.***

The process in which California's new mathematics *Framework* was conceived was discernibly politically motivated. The appointed committee was reconfigured with members who held viewpoints that matched political agendas and certainly was not grounded in how children best learn mathematics. Instead, they made decisions based on what worked for them and what was considered important in years past, not taking into consideration the demographic and economic changes that have occurred in California. Consensus was never reached by the appointed committee, but the working document was sent forward without public review or notification to all group members of the process (Jacob, 1999).

Originally the mathematics *Framework* document was to be revised; instead, it has been rewritten. Many inconsistent messages seem to be indicated (e.g., a variety of approaches should be used, but the best one is the traditional teacher explain/student practice). These inconsistencies will most likely permit teachers to choose what is familiar to them-the meaningless "traditional" content and pedagogy because they have not personally experienced any other approach. Many truly believe this is the way mathematics should be taught because this is all they know. While the mathematics education literature (Prawat *et. al.*, 1992; Sowell, 1989; Ginsburg & Baron, 1993; Cobb *et. al.*, 1991; Hope & Owens, 1987) cites the importance of having children construct knowledge from the concrete through the representational and finally to the abstract stage of understanding, there is minimal mention, at most, of the benefits of using concrete models to help children build mathematical understanding. Instead, the flawed Dixon report which is the research base for the new *Framework* purports to be a review of mathematics education, but in fact is an example of research biased to support the back-to-basics agenda (Jacob, 1999).

Research (Kloosterman, 1991; Kamii & Dominick, 1998) clearly documents that reverting back to having children memorize facts and algorithms will not empower children in building a firm foundation of

mathematical understanding that is critical for those who remain in the mathematics pipeline and eventually are able to capitalize on the benefits of so doing in the marketplace. In addition, children who find no value or understanding in what they are doing are the ones who will drop out of the mathematics pipeline by choice or force and end up being the victims of such an unjust system.

Powerful committee members outside the realm of mathematics education, for the most part, were able to literally write new documents in which children will be judged as succeeding or not succeeding in mathematics based on historical Anglo-Saxon standards. Never mind that the demographics in California has changed so drastically in the past 15 years that Anglos make up a minority of the population in southern California. Never mind that children do not learn by memorizing, practicing, and regurgitating meaningless rules. Ask any student who has experienced a "traditional" educational experience what it means to divide a fraction by a fraction, when it is useful or why "inverting and multiplying" works and a majority will have no clue. This even applies to mathematics majors! Should it be surprising that most students cram and memorize for a test and have no idea in two weeks how to do those same problems? Should it be surprising that prospective elementary teachers have weak mathematical understandings? By allowing only those students who live and persist in a "traditional" environment to succeed, then those who live and learn outside of that norm will surely not succeed and those lucrative positions that reward success in mathematics will not be accessible to the majority, who just happen to be people of color.

Furthermore, textbook adoption panels are reviewing materials for adoption, but what is being evaluated is the accuracy of the mathematics content, whether specific skills listed in the *Standards* document are addressed, whether the organizational aspects of the presentation are easy for teachers to follow and understand, and whether equitable access is given to all students. While these criteria appear noble, the process will simply become a check off list, since pedagogy issues are noticeably minimalized. Instead, districts will be allowed to choose from texts that meet the above criteria. It is not surprising that accepted texts can look very different and still meet the criteria. Who will be making the decisions at the district level? Guess which texts are easiest for teachers to follow? Which students will be negatively affected by this traditional "back to basic skills"' movement?

Critical educators must produce compelling evidence that the implications of the direction that mathematics education is moving in California is far greater than simply succeeding or not succeeding in mathematics; it affects the debilitating economic cycle that perpetuates a classist society. The disenfranchised will continue to blame themselves for their failure and will have fewer career choices because of their limitations in mathematics. The "haves" will continue to "have" and the "have nots" will continue to struggle in an inequitable classist society. Perhaps the political decisions are being made consciously or subconsciously precisely to keep the large numbers of people of color in a non-threatening place. Certainly their voices were becoming heard a bit too loudly for the comfort level of the dominant group.

Those who truly believe that all students deserve an equitable opportunity to succeed in mathematics must not allow this movement to discourage or silence them. Passionate dialogue, networking, and critical mathematics education must continue so those teaching mathematics at all levels understand why so many students remain disenfranchised from a discipline that has the possibility of offering hope and opportunities for improving the quality of their lives.

REFERENCES

California Department of Education. (1999). *Mathematics Framework for California Public Schools: Kindergarten through Grade Twelve*. Sacramento, CA.

Cobb, P., Wood, T., Yeckel, E., Nicholls, J., Wheattey, G., Tigatti, B., & Perlwitz, M. (1991). "Assessment of a problem-centered second-grade mathematics project." *Journal for Research in Mathematics Education* 22 (1), 3-29.

Ginsburg, H., & Baron, J. (1993). "Cognition: Young children's construction of mathematics." In Robert J. Jensen (Ed.), *Research ideas for the classroom: Early childhood mathematics* (pp. 3-21). New York, NY: Macmillan Publishing Co.

Hope, J. & Owens, 0. (1987). "An analysis of the difficulty of learning fractions." *Focus on Learning Problems in Mathematics* 9 (4), 25-39.

Jacob, B. (1999). Teaching mathematics for understanding after 1999 [e-mail]. Unpublished manuscript. Professor of Mathematics, University of California, Santa Barbara, CA and member of the 1997 Mathematics Framework Committee.

Kamii, C., & Dominick, A. (1998). "The harmful effects of algorithms in grades 1-4." In Morrow and Kenny (Eds.). *The teaching and learning of algorithms in school mathematics.* Reston, VA: NCTM.

Kloosterman, P. (1991). "Beliefs and achievements in seventh grade mathematics." *Focus on Learning Problems in Mathematics* 13 (3), 3-14.

Martin, B. (1997). "Mathematics and social interests." In A.B. Powell & M. Frankenstein (Eds.), E*thnomathematics: Challenging Eurocentrism in mathematics education* .(pp. 155-171). Albany, NY: State University of New York Press.

National Council of Teachers of Mathematics. (1998 draft document). *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Prawat, R., Remillard, J., Putnam, R., & Heaton, R. (1992). "Teaching mathematics for understanding: Case studies of four fifth-grade teachers." *Elementary School Journal* 93, 145-152.

Sowell, E. (1989). "Effects of manipulative materials in mathematics instruction." *Journal for Research in Mathematics Education.* 20, 498-505.

Stigler, J. & Hiebert, J. (1998). "Understanding and improving classroom mathematics instruction: An overview of the TIMSS video study." *Phi Delta Kappan* 79 (1), 14-21.

Circle

A circle goes round and round,
An end it has never found.
It is not a sphere,
It does not have a rear.
It has a diameter, circumference and no sides,
The moon is a circle which brings in the tides.
Columbus searched like a hound,
The world is definitely round.
Not an oval or a square,
You can find a circle almost anywhere.
Now this is the end,
which a circle cannot lend.
Now I must go, for which you know,
So, I hope you enjoyed the show!

Anna Palco

Multiplication

Oh, how I love to multiply,
Without multiplication, I think I'd die.
All my friends think I'm obsessed,
But they're not the one getting A's on their tests.
Through every problem, my knowledge expands,
I study to keep up with all its demands.
I practice so much, there's no time to play.
But that's fine with me, yes, it's quite all right,
Multiplication's so fun, never wrong, always right.

Molly Hager

Equations

Equation
a number
with a letter in its place
it is the letter you must replace.
There are plenty of ways
to find your answer
but all of it just depends
on what the problem is,
whose question you want to end.
I like to solve equations
because they're really no problem at all.
They're quick
and easy
and really cool
and soon become lots of fun.
And even though
they're algebraic
and at introduction they sound hard,
there's really nothing to them,
nothing really at all.

Benjamin Davidson

Research Papers in a Math Methods Course: A Window into the Life of Ramanujan

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An elementary mathematics methods course offers an opportunity to broaden the lens through which prospective teachers may view mathematics. Many students perceive mathematics to be clear and devoid of controversy. The answers are clearly right or wrong. It follows that those who do not enjoy mathematics may think of mathematics as a cold subject. The discipline to which they have been exposed is not the mathematical world in which stories, ideas, and developments have been acknowledged in an engaging and passionate sense. Ask these students to talk about mathematics, and one finds that they have mathematical interests of many sorts—disguised perhaps beneath the umbrellas of music, art, language, literature, science, and history. The mathematical world has become separated from the domains which people commonly identify as academic areas of interest or hobbies. Of course, there are those who like mathematics and may not appreciate the mathematical connections to the various other areas.

How can we effectively address such perceptions in a way that will enhance the mathematical appreciation of such students? Research papers[1] have been employed in my own teaching to address this concern. The selection of a topic for such a paper is a significant part of the experience. Some students are encouraged to pursue their interests in art, for example, to enter the realm of tesselations. Perhaps a project on Escher or some aspect of architecture will emerge. The predominance of female students in the programme raises an obvious topic: women in mathematics. Stories of Hypatia, Sonya Kovalevskaya, or Sophie Germain may become focal points of discussion. Mathematical concepts such as the introduction of zero, the consideration of Mayan arithmetic, various number systems, or algorithms for multiplication may become topics. Ideas such as Pascal's Triangle or the Fibonacci sequence may captivate students' interests. Others prefer to examine issues in mathematics education ranging from the use of calculators to gender equity or assessment practices. The fact that so many different topics are selected may be an awakening experience in itself. Writing a paper in a mathematics course represents a significant enough deviation from experience to justify some reconsideration of mathematics on the part of many students.

The remainder of this article is the work of Marlene Neff, a student in the elementary mathematics methods course (ED 3940) at Memorial University of Newfoundland during the winter of 1998. Marlene's paper focuses on the life of Ramanujan. It appears here in its entirety.

[1] Poster presentations have been used as altematives/ supplements to research papers in several semesters. The poster presentations tend to be more interdisciplinary in nature. Many students see them as more applied in that they may develop curriculum-based concepts that potentially connect areas of mathematics and science, for example. The poster presentations may also delve into mathematical history or other topics mentioned in the discussion of research papers.

"Anyone who has never made a mistake has never tried anything new."

--Albert Einstein

A Window into the Life of Ramanujan

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Srinivasa Ramanujan (1887-1920), although self-taught in mathematics, is considered one of the greatest mathematicians of this century. Born on December 22, 1887, in Erode, Tamil Nadu State, India, Ramanujan was one of India's greatest mathematical geniuses. He made substantial contributions to the "analytical theory of numbers and worked on elliptic functions, continued fractions, and infinite series;" yet, he had only a vague idea of what constituted a "mathematical proof" [1]. Though he "failed Fine Arts at the Madras University," he was eventually recognized as "a natural genius" by G.H. Hardy of Trinity College, Cambridge [1]. He died in 1920, at the age of 32, after a long illness. Ramanujan's path was obstructed by many obstacles: "poverty, a lack of a university education, the absence of books and journals, working in isolation in his most creative years" [2, p. 1]. Few mathematicians had to experience these kinds of obstacles. His theorems still confound mathematicians today, almost eight decades after his death.

Srinivasa (pronounced shri) Ramanujan Iyengar was born December 22, 1887, in his mother's parental home, 150 miles upriver from his parents' home. Srinivasa was his father's name, automatically given to him, but rarely used. Ramanujan means the "younger brother of Rama, the model of Indian manhood" whose story was handed down from generation to generation through "Ramayana, India's national epic" [3, p. 11]. This name was also picked as he shared other astrological similarities with the Saint Ramanuja, who lived in the 1100's. Iyengar was the "caste name, the branch of South Indian Brahmins" [3, p. 11], to which he and his family belonged. He became known as Ramanujan.

One of the challenges that Ramanujan faced occurred when he was two years old. A smallpox outbreak, which was responsible for "4,000 deaths in the Tanjore District," infected Ramanujan [3, p. 12]. Ramanujan managed to survive but carried the scars all of his life. His family met with other tragedies. His mother would lose three children before the birth of another son, when Ramanujan was 10 years old. His youngest brother was born the year he turned 17. Ramanujan was brought up as the centre of attention, similar to an only child.

There are stories of Ramanujan being a strong-willed child who acted out by rolling in the mud if he did not get his way. In the first three years of his life, he hardly spoke, although he did learn the "12 vowels, 18 consonants, 216 combined consonant-vowel forms of the Tamil alphabet" soon afterwards [3, p. 13]. His temperament caused him to have unexpected reactions to stress, as in breaking out with hives.

In October 1892, he began school. He was the child who was fond of asking questions and who challenged the teachers with questions such as, "Who was the first man in the world? How far is it between the clouds?" [3, p. 13]. There is not too much known about his childhood except that he liked to be by himself and that he was back and forth between schools because he disliked attending school. There is also mention that Ramanujan was obese, in a culture where "obesity was virtually unknown" [3, p. 14]. This obesity stayed with him into adulthood, until his illness.

Ramanujan's father was largely absent from his life, and rarely mentioned in the documents. He was an accountant for a cloth merchant in Kumbakonam, and in Indian society was left with "little role to play at home" [3, p. 18]. Though all of Ramanujan's relatives were "of high caste," they were very poor [4, p. 1]. His mother was described as a "shrewd and cultured lady" [3, p. 18]. She monitored his friends, his time and his decisions. When the time came, she found him a wife.

At the age of 7, Ramanujan was sent to the High School of Kumbakonam, and remained there for 9 years. His "exceptional abilities" began to appear before he was ten [4, p. 2]. He passed the primary examinations in English, Tamil, Arithmetic, and Geography, and came first in the district. Although his native language was Tamil, with the ascent of Britain in India, English was the language of the country's rulers and the "ticket of

admission to the professions" [3, p. 25].

Because Ramanujan's family was poor, they took in boarders who were studying at the Government College. These boarders acquired mathematics texts for Ramanujan, as well as taught him how to solve cubic equations. By the age of 12, Ramanujan had mastered Loney's *Trigonometry*, which included topics such as "exponential function, logarithm of a complex number, hyperbolic functions, infinite products, and infinite series" [2, p. 1]. At 15, Ramanujan was captivated by Carr's *A Synopsis of Elementary Results in Pure Mathematics*, with its compilation of 6000 theorems, which he had borrowed from the college's library. While *Synopsis* had given him direction, it had "nothing to do with his methods, the most important of which were completely original" [3, p. 45]. Carr's book was to have a great influence on his life.

Ramanujan was a minor celebrity during his school years. He received merit certificates and scholastic prizes for his mathematical marks. In December 1903 he took the matriculation examination of the University of Madras and obtained "a 1st class place" [2, p. 2]. In 1904, when the headmaster presented him the K. Ranganatha Rao prize for mathematics, he introduced him to the audience by stating that "if possible, he deserved higher than the maximum possible marks" because he was off the scale [3, p. 27]. He graduated from high school and entered Government College with the scholarship awarded to him.

> ***"Everyone was in awe of Ramanujan's mathematical gifts, but nothing came of them.***

By this time Ramanujan was completely engrossed in mathematics. He was not interested in studying any other subject. This caused him to "fail his English and Physiology examinations" at the end of his first year at the Government College at Kumbakonam and to lose his scholarship [2, p. 2]. He ran away from home, the first of numerous disappearances.

Ramanujan was very sensitive to any failure. Through the years there were incidents where he would become hurt or angry over relatively insignificant occurrences. Once when his friend scored one point higher than he on a math quiz, he became angry and would not speak to the boy for a long time, Later, in High School, Ramanujan saw how "trigonometric functions could be expressed in a form unrelated to the right triangles in which they were rooted," [3, p. 50]. When he found out that Leonhard Euler had predicted this 150 years before, he became so ashamed, he hid his papers in the roof of his house. When a close friend stopped writing, Ramanujan wrote the friend's brother to say "he is too sorry for his failure in the Exam to write to me" [3, p. 50]. Many other events through the years would cause him to flee or react in humiliation.

In 1906, Ramanujan decided to give college another try. He entered Pachaiyappa's College in Madras, and upon seeing his Mathematical Notebooks, which Ramanujan had begun to keep in 1904, the principal awarded him a partial scholarship. Unfortunately, Ramanujan again failed the examinations and lost his scholarship. Everyone was in awe of Ramanujan's mathematical gifts, but nothing came of them.

Ramanujan struggled out of school, without a degree, without a job, and without contact with other mathematicians. He immersed himself in his Mathematics, and devoted himself entirely into recording his results in his notebooks. Without any distractions, in some ways, these may have been the "most productive days of his life" [3, p. 65].

Ramanujan's mother must have decided that it was time for him to bring some focus to his life. In 1908, while visiting friends, she noticed a visiting young relative. She asked for the horoscope of this girl, and compared it to her son's. She decided that they were a good match and began the negotiations for the eventual marriage of Ramanujan and nine-year-old Janaki. Janaki did not see her intended husband until the wedding in July 1909. She was ten years old. Ramanujan's father, who was opposed to the wedding, did not attend. Janaki did not join Ramanujan for three years. During this time, while awaiting puberty, she prepared for her role as a wife.

Unfortunately, another health problem arose. Ramanujan developed "hydrocele, an abnormal swelling of the scrotal sac" [3, p. 72]. The solution was an

incision in the scrotal sac. The operation was simple, but because his family did not have the means for a doctor, the operation was postponed. Eventually, in January 1910, a doctor did the procedure for free.

This was a new stage in Ramanujan's life. Now that he was married, Ramanujan found it necessary to obtain employment. He travelled the rails and found jobs as a tutor. He began calling on influential friends. He was eventually sent to R. Ramachandra Rao, a wealthy mathematician, who was so impressed with the contents of Ramanujan's notebooks that he "offered Ramanujan a monthly stipend so that he could continue his mathematical research without worrying about food for tomorrow" [2, p. 2]. In 1911 Ramanujan's "first paper appeared in the *Journal of the Indian Mathematical Society*" [3, p. 82]. The paper posed the question, "What happens if n = infinity?" [3, p. 87]. No one explored this area more than Ramanujan.

In late 1912 he and Janaki began their married life, living with his mother and grandmother. Ramanujan then accepted a clerical position in the Madras Port Trust Office in 1912. Fortunately, the manager was a mathematician and encouraged Ramanujan to send his discoveries to English mathematicians. Ramanujan wrote to H.F. Baker and E.W. Hobson at Cambridge University asking for help. Both replied, no. On January 13, 1913, Ramanujan wrote to G.H. Hardy, who at thirty-five was "already setting the mathematical world of England on its ear" [3, p. 107]. Hardy replied, yes!

G.H. Hardy, a famous mathematician, was a fellow of Trinity College. He had been known to be sympathetic to the underdog. He had also been willing to "stray from safe, familiar paths" [3, p. 171]. When Hardy received Ramanujan's letter and theorems, he initially dismissed the letter. On a second look with his friend and colleague, J.E. Littlewood, the two men began to "appreciate the papers of a math genius" [3, p. 169]. Hardy would later say that "it was the strangeness of Ramanujan's theorems that struck him at first, not their brilliance" [3, p. 159].

Hardy urged Ramanujan to come to Cambridge in order that his "superb mathematical talents might come to their fullest fruition" [2, p. 3]. Because of his strong Brahmin convictions, which included not crossing the seas, Ramanujan declined. Gilbert Walker, a former mathematical lecturer at Trinity College, was sent to Madras to ask for support for Ramanujan as a research student. Ramanujan was paid "75 rupees per month for two years" beginning in May 1913 [3, p. 174]. He began sending a collection of theorems to Hardy. Ramanujan worked all hours, barely in contact with anyone, including Janaki. His sole mission was to pursue mathematics and to submit progress reports every three months. He continued his long distance correspondence with Hardy.

In 1914 E.H. Neville, who was sailing to India to lecture at the University of Madras, was asked by Hardy to convince Ramanujan to come to Cambridge. There are conflicting stories that Ramanujan had a vision in which the goddess Namagiri told him to travel to a foreign land. Whether this was "divine inspiration" or Ramanujan's method of obtaining permission to travel abroad, no one knows for sure [3, p. 189]. Some stories state that his mother had the vision.

While growing up, Ramanujan had lived the life of a traditional Hindu Brahmin. He wore the "kudimi, the topknot," his "forehead was shaved," he was "rigidly vegetarian" and followed the "rituals and ceremonies" of his religion [3, p. 130]. By sailing the seas, he would now be excluded from his caste: "friends, family would not have you in their homes," there would be "no bride or groom for the child," a "married child could not visit you," nor could you go into "temples, funerals" [3, p. 185]. Gandhi had suffered the same fate when he went to England for his education. On March 17, 1914, Ramanujan, wearing Western clothes and short hair, sailed for England.

The next three years were productive for Ramanujan. He and Hardy worked closely together and "profited immensely from each other's ideas" [2, p. 4]. All of Ramanujan's manuscripts would pass through Hardy's hands, who edited them for publication. The year of his arrival, 1914, Ramanujan had published one paper. By 1915, he had "a flood of papers published" [3, p. 208]. If Ramanujan had wanted recognition, he got it in 1915 when "nine papers appeared in the *Journal of the Indian Mathematical Society*" [3, p. 230].

The war came in 1914. In 1915-16, Madras debated whether or not to extend his scholarship and decided to extend it for one more year. In 1916 Ramanujan re-

ceived the degree that had escaped him so many times before. He received a B.A. by research. But, Ramanujan found it difficult adjusting to the English climate, and found it increasingly difficult to obtain the food he required to maintain his vegetarian diet. He became ill, and was soon spending most of his time in hospitals and sanitoria. He began to lose weight and energy and was eventually diagnosed with tuberculosis. Although very ill, he continued his mathematical work, and even apologized to Hardy for failing to do more.

Hardy recalled visiting Ramanujan in Putney while he was very ill. He remarked to Ramanujan that the cab he had arrived in had a number which was a "rather dull one" and hoped that it was not an "unfavourable omen" [4, p. 12]. The number was 1729. Ramanujan immediately answered, "No, it is the smallest number expressible as a sum of two cubes in two different ways. $1729 = 12^3 + 1^3 = 10^3 + 9^3$" [4, p. 12]. Ramanujan would also discuss the "ties between God, zero, and infinity" [3, p. 31]. When asked where he got some of his answers, he would often say, "The answer came to my mind" [3, p. 215].

Although Ramanujan's mind was still sharp, his body was now failing. He became picky about his food, complained constantly, and would not do what the doctors recommended. He continuously changed hospitals and doctors. During this time his letters from home ceased, and his isolation increased. During a short release from the hospital, Ramanujan had invited a friend and his fiancée, and her chaperone, over for dinner. He served soup, offered a second helping which was accepted by his friend, but when the fiancée and the chaperone refused, he left his home by taxi and did not return for four days. When asked by his friend why he had behaved in such a way, he replied that he was "hurt and insulted that the ladies did not want more" [3, p. 237]. For someone so self-confident about mathematics, Ramanujan still had an almost "pathological sensitivity to the slightest breath of public humiliation" [3, p. 50].

In the spring of 1917, Hardy wrote the University of Madras to tell them that Ramanujan had an incurable disease. The University made him a university professor at Madras, paying him "400 rupees per month, and 250 pounds per year fellowship" [3, p. 312]. He wrote back to tell them that it was too much and to donate some for educational purposes. Hardy worked successfully in having Ramanujan elected to the "Fellow of Royal Society" and the "Fellow of Trinity College, Cambridge" in 1918 [4, p. 6]. He could finally sign F.R.S. after his name.

The war prevented him from going home to India until March 1919. In 1918 Ramanujan had tried to commit suicide by throwing himself in front of a train, but was rescued when the conductor applied the brakes. Ramanujan was very ill when he arrived in India. A feud had developed between his mother and Janaki; therefore, Janaki had moved back with her brother and was not there to greet Ramanujan. He demanded that his mother send for her, which she did.

Ramanujan's health deteriorated quickly, but his relationship with Janaki improved. He was often depressed and sullen, and his mood became volatile. Janaki cared for him, and the two became closer than they had ever had a chance to be. Ramanujan's approaching death "inspired a final flurry of creativity impossible during normal times;" he sent a letter to Hardy, whom he had not written in more than a year [3, p. 323]. On April 26, 1920, Ramanujan lapsed into unconsciousness and died that morning.

The short life of Ramanujan cannot be understood without some appreciation for the mathematics that he loved and lived for. After his death many mathematicians, including G.H. Hardy, strongly suggested that his notebooks be edited and published. The first notebook was left with Hardy, and the second and third notebooks were donated to the University of Madras upon his death. The notebooks contained "new, interesting, and profound theorems that deserve the attention of the mathematical public" [2, Preface]. In 1929 G.N. Watson and B.M. Wilson began editing the notebooks, but the task was never completed until 1957, when an "unedited photostat edition of Ramanujan's notebooks was published" [2, Preface]. In 1986 Professor George Andrews of the Pennsylvania State University discovered "600 theorems on loose sheets of paper" which he termed "the Lost Notebook of Ramanujan" [1].

Ramanujan's notebooks served as a compilation of his results. He worked out most of his mathematics on a slate beforehand. In the notebooks it is clear that "infinite series abound throughout" [2, p. 6]. His love for

Bernoulli's numbers are also apparent, as they "appear in several of Ramanujan's formulas" [2, p. 7]. He developed "Magic Squares, an array of (usually distinct) natural numbers so that the sum of the numbers in each row, column, or diagonal is the same" [2, p. 16]. During most of the last year, before he went to England, he worked with the "general formulae in the theory of definite integrals" [4, p. 186]. Despite many ingenious results, "some of his theorems on prime numbers were completely wrong" [5].

In 1987, the centenary of Ramanujan's birth was celebrated all over the world. At the University of Illinois the celebrations included "a series of 28 expository lectures and several contributed papers that traced Ramanujan's influence to many areas of current research" [6, p. 1]. At Anna University, Madras, the University organized a number of "academic programmes throughout the centenary year and concluded the celebrations with an International Conference" [7, Preface]. Janaki, Ramanujan's widow, inaugurated the conference.

Ramanujan's notebooks would intrigue and frustrate whole generations of mathematicians. His life and works would captivate many. Kanigel wrote, " [the] more I learned, the more I, too, came under Ramanujan's spell" [3, p. 4]. Ramanujan's life leaves the reader captivated by an inexplicable force, Ramanujan's spell.

Ramanujan and Hardy's names would be "linked forever in the history of mathematics" [3, p. 253]. Hardy's remarks in *Ramanujan* give teachers food for thought:

> There was no gain at all when the College at Kumbakonam rejected the one great man they had ever possessed, and the loss was irreparable; it is the worst instance that I know of the damage that can be done by an inefficient and inelastic educational system [4, p. 7].

REFERENCES

1. Http://www.luc.ac.be/Research/NeuralNet/Activities/ramanujan.html. (1998).
2. Berndt, Bruce C. (1985). *Ramanujan's Notebooks - Part I*. New York: Springer-Verlag.
3. Kanigel, Robert. (1991). *The Man Who Knew Infinity: A Life of the Genius Ramanujan*. New York: Maxwell MacMillan International.
4. Hardy, G.H. (1959). *Ramanujan*. New York: Chelsea Publishing Co.
5. Http://www.groups.dcs.stand.ac.uk/~history/Mathematicians/Ramanujan html. (1996).
6. Berndt, Bruce C. (1989). *Ramanujan's Notebooks -Part II*. New York: Springer-Verlag.
7. Alladi, K. (1989). *Lecture Notes in Mathematics*. Berlin: Springer-Verlag.

Numbers

The animals were loaded 2 by 2
Some were yellow, some were blue.
The planets are in a row of nine
They change positions according to time.
365 days in a year
All in my calendar sitting right here.
4 score and 7 years ago
Lincoln made the South his foe.
Everything involves numbers like 1 and 2,
Including the earth and you.

Shea Ybarra

Circles
and circles are every-
where, but boys and girls do not
seem to care. They help us and help us
everyday, they are even the coins we have to
pay. they're on cars, buses and even a jet, there's
even one in the alphabet. You could roll 'em and
pull 'em and give 'em a tug, they are sometimes
the shape of a shell on a bug. A circle is
something extraordinary. If you asked
me if I liked them I'd have to say
"very."

Valentino Loiacono

Using Writing To Connect With Our Mathematics Students

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For the last two semesters, I have been experimenting with short writing assignments that are intended to keep a line of communication open between me and my students. I call these short papers "minute papers." Minute papers are written, short responses to one or more questions given to the students at the end of class. These papers are not a new idea, but I have found them very effective in developmental math classes.

EXAMPLE OF QUESTIONS AND TWO STUDENT RESPONSES

In a short paper, (1) explain what the main point of today's class was, (2) describe at least one thing you found interesting and (3) ask at least one question about today's lecture or activities.

Example 1

1. Conversions
2. I guess it's not really interesting, but when you gave us the answer to #3 on the quiz, I should have known that the vertical intercept was 4 and the x was 5. I didn't even come close to that answer...oh well, maybe next time.

(The student did know the procedure; she made only a small computational error on the quiz.)

3. No question for now except one I don't know if you can answer. Why do I have the feeling that I don't know what I am doing, but I really do?

(I responded to this question. I think it is a common feeling among hard-working beginning algebra students. I wrote, "I often ask my students thought-provoking questions about the material we are covering. I believe that if we have a deep understanding of the concepts, the mechanics are easier--and make sense! The 'whole picture' takes longer to grasp, but even before you have the whole picture, the understanding that you are developing is helping.")

Example 2

1. The main point of today's class was about radicals and squares.
2. I found some factoring ideas semi-interesting.
3. How do we know that 2^2 really equals 4?

(How could I not respond to this? I wrote, "There are certain mathematical statements that cannot be proven; they are just accepted as true. If you tweak the assumptions, you can get a mathematics that looks completely different but is in itself logically consistent. It's questions like this (what's true?) that makes mathematics interesting!")

WHY MINUTE PAPERS?

I started using minute papers in my classes after a discussion with other math instructors about how to make our mathematics classes more "humane." I was about to embark on a six-week, four day per week Intermediate Algebra class. My previous experience teaching such short courses was that within two weeks the students usually had a look of horror on their faces whenever I introduced a new topic, since they had not yet assimilated the previous material. I always felt a bit sadistic trying to teach so many topics in so little time. I wanted to do something different. The original intention of the minute papers was to create a forum for discussion. I believed that if the students took a few minutes to reflect on the material we had covered, they would be able to see the material as a whole and to see how it connected to material that we had studied previously. I didn't want the students to feel as if the material in the course was a torrent of abstract, useless and disjointed ideas passing them by. If a student felt completely overwhelmed he could say so. I also used their responses to assess when I needed to spend more time on a topic and when I could go on. This was particularly effective in a course that met daily. The students' questions also provided material for the next class.

HOW ARE THEY DONE?

I usually asked the same three questions: 1. What was the main point of today's class? 2. Describe at least one thing you found interesting, and 3. Ask at least one question about today's lecture or activities.

During a 16-week semester I also included periodic self-assessments. Again, I asked three questions: 1. Discuss your strengths as an algebra student, 2. Describe one topic that gave you difficulty at first but you have now mastered, and 3. Describe how you mastered this topic. I intentionally phrased these questions in the positive, although some students would write, "I have no strengths as an algebra student." Before the final exam I would add a fourth question to the self-assessment that gave the students an opportunity to discuss a topic that was still giving them difficulty.

I read and commented on these papers every night and returned them to the students at the next class meeting. I think the exercise would be pointless if I didn't return these papers promptly. Many students had a hard time asking questions. If a student asked no question, I would suggest something outlandish until they felt comfortable asking questions. Often I was asked questions about topics I knew nothing about such as baseball or which pickup truck is superior, but eventually all students asked questions and the vast majority of the questions had to do with mathematics.

OUTCOMES

Students claimed they benefited from reviewing the material at the end of class. (Examples to follow.) Students who were timid about asking questions during class time used the minute papers to ask questions that might otherwise not be asked. The minute papers were an excellent assessment tool for me because I had daily feedback from my students on what was taking place in the classroom. This alone was enough to make reading these papers worthwhile.

There were several unexpected and wonderful outcomes as well. I usually have a difficult time sorting out the students in my classes, especially learning names. But, after several minute papers, I felt I knew my students better than I had in previous terms; I knew their fears, their interests and their sense of humor, which is far better than eventually learning their names and getting to know only the students who approached me outside class. My students and I were in dialog with each other throughout the term.

Non-native speakers took longer than other students did to complete their papers, and one student remarked at the end of the term that she dreaded having to write in English. But this same student sought me out the following semester and told me how much the minute papers had helped her writing. She was taking a history course and was required to write essays on exams. The minute papers had helped her get over her fear of expressing her ideas on demand and in writing.

The best part of reading these papers was being able to respond to the students' questions. The brighter students who rarely had questions about mechanics often asked questions about how the topics we were studying related to other disciplines. I had one student who often discussed how strange it was that mathematics described the natural world, and he asked in many ways whether or not the universe was indeed mathematical. (How often can we discuss these questions in an intermediate algebra class?) I often was asked why I taught mathematics and why I found it interesting. (My responses to these questions were significantly longer than the students' papers!) Not only did I get to know my students, but my students got to know me as well.

STUDENT FEEDBACK AND SUGGESTIONS

I have used minute papers for the last two terms. For the last minute paper of the term, I ask the students to tell me what they thought of the minute papers. By this time, my students feel comfortable with me, and I believe the responses to be honest and genuine. Here are a few examples:

- I think the minute papers were a good idea, because it allowed us to reflect on what we did in class, and I was able to ask questions about something that had happened earlier in class. Instead of making the whole class go back to it, it allowed me to get my answer on the following day.
- I thought the minute papers were cool. They were a chance to ask a question about anything and to keep us thinking until the end of class.
- I thought the minute papers were great. I have to say that I thought it was a pain, but it made me

consider all the points of the class for the day. Plus, it made me pay attention that much more because I knew you were going to ask what we talked about.
- It was a good way to gather my thoughts and review at the end of class what we had learned.
- The papers made me realize my strengths and weaknesses.
- I wish I highlighted in my notes what was confusing about the lecture. It would have been more useful, and I would have been able to refer to specific problems.
- It also gave people a chance to ask obscure questions that were inappropriate at that time in class.
- I've taken math classes before where the instructor and I exchanged ideas for approximately two minutes. I definitely liked them.
- I didn't always have particular things to ask or say, but it was good to know there was a line of communication there.
- I think it encourages students to communicate with their teachers and forces them to think of information they may have been pondering...
- I was surprised by the time you took out of your own life to answer our problems.
- I think the minute papers are great! It's great to see you take the time to read about how we feel or any questions we may have. I must say I don't like to take the time to write them, but I am glad that I did.
- In my opinion, I didn't like minute papers because of the English, but on the other hand, I liked it because I could communicate with you easily. Especially when I had a lot of frustration, the paper helped a lot to express my feelings and idea.
- I especially appreciated the feedback from you.
- They made me ask questions I probably wouldn't have thought of.
- I titled things better in my notes because of it, which helped when studying for the final.
- Now I can understand how algebra can be applied to real world situations. Throughout the years of learning math, the teachers were asked the question "When are we going to use this?" They never directly answered the question, which I think held back the students on how math works. Either the teacher didn't know or didn't want to go into it.
- There were a few other instructors I wanted to hand one in to.

On the other hand....

- I didn't really enjoy doing minute papers. I actually thought they were a waste of time. I thought that if I had any questions or concerns, I could have asked you rather then write them down. Maybe others liked them, but I can't say that I did.

- The minute papers were painful, like getting a root canal and then chewing on tinfoil kind of pain.

IDEAS FOR USING THESE PAPERS IN THE FUTURE

Students have made several good suggestions they believe will help improve minute papers.

Of course, students all want credit for these papers. I have intentionally not "graded" these papers so that the papers would not be a source of further stress. But, I have decided to include the students' work on the minute papers as part of a participation grade next semester.

Some students feel that doing minute papers every class period is too much. Yet, I found that the papers were most successful in the courses that met four days per week. I don't think the papers are as effective when a class meets only two days per week. I still intend to have students write every class period.

Students tire of the same questions. Many students have suggested varying the questions. Including the self-assessment and asking for feedback on a new text were my only attempts at varying the questions last semester. One student made a good suggestion this semester. She suggested that I ask students to discuss their understanding of a particular type of problem or one concept in a minute paper. In fact, I used to ask questions like these on take-home writing assignments all the time, but I have gotten away from it. I plan to experiment with questions like these in this new format next semester.

Between Childhood and Mathematics: Word Problems in Mathematical Education

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How should we educate our children to help them develop into competent, productive members of the modern society? The purpose of these notes is to address this question. We shall concentrate on that period of life, which comes roughly to middle and high school, when childhood is over, but professional use of mathematics is not yet possible. This age seems to be critical for success vs. failure in rigorous abstract thinking: some students get prizes at olympiads, some are confused and frustrated. Our main point is that good teaching of word problems is essential at this period.

Since this article is about word problems, we need first of all to define the subject. To keep as close as possible to the exact meaning of the words, I suggest that a non-word problem is a problem, which is formulated using only mathematical symbols and technical words like "Solve the equation..." Correspondingly, a word problem is a problem which uses non-mathematical words to convey mathematical meaning. At the K-12 level there is not much room for sophisticated formalisms of professional mathematics, so non-word problems, which deal with formalisms, are necessary, but not exciting, exercises. No wonder that most interesting problems available at this level are word problems.

There is an important similarity between children's play and all aspects of modern culture: in both cases creative imagination is essential. On one hand, all life of children is a continuous play of imagination. On the other hand, all phenomena of modern civilization involve imagination. When we go to a theater or cinema or art gallery or read a book, we imagine certain events, but at the same time we know that they are not real. Modern mathematics is not an exception: imagination is essential not only to work in it, but even to understand it. It is only natural that school should not interrupt what is common for childhood and culture, namely creative play of imagination. When a teacher of geography tells her students "Today we shall travel across Africa," all normal children understand that this should not be taken literally: this will be an *imaginary* travel. Similar understanding takes place when a teacher of literature says "Today we shall spend in the company of Hamlet" or a teacher of biology says "Let us look inside a living cell". The function of school is to enlarge children's outlook, to teach them facts, images, ideas, laws, phenomena that go beyond their personal experience and everyday life. At school, as much as elsewhere. students are expected to have imagination and use it. Mathematics is not an exception from this rule. When a teacher says "Peter had ten apples and gave Mary three of them," all children understand that these are abstract Peter and Mary and abstract apples. This understanding is essential for children to study mathematics, which is a science about abstractions. Now look at the following problem:

A plane takes off and goes east at the rate of 350 mph. At the same time, a second plane takes off and goes west at the rate of 400 mph. When will they be 2000 miles apart?

I see nothing wrong with this easy problem. To my mind it is usable and even has some merits. For example, it may be used to demonstrate the idea of relative movement, which helps to solve it without algebra: in the coordinate system associated with one plane, the other moves at a speed $350 + 400 = 750$ mph, so the time needed to increase the distance by 2000 miles is $2000/750$ hour = 2 hours 40 min. However, a few years ago it was mentioned in *Mathematics Teacher* with the following pejorative comment: "Any normal student ought to ask, 'Who cares?' No one cares except the algebra teacher who assigns these problems and the student who wants a grade. Our curriculum is too crowded to allow us the luxury of such frills" [11, p. 159]. I am very worried by this comment, and it is a matter of principle. We can manage without this

or any other particular problem, sometimes we need to exclude something from curricula, but we should not *approve* asking "who cares?" instead of applying an intellectual effort, especially on the pages of an educational journal.

According to my experience, only a few students ask "Who cares?" instead of solving a simple problem, and these few students already are in trouble: mentally deficient or delinquent. Clearly, that "normal" student who asks back "Who cares?" does it because he *can not* solve it. This is really scary, especially if we remember that this is a college-bound student. I certainly don't want my children, indeed any children to be educated under such guidance. But perhaps instead of this problem we proposed some new, better one. Look at the following problem, which was presented as an example telling us why algebra is important to learn [12, p. 34]:

A batter hit a baseball when it was 3 feet off the ground. It passed 4 feet above the 6-foot-tall pitcher 60 feet away. It was caught by an outfielder 300 feet away, 5 feet off the ground. How far from the batter did the baseball reach its maximum height, and what was that height?

To solve this problem, we have to assume that trajectory of the ball is a parabola (that is, ignore air resistance), introduce some coordinate system and describe the trajectory by an equation, say

$$y(x) = k(x - b)^2 + m,$$

where y is the height and x is the distance from the batter along the earth (which is supposed to be flat). Then $y(0) = 3$, $y(60) = 10$ and $y(300) = 5$, whence we can find k, b and m. This problem is more difficult than the previous one, but I do not deem that it is better. In any case it is not more "real-world." As any school problem, it creates an imaginary situation, provides certain data about it and requires one to deduce the answer from these data. As usual, this imaginary situation is not real in the literal sense. How were all the heights and distances measured in the heat of the game? Why do we need to know the maximal height and how far from the batter it was reached? No answer is provided to these questions. This is normal and usual for traditional word problems, but in another article Zalman Usiskin called the traditional word problems *phony* and said that they are not needed because there are many "real applications" [11, pp. 158,159]. He should not use such a pejorative term even if he were right, but in fact he is not. Idealization of reality, reducing it to a definite formal system with a finite, strictly defined set of parameters and relations between them and asking all kinds of questions about this system, is the essence of scientific modeling, and there is nothing phony about it. What about "real applications?" We have just seen an example.

Why was such an ordinary, even somewhat cumbersome problem chosen for such an imposing purpose? Wait a little...notice that this problem involves *baseball*...many children love to play baseball...this suggests a guess: probably, the author hopes to convince them that algebra is important to learn because they will use it when they play baseball! The surrounding problems confirm my hypothesis: they are about such attractive topics as an around-the-world trip, a marching band and rock music. Clearly, they are intended to be interesting because of this. At this point I sharply disagree. To me a mathematical problem is interesting and educationally useful because of its intrinsic mathematical structure. I strongly disagree with the idea to attract students to mathematics pretending that it helps to play baseball, organize marching bands or enjoy rock music. This is a false promise.

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> ***...a mathematical problem is interesting and educationally useful because of its intrinsic mathematical structure.***

I have invented many problems and always made no secret that they were *mathematical* problems. First of all I cared about their mathematical meaning. To this I might add some fun. For example, I invented the following problem for the Russian School by Correspondence:

A mathematician who had a hat and a stick in his hands was walking home upstream a river with a speed which was one and a half times greater than the speed of the current. While walking he threw his hat into the river because he mixed it up with the stick. Soon he noticed his mistake, threw the

stick into the river and ran back with a speed twice that with which he had walked ahead. As soon as he caught up with the hat, he immediately got it out of the water and went home with the former speed. 40 seconds after he got his hat, he met his stick carried by the stream against him. How much earlier would he come home if he did not mix up his hat with his stick? [4, p. 8]

This problem was liked by some students and their teachers, although it is clearly not "real-world." One way to solve it is to denote v the speed of the current and t the time the mathematician spent running back, where time is measured in minutes. Then the distance he run back is $3vt$, the distance he went forward from getting the hat to meeting the stick is $3/2v \cdot 2/3 = v$, and the distance the stick moved back till he met it is $v(t + 2/3)$. So we can write the equation

$$3vt + v = v(t + 2/3),$$

where v cancels and we get $t = 1/4$ min. The time the guy lost consists of two parts: the time he ran back, i.e. 1/4 min, and the time he went forward the same distance, which is two times greater, that is 1/2 min. So the total time he lost is $1/2 + 1/4 = 3/4$ min. This problem has one interesting feature which the previous one does not: in solving it, we had to introduce an extra variable, in this case v, which cannot be found, but cancels. Alternatively, we could introduce a special unit of distance to make the speed of the current equal 1. Another class of problems that have this useful feature are often called work problems. This is an example:

Three workmen can do a piece of work in certain times, viz. *A* once in 3 weeks, *B* thrice in 8 weeks, and *C* five times in 12 weeks. It is desired to know in what time they can finish it jointly.

This problem was included by Newton into his textbook and cited by Polya [6, p. 47]. The solution is based on the well-known (unrealistic) assumption that each workman has a constant rate. We can take the "piece of work" mentioned in the problem as a unit of work and call it "job." Then *A*'s rate is 1/3 job/week, *B*'s rate is 3/8 job/week, and *C*'s rate is 5/12 job/week. When they work together, their rates add, and the total rate is

$$\frac{1}{3} + \frac{3}{8} + \frac{5}{12} = \frac{9}{8}$$

Then, the time they need is 1 job divided by 9/8 job/week, that is, 8/9 of a week. Why did Newton and Polya consider such problems valuable? This is an answer [6, p. 59]:

> *Why word problems?* I hope that I shall shock a few people in asserting that the most important single task of mathematical instruction in the secondary schools is to teach the setting up of equations to solve word problems. Yet there is a strong argument in favor of this opinion. In solving a word problem by setting up equations, the student translates a real situation into mathematical terms; he has an opportunity to experience that mathematical concepts may be related to realities, but such relations must be carefully worked out.

Pay attention that what Polya calls "real situation" is not real in the literal sense. Clearly, Polya took for granted that everybody has imagination and highly valued traditional word problems. He would be very astonished if somebody called them phony in his presence, and I completely agree with him: I believe that traditional word problems are very useful.

Another strange, but widespread, idea is that word problems are more uniform than non-word ones. For example, the influential "standards" [8, Summary of changes in content and emphases in 9-12 mathematics] recommend to decrease attention to "word problems by type" and never mention non-word problems by type or word problems not by type. This recommendation shows that the authors feel that there is something wrong with teaching word problems, but fail to analyze what exactly is wrong. They say nothing about how to teach them. Or, perhaps, the phrase "by type" means some bad manner of teaching? What about types of problems? They are everywhere. Give me a problem which you think is not by type, and I shall invent ten similar problems which will put it into a type. In fact, I often have to do this when I teach: first I solve a problem at the board, then I give a similar problem for all to solve in class, then I give a similar problem as a homework, then I give a similar prob-

lem on a test. All these stages (often more) are necessary, otherwise many students will not grasp the method.

In fact, at the K-12 level there are *many more* different word problems than non-word ones. Word problems enormously *increase* the variability of problems solved in the classroom. In addition to those limited formalisms of pure mathematics, which are available at the K-12 level, word problems bring a plethora of images, such as coins, buttons, matches and nuts, time and age, work and rate, distance and speed, length, width, perimeter and area, fields, boxes, barrels, balls and planets, price, percentage, interest and discount, volume, mass and mixture, ships and current, planes and wind, pumps and pools, etc., etc. It is an invaluable experience for children to discern those formal characteristics of these images, which should be taken into account to solve the problem. What is at least equally important, in my opinion, is that in solving word problems, children have to comprehend and translate into mathematics a multitude of verbs, adverbs and syntactic words indicating actions and *relations* between objects, such as put, give, take, bring, fill, drain, move, meet, overtake, more, less, later, earlier, before, after, from, to, between, against, away, etc. Although I say "children," I actually mean a wide range of ages, including college undergraduates, for whom all this may be quite a challenge [10]. How did that strange idea of uniformity of word problems come into existence? I think that some teachers and educators, too incompetent to cope with the richness of word problems, reduced them to a few types, and this secondary phenomenon, which went *against* the grain of word problems and came from incompetence rather that from potential of word problems, was mistaken more than once for an inalienable feature of word problems. For example, the influential "Agenda for Action" recommends [1, p. 3]:

> The definition of problem solving should not be limited to the conventional "word problem" mode.

What did the authors mean by the "conventional 'word problem' mode?" Perhaps, that uninspiring manner of teaching which still plagues classrooms, and which is caused by poor preparation of teachers? Who knows? In any case, they expressed their ideas in such an obscure manner that no meaningful action could be undertaken based on this recommendation. It is not a secret any more that some teachers of mathematics don't know enough mathematics. In this light the "who cares?" recommendation is especially dangerous because it may be used as a pretext by some teachers. Regretfully, [11] is not the only occasion when word problems are referred in a pejorative way. For example, the second chapter of an otherwise sound book [2] is filled with deteriorative jokes about word problems. Clearly, Morris Kline would not indulge in such frivolous mockery if he were not sure in advance that it would please some readers. Lately a member of an e-mail list proposed to *define* word problems as those given with the intention to evoke a knee-jerk reflex from students. When I objected that it is better to use the term "word problems" according to the meaning of the words, that is, apply it to problems that use words besides mathematical terms, this professional educator was very astonished and admitted that this idea was new to him. It seems that word problems were almost always taught so badly that most students could not separate word problems themselves from the dreadful manner of teaching. Ralph Raimi is one of those who made this distinction [7]:

> I was a tractable student and did what I was told, and they told me what boxes to put certain numbers into, for a limited range of problems, few enough to memorize. It was hard going, and I later realized how easy the problems were, but since I was told how to do them, and since I was rewarded with praise, that's what I did, totally without insight. Nor did the insight emerge as it does in language learning, when one puts words into sentences and inflections on verbs in a sort of continuous process of accretion. In my case algebra did not come to me that way, and what I learned later, that caused me to see how idiotic my high school exercises were, was not rooted in the

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...in solving word problems, children have to comprehend and translate into mathematics a multitude of verbs, adverbs and syntactic words indicating actions and* relations *between objects

> boxes I had learned earlier. The fault was not in the problems, nor in the "type" idea. The fault was in the teaching.

I spent the first forty years of my life in Russia, where presence, even abundance of word problems in mathematical education was always taken for granted. Larichev's textbook for 6-8 grades (13-16 years old) [3], which was used when I was there, contains a lot of word problems. At that time I thought that [3] was just an ordinary textbook. Now, after several years of teaching American college freshmen, many of whom get confused even by simple word problems, I am astonished by the high level and quality of Larichev's work. If graduates from American high schools could solve all the problems from his textbook, they would be prepared better than many of those who are actually sent to calculus. In particular, Larichev's book includes many historical problems, e.g. the following:

A flying goose met a flock of geese in the air and said to them: "Hello, 100 geese." The leader of the flock answered: "We are not 100 geese. If there were as many of us as there are and as many more and half as many more and quarter as many more and you flew with us, then there would be 100 of us." How many geese there were in the flock? [3, p. 27]

Do I need to mention that we solved this problem without calculators and in two ways—without algebra and with it? Russian textbooks for elementary school also contain plenty of word problems. These books are written with an eye to the future, and the problems prepare children to solve more difficult problems in the next years. Example:

A house is to be repaired and 150 window frames need to be painted. One painter can do it in 15 days and another can do it in 10 days. In how many days can the two painters do this job if they work together? [5, p. 190]

This problem may be considered as a preparation for Newton's problem. It follows straight from the data that the first painter makes 10 frames per day and the second one makes 15 frames per day. So together they paint $10 + 15 = 25$ frames per day. Therefore they need $150/25 = 6$ days. The crucial point here, as in all work problems, is to understand which quantity is additive. This is one reason why work problems are useful. Some students naïvely add 15 days and 10 days and come out with 25 days as the answer. It is essential how the teacher reacts to such wrong suggestions. She should quietly observe that this answer contradicts common sense: when two persons work together, they finish sooner that if only one of them worked. Here we get into the realm of teacher's competence, which is measured first of all by her reactions to wrong or partially wrong solutions. A skillful teacher encourages students to use common sense, which in this way gradually ripens into mathematical competence. In the subsequent grades students are proposed similar, but more difficult problems, so that their skills build one on another. In the 6-8 grades students are already proposed to solve problems with data denoted by letters, for example:

Two workers, working together, can fulfill a task in t days. The first worker can do this job in a days. How many days does the second worker need to do this job alone? [3, p. 166]

If word problems are so useful in Russia, why can't they be equally useful in America? There is a certain theory behind this. I shall call it no-transfer theory. This theory claims that children can not transfer ideas, methods and skills from one task, problem, situation to another and therefore it makes sense to teach them to solve only those problems which they will meet in everyday life. This fantastic theory is often taken for granted by American educators and is almost unknown to all other people. It was developed by Edward Thorndike about a century ago. In 1926 Thorndike published his influential book [9], where he claimed:

> Solving problems in school is for the sake of problem solving in life. Other things being equal, problems where the situation is real are better than problems where it is described in words. Other things being equal, problems which might really occur in a sane and reasonable life are better than bogus problems and mere puzzles.

Thorndike gives no examples of "bogus problems," but based on his argumentation one may conclude that this pejorative term refers to all problems which have no literal counterpart in everyday life. But then all modern mathematics is bogus! Based on his ideas,

Thorndike included in his book a chapter called "Unreal and Useless Problems", which starts as follows (p. 258):

> In a previous chapter it was shown that about half of the verbal problems given in standard courses were not genuine, since in real life the answer would not be needed. Obviously we should not, except for reasons of weight, thus connect algebraic work with futility.

Pay attention: Thorndike thought that whenever children are given a problem which they cannot meet in everyday life, they feel a sense of futility. All my experience as a teacher tells me that children's interest in mathematical problems is not determined by straightforward relevance to everyday life. It has much more complicated causes. A lot of my students were excited by various problems, whose wording was fantastic or jocular. In this connection let us consider the following problems:

Mary has forty coins in her piggy bank, all pennies and nickels, which total a dollar. How many pennies and how many nickels are there?

There are rabbits and pheasants in a cage. Altogether they have 100 legs and 36 heads. How many pheasants and how many rabbits are there?

What especially adds to the educational value of these problems is that they can be solved in various ways, even without algebra. For example, we can solve the piggy bank problem as follows. First we assume that all the coins are pennies. Then they total forty cents. This is sixty cents less than we need. Now observe that every time we substitute a penny with a nickel, the amount of money increases by four cents. So, to increase it by sixty cents, we need to perform this substitution 60/4 = 15 times. Thus we get 15 nickels and 40-15 = 25 pennies. We can check this answer by calculating 25 + 15 x 5 = 100 cents = 1 dollar. The rabbits-pheasants problem is included in [3, p. 90] as an "ancient Chinese problem." Polya included a similar problem in [6]. From one teacher I heard a charming way to explain its solution to children: Imagine that all the rabbits stand on their back legs. Then the number of legs standing on the ground is twice the number of heads, that is 72. The remaining number of legs is 28, and these are front legs of rabbits. So the number of rabbits is half of it, i.e. 14, whence the number of pheasants is 36 - 14 = 22. According to all my experience, normal children like these problems and don't ask "who cares?" or "when shall we apply this to everyday life?" Also, all normal children notice that in spite of their different imagery, the piggy bank problem and the rabbit-pheasants problem are similar, and having solved one of them helps to solve the other.

To Thorndike's credit he admits that some of those problems, which he calls unreal and useless, may be interesting for "pupils of great mathematical interest and ability" (p. 259). Many gems might be mentioned as examples of this, e.g. irrationality of $\sqrt{2}$ or the fact that there are infinitely many prime numbers. However, Thorndike mentions only a few historical problems, including the following:

The square root of half the number of a swarm of bees is gone to a shrub of jasmine; and so are eight-ninths of the swarm; a female is buzzing to one remaining male that is humming within a lotus, in which he is confined, having been allured to it by its fragrance at night. Say, lovely woman, the number of bees.

I can testify that in this Thorndike is right: there are students interested in such problems. When I asked Yuly Ilyashenko, who is a professor of mathematics now, how he became a mathematician, he remembered this problem. (It is included in [3, p. 167].)

Thorndike's ideas were criticized by several thinkers, including Vygotsky, who wrote [13, p. 233]:

> ...to refute the Herbartian conception, Thorndike experimented with very narrow, specialized and most elementary functions. He exercised his subjects in distinguishing of lengths of linear segments and then studied how this learning influenced their ability to discriminate magnitudes of angles. Of course, no influence could be found here.

Vygotsky conducted his own experiments, which showed that when dealing with higher mental activities, such as learning of arithmetic and native language, transfer takes place. In this connection Vygotsky spoke about another imporant notion: *mental discipline* (which he called "formal discipline"). The

notions of transfer and mental discipline are so closely connected that it is practically impossible to accept one and reject the other. It is well-known that the greater part of mathematics taught in high school has no straightforward application to everyday life. For this reason, when discussing the importance of mathematical education, we cannot avoid speaking about mental discipline embracing all the non-literal, non-direct and far-reaching results of schooling. Criticizing Thorndike, Vygotsky wrote [13, p. 233]:

> Partially the underdevelopment of the theory of formal discipline and mainly the inadequacy of its practical implementation for the tasks of the modern burgeois pedagogics led to demolition of the whole doctrine of formal discipline in theory and practice. Thorndike was the ideologist; in several works he tried to show that formal discipline is a myth, legend, that teaching has no remote influences, no remote consequences for development. As a result of his studies Thorndike came to a complete denial of existence of those interrelations between learning and development, which the theory of formal discipline correctly anticipated, but presented in a very ludicrous form. However, Thorndike's statements are convincing only insofar as they concern the ludicrous exaggerations and distortions of this theory. They do not concern and certainly do not destroy its kernel.

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...we want children to be able to solve problems with* arbitrary data *and answer* arbitrary *questions rather than only those which they meet in everyday life.

Thorndike's and Vygotsky's conceptions have quite different consequences for mathematical education and the usage of word problems in it. Let us list some of them.

1) If Thorndike is right and mental discipline does not exist, then problems solved in school should be identical with those which students have a chance to face in their present or future life. However, professions are very specialized now, and it is impossible to tell in advance who will follow which profession. More than that, even if we knew somehow that a certain student would become, say, a computer programmer, we still could not teach him exactly that computer language which he will use, because it is a safe bet that this language is not yet invented. Those who had been taught Basic had to program in Pascal or Fortran, and those who were taught Pascal, now program in C++. If mental discipline were just a myth, all their school time would be lost, but many people think that it was not. In fact, some people think that solving mathematical problems, say on geometrical constructions, also helps future programmers. This can be understood only if we accept the notion of mental discipline: if it exists, then transfer is possible and productive. Facing a new problem, children may exclaim: "This is analogous to the problem which we have solved, only with different words and numbers." In other words, they can notice intrinsic similarity between problems and transfer skills and ideas developed in dealing with some of them to solve other, more difficult ones. This is what I always try to achieve as a teacher.

2) If mental discipline does not exist, then word problems should be taken literally, at face value. For example, a coin problem makes sense only as related to dealing with real coins, a rabbit-pheasant problem is related only to rabbits and pheasants, etc. Data which are given should be the same as actually available in practice, and questions which are asked should be the same as those which we usually need to answer in everyday life. Problems in this case are grouped into types according to their paraphernalia, such as coin, rabbit or work. If, on the other hand, mental discipline exists, then intrinsic mathematical structure of the problems is most important, while coins, rabbits etc are just superficial external features. In this case we want children to be able to solve problems with *arbitrary data* and answer *arbitrary* questions rather than only those which they meet in everyday life. Without this, children will not be able to make the next step: from numerical data to data denoted by letters. This also means that numerical data do not need to be cumbersome, because they do not need to look like data taken from a concrete real situation.

3) If transfer is impossible, then interaction between

> mathematics and other school subjects, e.g. physics, is impossible, and it is not worthwhile to care about it. This is what we usually can observe in the public schools of America, where subjects are isolated from each other. If, on the other hand, Vygotsky is right and different school subjects interact, then it makes sense to coordinate curricula in mathematics and physics, so that mathematical notions will be applied in physics and vice versa. This is done systematically in Russian schools.

When I read Thorndike's "The psychology of algebra" [9], I got a strange impression. On one hand, it was clear that Thorndike was a hard, persistent worker. On the other hand, he seems to have had no idea of the essence of mathematics. All my classmates, all my students, all children whom I ever met, knew that problem apples were not real apples, but Thorndike did not know this! It is my impression that having spent much time experimenting with animals, Thorndike became convinced that living beings make efforts only in view of some material reward and uncritically transferred this idea to human beings. Every parent knows that children are spontaneously curious and have fantasy and love fairy tales and fantastic stories, but Thorndike seems not to know this. Every parent also knows that children enjoy *contrafactual* statements and images, often (inexactly) called "absurd" or "nonsense." Lewis Carroll and many other authors elaborated this idea with much success, but Thorndike argues as if he has never heard of it!

Today those ludicrous exaggerations associated with the notion of mental discipline, which Vygotsky mentioned, are almost forgotten. Their place is taken by equally ludicrous exaggerations of the opposite kind. An example: lately a new phrase was introduced into educational literature: "real-world problems." Nobody knows exactly what this phrase means, and different authors use it in different, sometimes contradictory, ways. In any case, it is clear that "real-world problems" are very different from that important and well-known part of mathematics, which is traditionally called applied mathematics. Applied mathematics needs precision, because it deals with hard reality, while problems presented as real-world are often vague and loose and have been said to have many answers (nobody ever said how many). To do applied mathematics one needs a lot of mental discipline, while usage of the phrase "real-world problems" often comes with the assumption that mental discipline is but a myth.

It is obvious that future mathematicians need to study mathematics. Let us ask another question: Why is mathematics important to learn for those who will not become mathematicians? Why is mathematical literacy important? One reason is that we read and write and calculate for our everyday practical needs. There is, however, another reason: a literate person is another kind of person than an illiterate one. Literacy and its analogs, such as mathematical literacy, are not just gadgets. They add new dimensions to personality. A literate person, in particular a mathematically literate person, not only answers old questions better, she asks new questions also. Mathematical literacy includes ability and habit to produce abstract *closures* which go beyond immediate necessity. Incompetent people often think that mathematical abstractions are difficult, and they are right in their own way, but abstractions would not be needed if they were not easy in some other sense. Indeed, to solve an abstract problem is easier (if you can do it) than to tamper with every particular case. This contrast is well visible in the case with word problems. For mathematicians they are so easy that some (like Morris Kline [2]) fail to recognize their importance. On the other hand, for people with undeveloped abstract thinking (some of whom regretfully are teachers of mathematics) word problems are enormously difficult. This is because every type of word problems is a small closure: as soon as you grasp the general idea, you can apply it to many particular cases. In this way word problems give some taste of abstract work to everyone who can cope with them. Let us teach all children to solve them.

REFERENCES

[1] *An Agenda for Action. Recommendations for School Mathematics of the 1980s*. NCTM, Reston, VA, 1980.

[2] Morris Kline. *Why Johnny can't add: the failure of the new math*. New York, St. Martin Press, 1973.

[3] P. A. Larichev. *Collection of problems in algebra. Part I for 6-8 grades*. "Uchpedgiz," Moscow, 1961 (in Russian).

[4] *Mathematical Olympiads by Correspondence*. N. B. Vasilyev, V. L. Gutenmakher, Zh. M. Rabbot and A. L. Toom. "Nauka," Moscow, 1986. (in Russian.)

continued on page 44

A Mathematics Manifesto: Think Differently! Think Quantitatively! Quantitative Awareness as a Fresh Thinking Cap

Miriam Lipschutz-Yevick
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Quantitative literacy among a large proportion of the population could make a powerful contribution towards improving both the inner and outer environment we share as one humanity. It will enhance people's self confidence and so lower their fear to participate in informed decision making.

The dissemination and the teaching of quantitative awareness and literacy to students, to children, to adults of all ages has become a matter of increasing interest. Several sessions, one of which was sponsored by the Humanistic Mathematics Network, were devoted to this topic at the recent meeting of the American Mathematical Society in San Antonio. This awareness can be elicited in many different guises and reveals aspects of our "life environment" previously hidden or ignored. Let us mention but a few of the imaginative ideas presented at the meeting as well as a few among the numerous contributions that have been made towards the use of simple mathematical tools aimed at gaining new insights into our "life environment."

MATHEMATICS IN TRADITIONAL DESIGNS

Integrating Mathematics and Culture in a Developing Nation (Dean E. and Susan C. Arganbright)

The students studied traditional patterns of design in weaving, in leathercraft, in tilings etc. from a geometrical standpoint. Their acquaintance with these ancestral creations triggered their enthusiasm in learning about abstract concepts such as linear transformations, periodicities, symmetry, etc., and stimulated them to use these to invent new designs.

MATHEMATICS ON THE PLAYGROUND

A Guide to Blazing a Math Trail (Mary-Margaret Shoaf)

A stroll with small children through a playground while observing the math all around. Thus the speed of descent on a slide translates into its steepness, the steepness into the slope and the slope into observing the relation between its length and height. The children's conceptual vocabulary is enriched with the word "slope."

MATHEMATICS IN ART

Art and Geometry: Proportion and Similarity (Catherine A. Gorini)

The speaker projected some familiar works of art revealing similarities and repetitions of geometrical patterns in the artistic composition. She thereby opened new vistas and enhanced the audience's appreciation of the more abstract elements entering into the work.

(for more, see Gorini's article of the same title on page 36)

MATHEMATICS IN POLITICAL DECISION MAKING

Information, Data, and Decisions (Deborah Hughes Hallett)

Statistical tools were taught to and applied by a group of government bureaucrats hailing from a variety of developing countries. One of the students' projects consisted of a sharp scrutiny of the use of the weighted averages of various indices (such as the G.N.P., life expectation and literacy) as a measure of development, the so-called H.D.I. (Human Development Index). An independently undertaken collection of relevant statistics revealed that this index is highly sensitive to the weighting (and so ordering) of the components. A relatively small change in these weights produces a massive change in the H.D.I. and so in the funds allocated to these various sectors. Equally blatant was the use of positive correlation between the H.D.I. and G.N.P. revealed in a regression analysis used by governmental agencies. This provoked a violent reaction due to the students' knowledge that the latter functioned as a component of the former to begin with. The students thus became strongly alerted to the danger of manipulation of data for political purposes. At the same time the importance of quantitative knowledge and tools to counteract false conclusions was made ever more evident.

MATHEMATICS IN VOTING SYSTEMS

Questionable Rules (Donald Saari)

This was a mathematical demonstration of the unreasonableness of declaring the candidate with the largest total vote as the winner in an election with more than two candidates. The sentiment of the electorate would be more accurately reflected by deciding on the winner by weighting the candidates' votes according to their rank in the the voters' (combined) preference. Professor Saari explained the combinatorics and discussed these alternatives with a group of fourth-graders using, say, the vote for class president as an example. These youngsters immediately concurred with his method of choosing, maintaining that the "winner take all" outcome would be unfair.

MATHEMATICS AND COINCIDENCES

Probabilities (Persi Diaconis)

In a lecture addressed to the general public, Diaconis applied what he labeled the "Birthday Tool"(i.e. the well-known estimate of the sizable likelihood of a coincidence of birthdays in a relatively small group of people), to the occurrence of events which we apprehend as coincidences in the course of our lives. He thus cleared away some of the confusion or obfuscation by others frequently created by such appearances. This simple mathematical technique once again offers a new thinking cap.

MATHEMATICS AND THE FEAR OF AUTHORITY

Does the Evidence of Authority prevail over the Authority of Evidence? (Shandy Hauk and Mark K. Davis)

The teacher would acquaint her math students with some of the weaknesses in the work of authority figures in the history of mathematics such as, say, Pythagoras, with the intent of freeing them from their fear of mathematics. They were thus encouraged to not be overawed by "established truth" but to trust in their own ability to think the subject matter through independently. It was pointed out that it took nearly fifteen hundred years to the eventual overthrow by Copernicus , Kepler, Galileo and Newton of the official but erroneous cosmology of Aristotle and Ptolemy, much of this delay caused by the fear of authority.

WRITING ABOUT THE MATHEMATICS

Writing things down as in a personal journal sifts one's thoughts, consolidates one's inner life and fits one's experiences into the pattern of one's life story. Similarly expressing the chain of reasoning developed in an attempt to solve a math problem or understand a math argument adds coherence to one's thinking and consolidates one's reasoning powers.

MATHEMATICS FOR LIFE AND SOCIETY

A Course in Quantitative Literacy (Miriam L. Yevick)

This course was developed and taught at Rutgers University to students with a defective mathematics background in lieu of the standard remedial one. Basic quantitative skills were extracted from applications relevant to the students' life environment: using powers of ten to estimate the total intake by the Port Authority during rush hour (and comparing the total accumulated net revenues to the original cost of construction); developing a "feel" of magnitudes by fitting items such as the number of hours worked in a lifetime, the number of pounds of meat consumed in one year in developed as against developing countries, the number of millionaires in the world, etc. into powers-of-ten slots; using trees to evaluate the number of individuals on which one's actions may leave a mark in a lifetime; using permutations to calculate the number of orderings of priorities in a national budget, etc. An adaptation of this course is currently being taught successfully to a group of seniors at The Windrows, a Princeton retirement community. They are encouraged to transmit this way of viewing the world to their grandchild as a token of their own concern and as a bridge to the future for both generations.

For both young and old the fear of mathematics was greatly reduced after they recognized their ability to acquire a quantitative thinking cap. They were also alerted to the potential of using math as therapy when encouraged to think about a math problem while engaged in daily tedious work, when overwhelmed by the stress of personal problems, or when experiencing insomnia or when suffering from physical pain.

SUMMARIZING SOME OF THE POTENTIAL POSITIVE CONSEQUENCES OF QUANTITATIVE AWARENESS:

1. An appreciation of a new dimension of experiencing the world.
2. The development of a sense of abstraction, generalization and precision of thought, hence an ability to think in larger units and so recognize patterns of personal and social problems in a global context.
3. Gaining an enhanced inner confidence by the mastery of a new tool with which to evaluate the "established truth," be it in politics, economics,

education...with a more critical stance.

4. Conquering the fear of math spills over in a lowering of fear in general and a potential to refashion one's habits of mind. It opens up the possibility of basic changes in one's "cosmology" on a personal and communal level.
5. It allows all to share in relishing the joy and beauties of mathematics.

A CALL TO ACTION

I listened to Public Radio while in San Antonio, and after hearing the news I was regaled several times a day by an instructive and exciting little astronomy lesson originating in the university's observatory. It occurred to me that we could do the same with a "Think Math!" lesson.

My suggestion is that we establish a website devoted to disseminating Quantitative Awareness. We would both teach and solicit comments and contributions of a kind similar to the type of examples we discussed previously, i.e. quantitative techniques to view such things as budgets, national and local priorities, election expenditures, environmental problems, etc.

A clear and humane vision is sorely lacking in most of the world's governing bodies (beholden mainly to moneyed moguls, corporate or individual) in the face of the numerous difficult problems facing our world . Thus all kinds of things have gone out of kilter even in the most advanced countries, such as health care, education, legal and political arrangements, environmental balance, demographics, distribution of wealth, let alone the allocation of resources. Many of us, as teachers, have seen the spark of understanding light up in the eyes of our students when they grasped a concept, then realized its use and that they themselves could apply it. It is becoming more necessary than ever that people at large be enabled to search for and advance solutions from below and figure things out at the local and global level. This requires a critical stance towards what one is "told from above," be it by the media or the politicians, coupled with a confidence in one's ability to think clearly, rationally and freely. Let's go for it!

Math is really great
It teaches you how to calculate.
You can learn stuff about math
Even when you take a bath.
Sometimes math can be strange
Like when learning about range.
Math can be real mean
When you are a teen.
You need to do multiplication
For analyzation.
a future with math is near
Please do not fear.
The things you learn about math today
Will bring you all the way.
I think math is really great
It teaches you how to calculate.

Eric Rasyidi

Adding is as easy as one, two, three.
Just add ten plus ten to reach twenty.
But as you really begin to try,
Addition becomes fun as can be.
Subtraction isn't much harder.
You don't have to be any smarter.
Instead of adding you take away.
You can subtract any day,
If you remember my simple starter.
Multiplying is like adding many times.
For example, let's use three dimes.
Instead of adding ten plus ten plus ten,
Just multiply three times ten.
Multiplication can be great sometimes.
Division is just shrinking numbers down.
When you divide there's no reason to frown.
You can divide in the rain.
You can divide on a noisy train.
Don't try dividing in a pool because you just might drown.

Marshall Sachs

Art and Geometry: Proportion and Similarity

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A course entitled "Art and Nature" has been the first course in the art major at Maharishi University of Management for several years. The main purpose of this first course is to introduce students to the field of art and creative expression. A significant component of the course is an ongoing project in which the students choose a natural site on or near campus (for example, "around the bridge over Crow Creek") and then observe the site, document their observations, and finally create a "map" of the site. This year the course included guest speakers from physics, biology, and mathematics. This paper will describe the lesson on mathematics that I presented to the students in the course.

There are, of course, many, many ways that mathematics can be related to art. I chose proportion and similarity for a number of reasons. My time was limited, and I knew that the mathematical background of the students would be uneven. I wanted to present ideas that students could actually use in observing their sites without any further study or instruction. In addition, similarity leads naturally to a discussion of fractals, and I wanted to include something that would give a flavor of current developments in geometry. The lesson eventually covered four topics: ratios of measurements, congruence, similarity, and fractals. We then looked at paintings selected by the instructor, Dale Divoky, to see examples of these concepts as used by different artists.

RATIOS

We can compare measurements in an absolute way, as when one measurement is four inches longer than another measurement, or we can compare measurements in a relative way, when we say that this measurement is three time as large as that measurement. Ratios are used to compare measurements in this relative way. A *ratio* or *proportion* of measurements is just the fraction formed by the two measurements.

A familiar example of a ratio is π, the circumference of a circle divided by its diameter, which is the same for all circles. However, the ratio of the circumference of a circle with radius r to its area is $2/r$, which changes as r changes. For a sphere of radius r, the ratio of the surface area to the volume is

$$\frac{4\pi r^2}{\frac{4}{3}\pi r^2} = \frac{3}{r}$$

and as r increases, this ratio gets smaller. That means that a large sphere has proportionately less surface area than a smaller sphere.

If we compare the ratio of surface area to volume for a sphere of radius r, which is $3/r$, to that of a cube of side s, which is $6/s$, we see that the ratio is smaller for the sphere. In fact, it is smaller for the sphere than for any other shape enclosing the same volume. Shapes such as soap bubbles, fruit, balloons, and over-stuffed suitcases tend to be spherical because of the presence of natural forces which minimize surface area for a given volume. The ratio of volume to surface area also helps explain why cells, which are constantly transferring molecules across the cell membrane, are small and have more surface for the given volume.

On the other hand, thin flat shapes have much more surface area per volume; paper and leaves are good examples, and their surfaces are useful for writing on or catching sunlight. This one insight into measurements and their ratios can help us understand how the functions and purposes of different objects are related to their shapes.

CONGRUENCE

Two objects are *congruent* if they have the same shape and size; see Figure 1. In this case, ratios of corresponding measurements will all be 1. There are many examples of approximate congruences in nature: our two

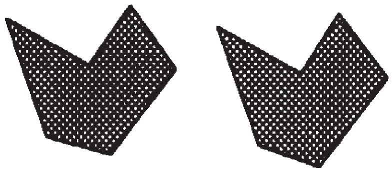

Figure 1
Two congruent figures

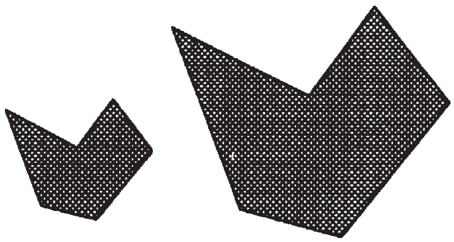

Figure 2
Two similar figures

hands, leaves on a tree, animals of the same species.

SIMILARITY

Two objects are *similar* if they have the same shape; this means that ratios of corresponding linear measurements of two similar figures will be the same. See Figure 2. Examples of similarity abound in nature. Plants and animals of the same species at different stages of growth are often similar. Crystals of the same substance are similar.

FRACTALS

A *fractal* is a figure that is similar to itself, or self-similar; this means that a part of the figure is (approximately) similar to a larger part or to the whole figure. The Koch snowflake is formed by adjoining similar equilateral triangles to the sides of an equilateral triangle; see Figure 3. The method of construction of the Koch snowflake assures that in the limit this process will give a self-similar figure. There are many examples of self-similarity in nature. Shapes like clouds or coastlines that look the same when scaled (up to certain limits, of course) are self-similar. Ferns are self-similar up to three of four levels of scaling. We see a more limited example of self-similarity in some trees, where the leaf or cone has the same overall shape as the tree itself.

IN ART

These mathematical concepts can be useful in the analysis of natural objects or works of art and can serve the artist in the design of a project. We might begin by asking questions like the following.

1. What are some of the more important or striking objects?
2. What are the overall proportions of a particular object? Why is the object shaped in this way?
3. What shapes predominate? What overall kinds of proportions do the predominant shapes have? Examples of overall proportions are large and flat, long and thin, round or spherical. How do these shapes create an effect in the observer?
4. Are any of the shapes similar or congruent? Are the objects corresponding to these shapes naturally similar or congruent?
5. Are there self-similar shapes? What feeling does this create?

STUDENT RESPONSE

Overall, I felt that the lesson was a success. The students brought out good observations from their side, and our analysis of about 20 paintings brought out all of the mathematical ideas we had discussed. Dale Divoky reported that in the analysis of their sites, several of the students were using the idea of similarity. One student kept finding similarity between more and more different objects and used it as a theme in his final "map" of his site. Dale felt that the main value of the lesson was to bring the attention of the students to the ideas of congruence, similarity, and self-similarity so that they could see them when making their observations. He said:

> Often it is the case that we only see what we know to be there or assume to be correct. Knowledge, therefore, is most valuable for observing the world around us (or within us). The knowledge of ratios, similarity, and fractals were of great value to the students in the Art and Nature course. Not only were their sensibilities heightened, but their aesthetic response became a more insightfully creative response.

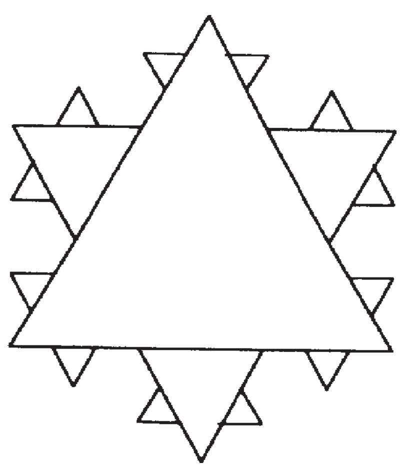

Figure 3
The first three stages in constructing the Koch snowflake.

Analogic Thinking in Science and Math

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One of the great pleasures of my position in a nationally known writing-across-the curriculum program is discovering in many scientists a deep appreciation for humanistic thinking. The science wars wage on in the background, and I do find plenty of evidence of rifts between the "two cultures" that C. P. Snow described a half century ago. Nonetheless, there are on our campus many mathematicians and scientists who not only harbor all sorts of artistic talents, but also call upon their students to use language and to think imaginatively. Long before Professor Dennis Sentilles' calculus course was formally designated "writing intensive," he had asked his students to write. That is, Sentilles recognized the power of language to help students conceptualize the mathematical procedures they were working through. His most noteworthy assignment, now a staple in his writing intensive sections, asks students to compare differential calculus to a videotaped tennis game. Students use the extended metaphor (see "A Leitmotif for Differential Calculus," facing page) to explain the nature and measurement of time and motion and their representations from practical, cognitive, scientific, and mathematical points of view.

Intrigued by this professor's assignment, I wanted to review other scientists' use of analogic thinking and to investigate, however informally, some students' responses to analogic thinking. Following is a brief tour through the history of analogic thinking in science as well as a discussion of analogic thinking as reported by six students, three from Sentilles' calculus classes and three from a writing-intensive genetics class that also foregrounds language and imaginative thinking.

SCIENTISTS AND LANGUAGE

Scientists have typically defined scientific writing in terms of the other: It is not literary. It is not ambiguous, expressive, personal, or persuasive. It certainly does not favor metaphor. This prejudice against "literary language" was strong in 1660, when members of the first British society of scientists denounced "all amplifications, digressions, and swellings of style" and called for a return to a "primitive purity" of language. Instead of the "superfluity of talking" that has "overwhelm'd most other Arts and Professions," the new sciences demanded a "naked, natural way of speaking; positive expressions; clear senses; a native easiness" (as quoted by Locke 4 and Bizzell 642).

Three hundred years after the founding of the British Royal Society, many scientific style manuals still pan any use of metaphor or figurative language. Sentilles' use of an extended analogy in calculus might be suspect except for its heuristic or pedagogical value. Analogies might be useful for communicating something to a broad or popular audience, but many scientists would still argue that analogies have little place in discovery or in communicating to a specialized audience. A look at the history of science suggests otherwise, though: analogic thinking has been important both in the discovery and the communication of knowledge, as well as in the more obvious role of teaching. An informal protocol/interview analysis of six students suggests that analogic thinking may be valuable, not so much because it bridges old and new concepts and expedites learning, but because in many cases it disrupts and slows down learning.

ANALOGIC THINKING IN SCIENCE

Investigations into the role of analogy in science have not been limited to pedagogy. Philosophers and historians of science have also studied the role of analogy at the points both of scientific discovery and scientific argument or justification. In the nineteenth century, physicist and mathematician Henri Poincaré as-

A Leitmotif for Differential Calculus

Imagine being out on the tennis court with the ball rising toward you. Differential calculus is the mathematics that describes and measures change in such an ever changing "time-ball" system. One can use this easily imagined setting to elucidate a leitmotif for differential calculus along the following conceptual theme line, where f(t) is the height of the tennis ball at time t:

Computation < ——-— Abstraction < ———— Life/Reality —-— > Cognition

Geometry of Graph	Math Model	Videotape	External Reality	Cognition
Graph of *f*	The function *f*	The whole videotape	The path of the ball	The time-ball "system"
One point on the graph: *(t, f(t))*	*f(t)*	Image on frame taken at time *t*	Where the ball is at time *t*	State at time *t*
Horizontal axis	Domain of *f*	Length of time spent taping (# frames)	How long the ball is in flight	Duration of events
Vertical axis	Range of *f*	All frames on the videotape	All the different positions of the ball	All individual states of the system
Vertical increment in graph	Difference in function values: *f(t+h)-f(t)*	Change between images	How much the ball rises between two moments	Change-of-state
Horizontal shift or increment	Difference in underlying variable: *(t+h)-t=h*	Time between frames	How much time changes	Time span
Slope of secant (two-point) line to the graph between *(t, f(t))* and *(t+h, f(t+h))*	Average rate of change: $\frac{f(t+h)-f(t)}{h}$	Sense of change between images	How quickly the ball appears to rise between two separated moments	Average rate of change (rise): one's sense of motion over a span of time
Slope of tangent line at the point *(t, f(t))*	$f'(t)=\lim\limits_{h \to 0}\frac{f(t+h)-f(t)}{h}$	Video motion seen at time *t*	How fast the ball is rising at time *t*	Rate, or change-of-state, at the moment *t*
All tangents to the graph of *f*	The derivative of *f*: *f'* as a new function	Viewing the video	The flight of the ball	The motion, or flow, of the system

serted that "logic, which alone can give certainty, is the instrument of demonstration; intuition is the instrument of invention," and he credited analogy with being the guide to mathematical invention. The Italian rhetorician Giambattista Vico made similar claims two hundred years earlier. Many scientists, including Humphry Davy, Robert Hooke, Johannes Kepler, Antoine Lavoisier, and Robert Oppenheimer, have also acknowledged the role of analogy in discovery or in intuition (Leatherdale). Perhaps the most famous scientific analogy is Friedrich August Kekule's account of dreaming about a serpent biting its own tail just prior to his discovering the structure of the benzene ring:

> During my stay in Ghent, Belgium, I lived in a fine room on the main street. I sat in this room and wrote on my textbook, but could make no progress—my mind was on other things. I turned my chair to the fire and sank into a doze. Again the atoms were gamboling before my eyes. Little groups kept modestly in the background. My mind's eye, trained by the observation of similar forms, could now distinguish more complex structures of various kinds. Long chains here and there were firmly joined; all winding and turning with snake-like motion. Suddenly, one of the serpents caught its own tail and the ring thus formed whirled exasperatingly before my eyes. I woke as by lightning and spent the rest of the night working out the logical consequences of my hypothesis (qtd. by Leatherdale 20).

Astonishing as some accounts of analogic thinking are for scientific discovery, they are less controversial than the accounts of analogy in scientific argument, particularly in scientific induction. While Aristotle cautioned against argument by analogy (as many logicians have since), Francis Bacon recognized the importance of analogy to scientific argument. John Maynard Keynes further credits Bacon with distinguishing between positive and false analogies. Twentieth-century philosopher of science Mary Hesse modifies Bacon's distinction between positive and false analogies by examining the positive and false elements within any given analogy. Within any one analogy are both positive and negative components. The predictive power of analogic thinking comes, according to Hesse, from a third element, the part of the analogy about which scientists are still undecided. This distinction is similar to one made by Mike in the discussion below: students might be irritated by the false elements of an analogy, but constructively troubled by a "third element," the part that slows them down and causes them to mull over the concept. Few scientists or philosophers of science deny that analogies offer a heuristic value—in the classroom or in the profession, but there is less consensus about the necessity of analogy for scientific explanations. Hesse, among others, argues that analogy is necessary for scientific argument.

Philosophers have asked parallel questions about the role of analogy in language. Friedrich Nietzsche's radical assertion that all language is metaphoric (and, therefore, analogic) has become commonplace in the twentieth century. Postmodernists have largely dismissed the cautionary hedge in I. A. Richards' comment, "Even in the rigid language of the settled sciences we do not eliminate or prevent [metaphor] without great difficulty" (92). I contend that we in the late 1990s need to revisit Richards who, on one hand, denounced "the one and only one meaning superstition" and boldly asserted that "metaphor is the omnipresent principle of language" and, on the other hand, recognized greater rigidity and stability in the language of science.

To the degree that philosophers and scientists agree that metaphor and analogy play a vital role in science, they aren't entirely celebratory. Turbayne cautions that victims of metaphor are trapped unwittingly by prevailing metaphors, much as Thomas Kuhn argues that the prevailing metaphors in a given paradigm both shape and limit scientists' thinking.

However, the "problem" areas of analogies might be prime sites for "disequilibrium," Jean Piaget's term for the tension between the known and unknown that motivates leaming—in this case, learning about science and learning about language. This is the concept that Robert Mayer builds upon in "The Instructive Metaphor: Metaphoric Aids to Students' Understanding of Science" (1993).

ANALOGIC THINKING IN NOVICE SCIENTISTS

Mayer is not the first to think about the role of a particular kind of analogy, metaphor, in learning. The 1970's marked the "cognitive turn" in psychology and

in "metaphorology," a time in which psychologists, cognitive linguists, anthropologists, and literary theorists widely accepted the premise that metaphor is not just a marker of deviance (genius), as Aristotle believed, but is common (by degree) to all thought. Cognitive linguists explored not only linguistic structures in a text, but also the ways in which a reader processed them. In the following quarter of a century, sociolinguists have called attention to the importance of context and social relations in discourse.

In a 1994 study of student processing of metaphor, *Understanding Metaphor in Literature*, Gerard Steen analyzes the ways in which students of literature process both potential metaphors (linguistic structures identified by experts as metaphor) and realized metaphors (cognitive reconstructions of potential metaphors). As a discourse analyst working in the realm of pragmatics, Steen assumes that the reader, the text, and the context are all constituents in the study, but that the reader is at the center of the investigation. He assumes that a reader's goals are partly socially determined and that discourse communities share certain regularities and conventions. In this study Steen attempts to move reception theory from the text to the reader as a locus of discourse analysis. Steen first asked students to underline metaphors in a text to see if students recognized as potential metaphors the same linguistic structures as those identified by expert readers, in this case a panel of literature professors.

Sharing scholarly debts to pragmatics and discourse analysis, I wish to explore the relevance of Steen's inquiry to science literacy studies. I broadened and altered the scope from a study of literature students' processing of metaphor to a study of science students' processing of analogy. Over a dozen students participated in this study, but I focused on six, three from the calculus course described above and three from a writing-intensive genetics course. Slightly modifying Steen's methodology, I asked students first to underline analogies in each of three texts and explain them in a taped interview afterwards, and secondly to "think aloud" as they orally read three different passages. In each of the two sessions, the underlining/explanation session and the think-aloud/explanation session, students responded to an excerpt from a work of popular science written for a general audience, an excerpt from a science text (*Human Genetics*) written for college students, and an excerpt from a science journal written for experts in neurophysiology. My initial question, to what degree do student readers of science think analogically, developed into the following six questions as the interviews took place:

- Do these six students recognize as potential analogies the same linguistic structures identified by experts?
- Do these six students process potential analogies analogically? (That is, are potential analogies read literally or figuratively, and, if figuratively, in what ways?)
- Do these six students find analogies helpful in understanding the content?
- What happens when an analogy breaks down, as most analogies eventually do?
- To the degree some analogies bridge new and old information, how does the bridging work?
- Does analogic thinking lead to greater insight into the nature of language?

In general, there was a wide discrepancy between potential and realized analogies. Most of the students only realized or reconstructed the potential analogies when they talked or wrote about them. These students found some analogies much more helpful than others, but all of the students interviewed affirmed the potential instructional value of using analogic thinking in the sciences and of having qualitative learning precede quantitative learning. And, for most students, the interview project led them to think about the language of science in ways that had never before occurred to them. The not-so-literal dimension of language is more pervasive than most students had realized. This awareness, in turn, did lead a few students to think about the ways in which science is "made" in new ways, but it did not cause them to question the value of science. These conclusions, along with the success of the written assignments foregrounding analogy in the calculus course, point to the value of making deliberate use of carefully selected analogies in the sciences.

POTENTIAL ANALOGIES

With one exception, students preferred the popular science genre to either the text excerpts or to the technical academic articles, and they attributed their preference to the abundant images and comparisons in the popular science writing. Students made comments such as "It got you interested" and "That was help-

ful; I probably wouldn't've understood it [the article] without them [the analogies]" — or "It gave me something concrete to hold in my mind." In the first popular science excerpt, entropy was compared both to an engine running out of gas (a conventional analogy used in most physics courses) and to a casino closing down (a novel and productive analogy for all six readers). Even when students claimed to enjoy excerpts with many "potential analogies," though, they didn't always identify the analogies as such. Five of the students rarely identified as analogies anything other than similes or phrases that were announced by tags such as "...is like." None of the students, for example, identified "cDNA library," "transcription," "editing," or "palindrome" as part of an extended linguistic analogy, even though they could readily identify more terms in the same group once the extended analogy had been pointed out to them. The distinction made here between "potential" and "constructed" analogies is affirmed by the students' "monovalent" reading of many conventional analogies (by a literal reading of a conventional analogy). With the exception of the same student (Steve), the undergraduates disliked the technical article, which made little blatant use of analogy.

Steve, the most advanced of the calculus students, expressed decided appreciation of analogies, but constructed his own analogies with or without the prompt of the "potential analogy." As he put it, "I'd almost say that any time I see something that I'm familiar with, the whole index [of mind and memory] is opened up, and I can pull out my file card and say, ah, here's one!" The lack of "potential analogies" in the technical article did not bother him because he was rifling his own mental files, including many "received" analogies from other texts and lectures. He liked the technical article precisely because it was the most foreign to him, because it challenged him the most to construct his own analogies or to recall analogies from memory. In the genetics text excerpts, also, one phrase after another would elicit an analogy not present in the text structure but present in Steve's memory from a drawing on the blackboard in a previous course or from a picture in an old textbook.

CONSTRUCTED ANALOGIES

Although Steve was the most active reader—the most ranging in his connections beyond those presented in the text—all six students constructed analogies when given an opportunity to write or talk about them. Some analogies were grounded in "potential analogies," those text structures that expert readers would identify as analogies; other analogies were comparisons between seemingly-literal information in the text and something in the students' experience. For example, few students read "blind watchmaker" as much other than a placeholder for an unfamiliar idea in the think-aloud interviews, but the more they talked, the more they began to make sense of "blind" and to sort through similarities and differences in the sonar capacity of bats (a result of chance and evolution) and the sonar capacity of machines (purposefully designed by engineers).

> "
> ***...if you force people to make analogies or have an analogy set up for them, the fundamental parts of calculus won't be glossed over so much, but will be used and understood. --Steve***

The three calculus students had also just completed the course in which they were asked to explain in writing an extended analogy of a videotape of a tennis game. As indicated earlier, the professor of this course, Dennis Sentilles, compares calculus to an ever-changing "time-ball system." A function is compared to the whole videotape of the path of a tennis ball, and the domain of f is compared to the length of time the ball is in flight (the number of frames on the videotape) and the range of f is compared to all the different positions of the ball (or all the frames on the videotape). All three students found this approach to calculus revolutionary and constructive. Steve, who claimed to have an intuitive understanding of the equations, found himself re-defining and clarifying ideas that had already made some sense to him. He found the videotape analogy indispensable and, when asked if he would teach in the manner of Sentilles, responded, "most definitely...I think if you force people to make analogies or have an analogy set up for them, the fundamental parts of calculus won't be glossed over so much, but will be used and understood." That, unfortunately, is what Tom described as being the case in previous courses: "Before I went in there, I just got pushed through calculus, and I didn't really learn to conceptualize it." Learning to think qualitatively and not just quantita-

tively proved to be revolutionary for him in other courses as well. "After this course, after I started in this course, my grades shot up because I would sit back and look at something and say, 'Okay, I can't get this, why? What are we doing?' And after a while, I'd say, 'ah, that's why!'" Mike, the third calculus student, also said that "for me, it was the analogies that made my understanding. I couldn't just throw those out." All three of them, though, identified the writing process as the place where the received analogy started to make full sense, where the "potential analogy" created by Sentilles became a "constructed" or "reconstructed" analogy in their own minds. Two tentative conclusions might be drawn from this: first, potential analogies might not offer much if students aren't asked to play with them, if students aren't given the time and resources to reconstruct them. Second, all analogies are not equally valuable: Sentilles had experimented with many other potential analogies before settling on the time-ball analogy that proved to be powerful for a wide variety of math students.

ANALOGIC BREAKDOWN

Steve identified "negative analogy" or inaccurate comparisons in both the genetic text and popular science excerpts. He conceded that these limitations could lead to misunderstanding. Instead of ending the analogy, though, he felt readers should keep extending the analogy: "Keep re-defining, keep talking." Here, I would like to distinguish between those analogies that fail because the student already possesses a more refined understanding of the concept and those analogies that are troubling, provocative. Most students who found some analogies too simplistic simply skipped over them. In the *Human Genetics* text, numerous similes were used, and some simply failed for students who had considerable coursework in biology. For example, most students found useful a comparison of Vitamin D and a faulty receptor to a ferry unable to dock, but advanced biology students found other similes limited. However, if an analogy was troubling (not simplistic, but troubling), most students found even the troubling part stimulating. Mike and Steve both objected to some of the emotional implications of the casino analogy for entropy; neither wanted to view entropy as something "bad" or something to lament. Mike didn't fault the writer, though, for an imperfect analogy is thought-provoking, slows a reader down: "Because if he didn't have words in here like that you'd just read it and go on...but then he used words that have a lot of different meanings, or could have...and you have to think...and it makes you mull it over." In other words, the "meaningful calculus" sought by professors such as Sentilles or the "instructive metaphor" sought by learning theorists such as Richard Mayer comes about by gaps that motivate new learning.

BRIDGING OLD AND NEW INFORMATION

All six students were quick to credit analogies with getting them interested in new material. Only one of them expressed much interest in entropy, but all of them found the article about entropy quite interesting. It appears, though, that analogies function even more effectively by breaking up a bridge, by creating a hurdle, or slowing down a train of thought. In both cases, the analogy often functioned as a placeholder, a space for a concept that would become better understood in time. Several of Mike's comments pointed to still another function of analogy: a bridge not from the unfamiliar to the familiar, but from the now-thoroughly-understood to memory. In other words, analogies can function as a way of compressing and repackaging already-understood concepts for long-term storage.

INSIGHT INTO LANGUAGE

The more I gave these six students an opportunity to explain themselves, the more they realized that they were dependent by degree on analogies. Put another way, the more they tried to remove themselves from analogic thinking, the more they realized they couldn't do it. They began to realize that words and concepts are born and grow and change, and most found it impossible to express scientific concepts in absolutely value-free language. Many had never before thought about the etymology of conventional vocabulary such as "bacteria" (little staffs). Several returned to the second interview with examples of analogies they had found in other science texts. Although students gained insight into language, their insight was a byproduct of their learning about something—entropy or natural selection or transcription or neural transplants. They weren't bashing science, but were gaining insight into it.

CONCLUSION

Writing across the curriculum (WAC) programs, such as the one in which I work, have largely substituted a belief in linguistic positivism (which treats language

as if it were a transparent medium and writing skills as if they were generalizable across all contexts) with a belief that language can never be completely "clear," can never be completely rid of analogy, and, even if it could, it shouldn't. As scientists and humanists work together to better understand the languages and conventions that do characterize our disciplines, we may also better understand each other.

REFERENCES

Bizzell, Patricia and Bruce Herzberg, eds. *The Rhetorical Tradition: Readings from Classical Times to the Present*. Boston: St. Martin's P, 1990.

Dawkins, Richard. *The Blind Watchmaker: Why the Evidence of Evolution Reveals a Universe Without Design*. New York, Norton: 1987.

Hesse, Mary. *Models and Metaphors in Science*. Notre Dame, IN: U of Notre Dame P, 1966.

Kuhn, Thomas S. "Metaphor in Science." in *Metaphor and Thought*, Ed. Andrew Ortony. Cambridge: Cambridge UP, 1993.

Leatherdale, W. H. *The Role of Analogy, Model, and Metaphor in Science*. New York: American Elsevier Publishing Co., 1974.

Lewis, Ricki. *Human Genetics: Concepts and Applications*. Dubuque, IA: W. C. Brown, Publishers, 1994.

Locke, David. *Science as Writing*. New Haven: Yale UP, 1992.

Mayer, Richard E. "The Instructive Metaphor: Metaphoric Aids to Students' Understanding of Science." in *Metaphor and Thought*, Ed. Andrew Ortony. Cambridge: Cambridge UP, 1993.

Ortony, Andrew. *Metaphor and Thought.* 2nd ed. Cambridge: Cambridge UP, 1993.

Richards, I. A. *The Philosophy of Rhetoric*. Oxford: Oxford UP, 1964 (1936).

Russell, David. *Writing in the Academic Disciplines, 1870-1990: A Curricular History*. Carbondale, IL: Southern Illinois P, 1991.

Sentilles, Dennis. *Applying Calculus in the Economic and Life Sciences*. Belmont, CA: Wadsworth, 1989.

Snow, C. P. *The Two Cultures: And a Second Look* (includes original 1956 essay). Cambridge: Cambridge UP, 1964.

Steen, Gerard. *Understanding Metaphor in Literature: An Empirical Approach*. New York: Longman, 1994.

Turbayne, Colin Murray. *The Myth of Metaphor*. New Haven: Yale UP, 1962.

Between Childhood and Mathematics: Word Problems in Mathematical Education

continued from page 32

[5] *Mathematics 4. Textbook for the fourth grade of elementary school*. Ed. by Yu. M. Kolyagin. "Prosveshenie," Moscow, 1992 (in Russian).
[6] George Polya. *Mathematical Discovery. On understanding, learning, and teaching problem solving*. Combined edition. John Wiley & Sons, 1981.
[7] Ralph A. Raimi. E-mail communication.
[8] *Curriculum and evaluation standards for school mathematics*. NCTM, March 1989.
[9] Edward L. Thorndike a.o. *The psychology of algebra*. The Macmillan Company, New York, 1926.
[10] A. Toom. "How I Teach Word Problems." *Primus*, v. VII, n. 3, September 1997, pp. 264-270.
[11] Zalman Usiskin. "What Should Not Be in the Algebra and Geometry Curricula of Average College-Bound Students?" *Mathematics Teacher*, v. 88, n. 2, February 1995, pp. 156-164.
[12] Zalman Usiskin. "Why Is Algebra Important To Learn?" *American Educator*, v. 19, no. 1, Spring 1995, pp. 30-37.
[13] L. S. Vygotsky. *Thought and Speech. Collected Works in 6 volumes*, Moscow, "Pedagogics," V. 2, 1982 (in Russian).

A Collection of Ideas on Systems and Their Extensions

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Systems beget prejudice. Closed systems beget malicious prejudice. What's inside is familiar, comfortable. What's outside is not--it's considered unimportant, irrelevant, sometimes even nonexistent, and invariably it's given a **derogatory** name. But if perchance the system gets extended, and the extended system somehow or other, not only continues to work, but even works better, then a new closed system is born--a new, improved, superior system that is then given a *laudatory* name.

NUMBER SYSTEMS

The counting numbers 1, 2, 3, ..., also formally called the natural numbers, are the subject of Kronecker's famous quote, "God made the natural numbers, all the rest are the invention of man." They are familiar, they represent real things in the world. They are indeed natural. The numbers -1, -2, -3,..., in some sense a mirror image of the natural numbers, are very unnatural. They don't represent things which can be pointed at, and they are spoken of disparagingly as **negative** numbers. One way to justify calling them numbers is the following. An equation with an unknown can be considered as a question. "$x + 1 = 2$?" asks "What number upon adding 1 to it results in 2?." The answer is the number 1. "$x + 2 = 1$?" however, has no answer in the natural number system. By going outside the system one can posit -1 as the answer. In fact, by combining the naturals and the negatives with the number 0 as connecting link, namely ..., -3, -2, -1, 0,1, 2, 3, ..., this new system is powerful enough to answer the question "$x + a = b$?" for any numbers a and b in the system. This was impressive enough that it came to be called the *integer* number system. Integers have integrity. They are all honest-to-goodness numbers.

There are questions, however, like "$2x = 1$?" which have no answer in the integer number system. The answer 1/2 is a ratio of two integers, and, because it is not an integer, that is, a whole number, it was spoken of disparagingly as a **fraction**, namely a broken, or fractured, number. The fractions are not like a mirror image of the integers but are thought of as sitting in the spaces between the integers. By extending the integers to include the fractions, the resulting system became powerful enough to answer the question "$ax = b$?" for b any number in the system and a any number other than 0. It all seemed so reasonable to consider fractions as bona fide numbers, that the extended system came to be the *rational* number system.

The rational number system, in its turn, failed to answer many questions, some as simple as "$x^2 = 2$?" The answer in this case can be represented by the diagonal of a square whose sides have length 1, and this answer, $\sqrt{2}$, is of course spoken of disparagingly as an **irrational**. The irrationals are thought of as being distributed among the rationals, namely along the same number line, which at first is hard to conceive of, since the rationals by themselves are "dense" in the sense that between any two rationals, no matter how close, there is another rational. Even worse, it turns out the irrationals are "infinitely denser" than the rationals. By extending the rationals to include the irrationals, not only is the question "$x^a = b$?" answered for many combinations of a and b, but also many other kind of questions (e.g. "$dx = c$?" where d is the diameter and c the circumference of a circle). The irrationals turned out to be very useful, and because it seemed at the time that no more numbers could be squeezed into the number line, the combination of rationals and irrationals came to be called the *real* number system. Irrationality had bestowed upon it the mantle of reality.

These intimations of reality however did not provide omniscience, for there were still questions with no answers, such as "$x^2 = -1$?" The answer $\sqrt{-1}$, not being real, was spoken of jeeringly, one might say, as being **imaginary**, and for that reason was given the special symbol i. This imaginary number did not fit anywhere on the real number line. Further, multiplying i by a real number y suggested an imaginary number line iy which intersected the real number line at 0,

and adding a real number x to iy, i.e. $x + iy$, produced a number plane, which turned out to be so incredibly useful in solving a myriad of complex problems (from those in formal mathematics to those in theoretical physics) that the numbers $x + iy$ came to be known as the *complex* number system. This extension transcends the real number system in a way quite different from the previous extensions, for in this case a new dimension has been added to the number system. The complex plane is not the only way to extend the real number line. The answer to "$|x| = -1$?" is given a symbol h and is spoken of as being **hallucinatory**, with the numbers $x + hy$ making up the *perplex* number system [1]. Also, the answer given to "$x^2 = 0$ and x not equal to 0?" might be given the symbol ℓ and be spoken of as being **ludicrous**, with the numbers $x + \ell y$ making up the *ethereal* number system [2]. These two number systems, while not as useful as the complex number system, nevertheless do have their uses, and lend credence to the idea that having more than one extension to a system doesn't mean they should compete with each other as to which is the correct one. As inventions of man, all are, so to speak, on the same footing.

PEOPLE SYSTEMS

People systems are closed by borders, which might be of various kinds, such as geographical, class, gender, and theological. Extensions of people systems can thus occur in various ways.

The Athenian city-state was a system in which the citizens of the state lived comfortably with the Athenian culture providing gracious living in an environment of theater, music, art, and philosophical discussion. Due to the lack of these things, the neighboring city-state was, in a sense, a mirror image of the Athenian one, and the word **Spartan** was used disparagingly by Athenians. There were questions, however, which had no answer when Athenians restricted themselves to their own state, such as "What military unit can serve as a rapid reaction force, with the ability to cover fifty miles in a day and survive for two weeks with no backup support?" By joining Athens and Sparta (and other city-states) together, the resulting combination became much more powerful, questions like the foregoing could now be answered, and all citizens of the new system were proudly known as *Hellenes*, or *Greeks* as we would now say.

If one were to ask "Who does the cooking, carrying and cleaning?", the answer would not be a Greek citizen, but rather a **slave**, who was not a citizen and who was spoken of disparagingly. One might further add that slaves were distributed in the spaces between citizens. It took a long time for the Greeks, and for the countries of the world in general, to extend full citizenship (implying freedom) to those who do the menial tasks of society, but there is a general consensus that doing so increases the well-being, strength, and productivity of that society, and this seems especially so when citizenship means the right to vote as in a genuine *democratic* system.

With the abolition of slavery, there still remained a large class of people which society depended greatly on, and yet were usually spoken of in a disparaging manner, namely **women**. They were distributed roughly equally in the spaces between men, but were not considered their equals in that they were not granted the same rights and privileges. Given the increase in power and effectiveness with previous extensions, it is an expectation of many that the same will be a consequence of extending equality to all people in a society, but achieving an *egalitarian* system is presently in all societies an ongoing struggle.

The first society considered consisted of humans, and the subsequent extensions all confined themselves to what might be called the human plane. There are questions asked, however, for which no human is an answer, such as "Who makes the thunder, the earthquakes, the floods and the droughts?" To pose an answer to such questions means transcending the human plane by adding a new dimension, enabling gods of various and diverse sorts to appear. Initially, these gods are more like **adversaries**, who humans could just as well do without, but often a relationship develops and a god becomes a protector and enabler for a particular group of humans, resulting in a *theocentric* system. This has a powerful effect on that group, granting it cohesion, resolve and purpose, which ends up affecting the people's daily habits of working, eating, playing, and worshipping, as well as their hopes and fears, their loves and hates, their courage and fortitude and peace of mind in the midst of life's troubles. The fact that different pantheons of gods are posited by different groups shows that there is more than one

continued on page 50

Book Review: *Strength in Numbers* by Sherman Stein

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Strength in Numbers. Sherman Stein. John Wiley and Sons, Inc., New York: 1996.

This is a charming and wonderful book that benefited from the comments of students and teachers of junior high schools. The author leads us along a path of discovering the joy and power of mathematics in everyday life. The presentation is simple and non-threatening. The mysteries of dividing by fractions, why a negative times a negative is positive, cool numbers, hot numbers, false precision, are discussed and commented on. The mathematics that are needed for various occupations are reviewed.

Some misuses or false precision are examined. In 1962 the citizens of the San Francisco Bay Area were asked to vote for the largest municipal bond issue in history to pay for the Bay Area Rapid Transit system. They were told that by 1975 there would be 258,496 riders daily. That number assured a profit of 13 cents a ride, enough to cover all expenses. It turned out that in 1975 there were only 135,000 riders a day, which meant a loss of $1.31 a ride.

Where did the figure 258,496 come from? That number with its six figure precision reassured and intimidated. How could a number given so precisely not be correct?

Reforms in teaching math are also surveyed, beginning in the early years of the century. "these reforms spring forth even though there is no agreement on the cause of the problem. It is as though a doctor keeps plying patients with a variety of pills without ever figuring out what ails them." Stein reviews the reform efforts of L.P. Benezet in the 1930's and SMSG in the 1960's. He also comments on the NCTM Standards and the California Framework. He is not optimistic.

Parts two and three are about the methods of mathematics, infinite series, fractions, finding a curved area, the ratio of the circumference to the diameter of a circle and other interesting concepts.

The author explains the beauty and mysteries of mathematics in a clear manner that will be welcomed by parents and their children.

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One plus one
two plus two
Those I can do.
Three plus three
four plus four
It's not a bore.
Five plus five
six plus six
Those I can mix.
Seven plus seven
eight plus eight
Math is my fate.
Nine plus nine
ten plus ten
Those I can do in pen.
Addition is easy
I'm not being silly
I can do them freely
I'm not joking really.

Mae Talle

An Informal History of Classical Rhetoric for Mathematicians (Plato and Aristotle)

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Rhetoric, "the art of persuasion," gets a bad rap among workers in the mathematical sciences. This is unfortunate, since although mathematics is concerned primarily with the demonstration of formal truth, mathematicians do live in the world and need to concern themselves with persuading students, business agents and others not on the mathematical team of the importance of their enterprise. The natural revulsion among keepers of the flame of formal truth for the dirty instruments and forms of public persuasion is a handicap (not fatal yet, perhaps, but a handicap nonetheless) in attracting students and obtaining public support. Thus, a brief history of the origins of rhetoric and its relation to the development of logic—mathematical and otherwise—can be useful in providing a view of the relation between those two integral arts of "fixing" belief, to use the term favored by the philosopher and mathematician C.S. Peirce.

Rhetoric had its institutional roots in the political chaos in Greece around 400 B.C. following Sparta's defeat of Athens and the temporary collapse of democracy. Although the models of expression favored by rhetoricians trace back to Homer and the oral tradition of heroic action and expression, the availability of tutoring in rhetoric is tied to the period of the restoration of democracy after the oligarchic tyranny following the defeat of Athens. During the tyranny, arbitrary seizure of property and assassination were common, and, as in recent history in South Africa and Central America, injustices filled lists waiting for redress.

With the restoration of the democracy, courts were established to hear claims and grievances. These courts, in the Athenian pattern, did not have established officers, such as judges, prosecutors, and licensed defense lawyers. They were made up of groups of citizens—as many as fifty—who listened to the grievances of citizens and the defenses of those accused, and delivered judgment on the spot. In this situation, cleverness in speaking would provide a strong advantage for plaintiffs and accused, and as in courts today, the expression of pain and suffering would carry a lot of weight. Not surprisingly, some entrepreneurial individuals offered lessons in speaking to get maximum benefit from these situations. The best known of these individuals are Corax and Tisias, to whom the art of rhetoric is traced by Aristotle.

Imagine the typical case. The "jury" of 50 men sits in an open theater of sorts while the plaintiff explains how his brother was beaten to death and his home seized by the defendant. In his complaint he goes into great detail about the agonizing pain his brother suffered, the misery of his brother's wife and children in the loss of their father, the humiliation of their loss of privilege and income, the jealous glee of the executors of the forfeiture and the illicit pleasure they take in the property to which they have no right—all of this aimed at arousing the active sympathy and outrage of the jury. The defendant parries in the same terms, pointing to injustices committed by the dead brother that earned him his fate, the insult and pain caused by the present claim, the well-known skulduggery of the plaintiff, and how the misery of the defendant's wife, children and father was due to the wretchedness of the home forfeited to the defendant which has since been transformed through considerable effort, expense and good will within the last ten years into a location of public hospitality.

Since such skill in arguing had a substantial value, it became a major part of the educational framework. Teachers of the youth of Athens were expected to provide experience and training to prepare men to defend themselves against complaint, much as it was expected that young men would learn to defend themselves with weapons. It is hardly surprising then that the youthful Plato, in observing such processes, would be appalled that this could be considered a form of Justice, a search for Truth. And so rhetoric as an art became the target of special scorn as an educational

discipline because of its claim to discover truth and further justice. In his dialogues on rhetoric, specifically the *Gorgias* and *Phaedrus*, Plato has Socrates attack common rhetorical practice as training in making the worst appear the better cause. This competed with training in reasoning, which he called dialectic. (Although this is a yawing term, cruising across philosophical trade routes, it can generally be taken to mean discussion logic, or finding questions for exploratory discussion.) Plato wanted to ask questions and shape definitions to clarify and comprehend Truth, Justice, Virtue; in his Academy the mode of dialectic was the foundation of education.

Fortunately for the western world, one of Plato's students and teaching assistants at the Academy was Aristotle, who took a different view. Aristotle was interested not just in the essential truth and falsity of methods of thinking and knowing, he was also interested in describing how things worked, practically. Aristotle's *Rhetoric* seems to have been an outgrowth of his discussions of rhetoric at the Academy. (Plato must have decided that the Academy needed to have instruction in rhetoric to attract students, but he was darned if he was going to do that, so he gave the job to this talented junior staffer.) Aristotle's approach was neither to teach merely the Coraxian tricks, or critique the practice of such tricks, but to ask: what kind of thing is rhetoric? And his explanation in the *Rhetoric* and the larger *Organon*, of which it is essentially a part, can be read as an answer to Plato's concerns, an answer that in a sense follows Plato's method.

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...it is necessary to distinguish between persuasion relative to the community at large, and persuasion within a field of shared assumptions in a particular investigation...

"Plato!" he seems to say, "hold on a minute—you are getting some things mixed up! First of all, you define rhetoric in terms of the example of these court proceedings. That's far too narrow a definition. Rhetoric should be described in terms of its situation and its appeal. Those court proceedings are examples of speech in judicial situations emphasizing an appeal to audience emotion. The jury isn't trained, they have no standards of justice, no code to enforce, so naturally the most effective appeal is to their sympathy and anger. But surely that doesn't define the only type of speech a citizen needs to do." In a wonderfully simple yet profound set of categories, Aristotle establishes an observation that speech situations are either judicial, deliberative, or ceremonial (epedeictic). Judicial speech aims at accusing someone of crimes they have committed in the past; deliberative speech aims at recommending policy for the future; and ceremonial speech aims at praising or blaming a person for their character. These are the only types of speech that operate with the public at large. Within any of these categories (think: rows), one may use one of three different appeals (think: columns) that are grounded in the speaking situation. One may appeal to the feelings and perceptions of the audience (pathos), as in the court speeches that so appalled Plato. But one may also base an appeal on one's own character or reliability (ethos), or on the argument or evidence (logos). In response to Plato's identification of rhetoric with basically a single mode of speech (judicial aim with emotional appeal), Aristotle's definition of rhetoric is a full grid with lists of terms expanding each of the nine cells of the grid. Aiming at comprehensiveness of description, he expands the audience appeal, for instance, from appeals to anger and sympathetic sadness to a full description of the kinds of associations and perspectives different audiences tend to share. One might say that in so doing, he wrote the first descriptive psychology. For example, for old men, happiness takes the form of protection or successful children while for young men it might be challenge or opportunity to achieve honor . So if you are speaking to an audience of senior citizens, be aware of this difference in shaping your material. Young men tend to pay more active attention to personal slight than old men, who feel more secure in their established reputations. This sort of listing-out description easily expands into an outline that fills several hundred pages without very much analysis or detailed development.

But defining rhetoric is only part of Aristotle's response to Plato. Secondly he points out that it is necessary to distinguish between persuasion relative to the community at large, and persuasion within a field of shared assumptions in a particular investigation—i.e., within a discipline. Within a field of exploration,

individuals share common terms and assumptions from which they discover new truths or facts, and within which they give value to certain observations. Since those assumptions, truths, and facts are not universally known, the truth of in-discipline reasoning is not self-evident to society as a whole. Assumptions may be passed on with the authority of the discipline, and thus may not be questioned by the masses, but that is due to the enforceability of authority, not to their reasonability. Aristotle's presentation of logic in the *Organon* and the *Rhetoric* assumes that Plato's model for reasoning is the basis for the development of a discipline, a science. If one assumes, as scientists and mathematicians too frequently do, that the principles and assumptions of scientific discourse are self-evident in the public arena, one tends to lose debates.

In his discussion of Logic in the *Organon* (Topics, Categories, Prior and Posterior Analytics), Aristotle retains this distinction, setting up syllogism as a method of reasoning from demonstrated premises to demonstrable conclusions, and setting up dialectic (the basis of logos in the *Rhetoric*) as discussion from common assumptions and opinions that are simply accepted without needing to be demonstrated. Thus, in doing rhetoric, you argue from your audience's opinions. In science, you argue from demonstrated truths. However, one must note that just as some audience opinions may be wrong (and the ethical character of the speaker may be sacrificed in the long run if audiences perceive him/her to be relying on audience beliefs that he/she knows are wrong), demonstrated premises may in the future be discarded by a scientific community -alchemy, for example. Aristotle points out that rhetorical argument aims to persuade audiences, not to do science. Rhetoric (and dialectic) is about public speaking (and informal discussion), not about understanding the nature of the mind, insects, the weather, or morality.

However, Aristotle does point out that rhetoric and dialectic can play a role in discovering truth. These subjects are useful in education (= propaedeutic), and they can be useful in assessing first principles or premises. The problem with first principles is that they have not been demonstrated to be true. Rhetoric and dialectic cannot demonstrate their truth—nothing can. But rhetoric and dialectic can assist in comparing the meaning and effect of statements of first principles, and there are advantages in being able to do that. It is easy to explain the particular balance between dialectic and syllogism in the medieval age given its prescientific situation and the dominance of religious perspectives in education and social understanding. The classical model for educated discourse provided in Aristotle is a complex weaving of social practice and theoretical understanding that values both. Since Descartes and Bacon, the balance in our mode of discussion has been shifting toward emphasizing and valuing scientific rather than rhetorical reasoning. As a result, educated discourse has become more arcane and alienated from the common discourse. The classical model of the Greeks provides a guide to righting this balance with the assumption that any educated person needs to be able to operate in both public and within-discipline modes. Not being able to do so constitutes a cultural handicap which we must define our educational principles and educational principles and methods to correct.

A Collection of Ideas on Systems and their Extensions

continued from page 46

way to extend the human plane in a new dimension.

This leads, almost invariably it seems, to competing claims as to which extension is the correct one. This is rather hard to avoid when various of these posited gods each reveal to a chosen messenger on earth that it is the one true god and that all others are the invention of man.

REFERENCES

1. Fjelstad, Paul, "Extending special relativity via the perplex numbers," *Amer. J. Phys*. **54** (5), 416-422 (1986).
2. Fjelstad, Paul, "Limitless calculus via Ethereal Numbers," *UMAP J.* **15**(2), 161-174 (1994).

Book Review: *What Is Mathematics, Really?* by Reuben Hersh

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What Is Mathematics, Really? Reuben Hersh. Oxford University Press.

The mad oddball maverick strikes again! This is a great book. What, you may ask, makes a book great? Here are my criteria:

1) I learn a great deal from it.
2) The book gives me new insights that change or reinforce previous viewpoints (prejudices?).
3) I agree with almost everything the author says.
4) It stimulates thinking.
5) It is written with clarity and with conviction.

In my opinion, Reuben satisfies all these in this book. What more could one ask for, except maybe a chocolate donut? Criteria number five is one which is not often met in books about mathematics and in mathematical writing in general. Recently reading two forwards to a book, one written by Reuben Hersh and one by another writer, reminded me of the great range in this quality and its importance in making reading pleasurable. Reuben's was far superior in this respect; it was like night and day. One wonders, sometimes, if mathematicians equate obscurity with scholarliness. Reuben has a pleasant, conversational style which is very refreshing in mathematical writing.

Is this what mathematics is—really? Well it's much closer than what we've seen before. It is a giant step in that direction; it's probably within an epsilon distance, even. I have always felt that "philosophical" concern was much like contemplating our mathematical navel. Also, I have often used Albert Einstein's quotation, "Is not all of philosophy as if written in honey? It looks wonderful when one contemplates it, but when one looks again, it is all gone. Only mush remains." You can imagine my consternation when I found out that this whole book was about the philosophy of mathematics, and I had paid good money for it. Is that what mathematics really is? Come on! Well, it turns out that it is, and it was well worth the money. Reuben builds a convincing case for this. Whether or not we articulate it or even think about it as such, we all have a philosophy of mathematics that guides us in our teaching and mathematical work. He moves the philosophy of mathematics away from the realm of navel contemplation.

An important part of the book is a thorough discussion of the sundry philosophies of mathematics. The history of these philosophies is important in understanding the evolution of mathematics, but categorizing mathematicians, i.e., labeling them as belonging to one of these categories, bothers me. Labeling, or categorizing individuals, is a distancing phenomenon. It tends to be divisive. Actually, I think that is a part of the humanistic philosophy, that we must be cognizant of all the philosophies of mathematics and their contributions to mathematics and our culture in general. Reuben very convincingly argues that all the different philosophies, except the humanistic one, fail to satisfy the criteria of THE PHILOSOPHY OF MATHEMATICS that he puts forth, but he does recognize their importance in the evolution of mathematical thought.

If, many years ago, someone would have asked me what kind of mathematician I was, formalist, Platonist, intuitionist, foundationalist, I would have answered yes. There are certain ideas in each of these that I subscribe to, and I think that is consistent with what Reuben is preaching. Actually, I guess I have always been a humanist as described here. However, I didn't know it and didn't know there was such a thing until I attended an open forum on Mathematics as a Humanistic Discipline at the International Congress of Mathematicians at Berkeley in 1986. Although as a student, I really ate up the courses that were heavy with formalism and foundations; I had great admiration for the writings of Raymond Wilder, Morris Kline, Edward Kasner, James Newman, Edna Kraemer, Lillian Lieber, and the like. So, at heart, I guess I was and am essentially a humanist. Reuben describes humanistic mathematics as social-cultural-historical. Doesn't cultural say it all? Perhaps that is not obvious

to all. The book has lots of advice for philosophers of mathematics, such as "Give up the illusion of mathematical precision, aim for insight, enlightenment." This is very good advice for all who work with mathematics. He also has some good advice for teaching mathematics, such as how to use proof in the classroom, not to convince so much as to help explain and as an aid to understanding. Maybe I'm reading something into this, but I think his discussion seems to support my criticism of most "liberal arts" math courses and textbooks. They seem too concerned with "doing" mathamatics rather than aiming for understanding the thought processes in the evolution of mathematics. There is a great deal of excellent advice in this book for anyone involved in teaching mathematics, which is somewhat surprising since it comes from someone who used to teach the "wrong stuff."

There is a nice discussion on the meaning of numbers which exemplifies the fuzzy thinking that can result from using a simple term in more than one way without clarifying how it is used. For example, the number 2 as used as an adjective, as in 2 bananas and as a noun, as in a number, the consequent of 1 a la Peano; it has a multiplicative inverse; it is an integer, a rational number, a real number, a complex number. Students learn numbers as adjectives, and then we start using them as nouns without discussing this difference. This leads to confusion and fuzzy thinking. All this discussion of nouns and adjectives, you would think that this author is some kind of English major. However, mathematics is primarily a language, and it would behoove all of us to keep that in mind—always. There is much discussion (argumentation) in the philosophy of mathematics that is merely semantics, lack of clarity, fuzzy thinking. There is much ado about whether the things of mathematics are "objects," are "real", whether mathematics is (are?) "invented" or "discovered." Those discussions sound so erudite, but I think that is the sort of thing Einstein was alluding to. Perhaps I am being naïve, but who cares about all those types of discussions? The things of mathematics are ideas. Ideas are real, and what is important is that they are useful and important and have been and will be essential in the development of our culture. There is much discussion of philosophy of mathematics in this book that is just that, the importance of mathematics in the development of the culture.

There is an excellent discussion on the myths in mathematics. You should read this for yourselves, and we all should keep them in mind. We too often fall victim of these myths, and that leads us to less than adequate presentations. The author states,"the standard exposition purges mathematics of the personal, the controversial, the tentative, leaving little trace of humanity in the creator or the consumer." Of course, he has fallen victim to a myth himself in assuming there is a "standard exposition." This, however, does not detract from the importance of what he is saying. Expositions in mathematics far too often are as he has stated. Let's work on that, shall we?

In my opinion, in attempting to discuss what mathematics is in terms of humanistic views, our language fails us, although Reuben does it much more elegantly than anyone else. It seems that couching the discussion in philosophical, technical, scholarly language distances us from what we really believe. The essence of mathematics is its open-mindedness. Mathematics is different things; the key is to be open-minded and tolerant of the views of others.

The author makes a very cogent statement, "Mathematics is a lawful, comprehensible evolution from a basic core. It develops in response to internal strain (here a definition would help) and external pressure." Raymond Wilder referred to these as "hereditary and environmental stress." Perhaps this would suffice as the definition that the author says would help. If not, and even if so, a reading of Wilder's "Evolution of Mathematical Concepts" would be of interest.

Reuben's use of the blind dudes and the elephant as a metaphor for the sundry views of mathematics is nicely done. It seems to me I have seen this before, but Reuben does it much better than anyone.

I think I have earned the right to be a curmudgeon so I am going to voice a wee criticism. I think Reuben has overdone it a bit relating mathematical philosophies with political philosophies. It seems to me like belaboring the obvious, but he made his point, and maybe others are not so hard to convince as I.

Did I say this is a great book? I highly recommend it. I believe it should be required reading along with Wilder's "Evolution of Mathematical Concepts" and Morris Kline's "Mathematics in Western Culture" for any student in a mathematics program.

In Future Issues...

Paul Alper and Yongzhi Zang
Tea Tasting and Pascal Triangles

Paul Fjelstad
Mathematics Is an Art

Bill Marion
Teaching a Humanities Course: A Mathematician's View

Kathleen Shannon
Calculus for the Liberal Arts: A Humanistic Approach

Rina Zazkis
Divisibility: A Problem Solving Approach Through Generalizing and Specializing

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