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A SEMILINEAR WAVE EQUATION WITH SMOOTH DATA AND NO RESONANCE HAVING NO CONTINUOUS SOLUTION

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Abstract. We prove that a boundary value problem for a semilinear wave equation with smooth nonlinearity, smooth forcing, and no resonance cannot have continuous solutions. Our proof shows that this is due to the non-monotonicity of the nonlinearity.

1. **Introduction.**

Here we consider the hyperbolic boundary value problem

$$\begin{cases} \Box(u) + g(u) = p(x,t) = p(x,t+2\pi) = p(x+2\pi,t) & x,t \in \mathbf{R} \\ u(x,t) = u(x,t+2\pi) = u(x+2\pi,t) & x,t \in \mathbf{R}, \end{cases} \tag{1}$$

where $\Box$ denotes the D'Alembert operator $\partial_{tt} - \partial_{xx}$,

$$g(t) = \tau t + h(t) \quad \text{with} \quad \tau \in (0,\infty) - \{k^2 - j^2; k,j = 0,1,\ldots\}, \tag{2}$$

and $h : \mathbf{R} \to \mathbf{R}$ is a differentiable function with support in $[0,D]$ and such that

$$h(D/2) < -\tau D/2. \tag{3}$$

Thus, for some $t \in (0,D)$, $g'(t) < 0$.

The wave operator $\Box$ subject to the boundary conditions in (1) has discrete spectrum. It is given by $\sigma(\Box) = \{k^2 - j^2; k,j = 0,1,\ldots\}$. All the eigenvalues have finite multiplicity except for 0 whose eigenspace is spanned by

$$\{\alpha_{k,k}, \beta_{k,k}, \gamma_{k,k}, \delta_{k,k}, ; k = 0,1,2,\ldots\}, \tag{4}$$

where

$$\begin{aligned} &\alpha_{k,j}(x,t) = \sin(kx)\cos(jt), \beta_{k,j}(x,t) = \sin(kx)\sin(jt), \\ &\gamma_{k,j}(x,t) = \cos(kx)\cos(jt), \quad \text{and} \quad \delta_{k,j}(x,t) = \cos(kx)\sin(jt). \end{aligned} \tag{5}$$

In [2] it was shown that if g is monotone and $\lim_{|t|\to+\infty} g(t)/t = \tau$, the boundary value problem

$$\begin{cases} \Box(u) + g(u) = p(x,t) = p(x,t+2\pi) & (x,t) \in (0,\pi) \times \mathbf{R} \\ u(0,t) = u(\pi,t) = 0 \quad t \in \mathbf{R} \\ u(x,t) = u(x,t+2\pi), \quad (x,t) \in [0,\pi] \times \mathbf{R}, \end{cases} \tag{6}$$

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has a weak solution in $L^2([0,\pi]\times[0,2\pi])$. A related result for systems of equations is found in [1]. Also in [2] it is shown that if, in addition, there exists $\epsilon > 0$ such that $g'(z) \ge \epsilon > 0$ for all $z \in \mathbf{R}$ then such a solution is of class C^∞ when p is of class C^∞. Here we prove that such a result cannot be extended to (1) when g is nonmonotone. In fact we show that *the lack of monotonicity prevents even the existence of continuous solutions regardless of the smoothness of of p.*

Studies of (6) for non-monotone g may be found in [8] and [5] where is it proved that it has a solution for p in a dense set of $L^2([0,\pi]\times[0,2\pi])$. In [4], also for non-monotone g, sufficient conditions for the existence of a solution in the Sobolev space $H^1([0,\pi]\times[0,2\pi])$ are given in terms of the components of p in the kernel and range of the operator $\Box$. Here $H^1([0,\pi]\times[0,2\pi])$ denotes the Sobolev space of square integrable functions in $[0,\pi]\times[0,2\pi]$ having first order partial derivatives in $L^2([0,\pi]\times[0,2\pi])$ and satisfying the boundary condition in (1). Extensions of this result to cases where the period 2π is replaced by a number such that all the eigenvalues have infinite multiplicity were are found in [3]. For additional studies on solvability of equation (6) with multiple eigenvalues of infinite multiplicity the reader is referred to [7]. For a survey on boundary value problems for semilinear wave equations we refer the reader to [6].

2. **Preliminaries and statement of main result.** Throughout this paper $\Omega = (0,2\pi)\times(0,2\pi)$, We denote the norm in $L^p(\Omega)$ by $\|\ \|_p$. We let N denote the closed subspace of $L^2(\Omega)$ spanned by $\{\alpha_{k,k}, \beta_{k,k}, \gamma_{k,k}, \delta_{k,k}; k = 0,1,2,\ldots\}$, see (4). That is, N is the null space of the wave operator $\Box$ subject to the boundary conditions in (1). We let H denote the *Sobolev* space of functions u that are 2π-periodic in both x and t, and such that u as well as its first order partial derivatives belong to $L^2(\Omega)$. The norm in H is denoted by $\|\ \|_{1,2}$. We let Y denote the subspace of H of functions y such that

$$\int_\Omega y(x,t)v(x,t)dxdt = 0 \quad \text{for all} \quad v \in N. \tag{7}$$

We say that $u = y + v \in Y \oplus N$ is a weak solution of (1) if

$$\int_\Omega \{(y_t\hat{y}_t - y_x\hat{y}_x) - (g(u)-p)(\hat{y}+\hat{v})\}\, dxdt = 0, \tag{8}$$

for all $\hat{y} + \hat{v} \in Y \oplus N$. Our main result is:

Theorem 2.1. *There exists $c_0 \ge 0$ such that if $|c| > c_0$, and $p(x,t) = c\sin(x+t)$ then (1) has no continuous weak solution.*

Corollary 2.2. *There exists $c_0 \ge 0$ such that if $|c| > c_0$, and $p(x,t) = c\sin(x+t)$ then (1) has no weak solution in $H^1([0,2\pi]\times[0,2\pi])$.*

The corollary follows immediately from the theorem since every element u in $H^1([0,2\pi]\times[0,2\pi])$ may be written as $u = y+z$ with $y \in Y$ and $z(x,t) = z_1(x+t) + z_2(x-t)$ with $z_1, z_2 \in H^1([0,2\pi])$. Since the elements in $H^1([0,2\pi])$ are continuous function, z is continuous. Hence it cannot be a solution to (1).

3. **Regularity.** Let $u = y + v$ be a weak solution to (1). We write $\alpha(x,t) = \sin(x+t)$, $v = a\alpha + w$, $a \in \mathbf{R}$, and $w = \bar{v} + z$ where

$$\int_\Omega \alpha w dxdt = 0, \quad \text{and} \quad 4\pi^2\bar{v} = \int_\Omega v dxdt = \int_\Omega w dxdt. \tag{9}$$

Since $z \in N$ we may write $z(x,t) = z_1(x+t) + z_2(x-t)$ with z_1, z_2 2π-periodic functions such that

$$\int_\Omega z_1(x+t)dxdt = \int_\Omega z_2(x+t)dxdt = 0. \tag{10}$$

Lemma 3.1. *Under the above assumptions,* $\|z_i\|_\infty \leq 3\|h\|_\infty/\tau$, *and* $|\bar{v}| \leq \|h\|_\infty/\tau$.

Proof. Taking $\hat{y} = 0$ and $\hat{v} = \alpha$ in (8) we have

$$\int_\Omega (\tau a\alpha + h(u))\alpha dxdt = \int_\Omega c\alpha^2 dxdt. \tag{11}$$

This and $\|\alpha\|_2 = \sqrt{2}\pi$ yield

$$|\tau a - c| \leq 2\|h\|_\infty \tag{12}$$

For b positive odd integer, it is easy to see that $\bar{z}_1(x,t) = z_1^b(x+t)$ and $\bar{z}_2(x,t) = z_2^b(x-t)$ are in N. Hence, taking $\hat{v} = \bar{z}_1$ in (8) we have

$$\begin{aligned} \tau\|z_1\|_{b+1}^{b+1} &= -\int_\Omega (h(u(x,t)) + \bar{v}\tau - (c-\tau a)\alpha(x,t))z_1^b(x,t)dxdt \\ &\leq 3\|h\|_\infty |\Omega|^{\frac{1}{b+1}} \left(\int_\Omega |z_1(x,t)|^{b+1} dxdt\right)^{\frac{b}{b+1}}, \end{aligned} \tag{13}$$

which yields

$$\tau\|z_1\|_{b+1} \leq 4\|h\|_\infty |\Omega|^{\frac{1}{b+1}}. \tag{14}$$

Since b may taken arbitrarily large and $\|z_1\|_\infty = \lim_{b\to\infty} \|z_1\|_{b+1}$ we have

$$\tau\|z_1\|_\infty \leq 4\|h\|_\infty. \tag{15}$$

Similarly $\tau\|z_2\|_\infty \leq 4\|h\|_\infty$. Since

$$4\pi^2\tau|\bar{v}| = \tau|\int_\Omega w(x,t)dxdt| = |\int_\Omega h(u(x,t))dxdt| \leq 4\pi^2\|h\|_\infty, \tag{16}$$

the lemma is proven. □

Lemma 3.2. *There exists* $K > 0$, *independent of* c *such that if* $u = y + v \in Y \oplus N$ *is a weak solution to* (1) *then* $|y(x,t)| \leq K\|h\|_\infty$ *for all* $(x,t) \in \Omega$, *and* $\|y\|_{1,2} \leq K$.

Proof. Let

$$\begin{aligned} y &= \sum_{k\neq j} a_{kj}\alpha_{k,j} + b_{kj}\beta_{k,j} + c_{kj}\gamma_{k,j} + d_{kj}\delta_{k,j} \quad \text{and} \\ P_Y(h(y+v)) &= \sum_{k\neq j} A_{kj}\alpha_{k,j} + B_{kj}\beta_{k,j} + C_{kj}\gamma_{k,j} + D_{kj}\delta_{k,j}. \end{aligned} \tag{17}$$

Since $\|P_Y(h(v+y))\|_2 \leq \|h(y+v)\|_2 \leq 2\pi\|h\|_\infty$, $a_{kj} = A_{kj}/(k^2 - j^2 + \tau)$, $b_{kj} = A_{kj}/(k^2 - j^2 + \tau)$, $c_{kj} = C_{kj}/(k^2 - j^2 + \tau)$, and $d_{kj} = D_{kj}/(k^2 - j^2 + \tau)$, by Parseval's identity we have

$$
\begin{aligned}
|y(x,t)| &= \Big|\sum_{k\neq j} a_{kj}\alpha_{k,j}(x,t) + b_{kj}\beta_{k,j}(x,t) + c_{kj}\gamma_{k,j}(x,t) + d_{kj}\delta_{k,j}(x,t)\Big| \\
&\leq \left(\sum_{k\neq j} A_{kj}^2 + B_{kj}^2 + C_{kj}^2 + D_{kj}^2\right)^{1/2} \left(\sum_{k\neq j}\frac{1}{(k^2-j^2+\tau)^2}\right)^{1/2} \\
&\leq 2\pi\|h\|_\infty \left(\sum_{k\neq j}\frac{1}{(k^2-j^2+\tau)^2}\right)^{1/2} \\
&\equiv K_1\|h\|_\infty,
\end{aligned} \tag{18}
$$

where we used that the last series in (18) converges. Similarly

$$
\begin{aligned}
\|y\|_{1,2}^2 &\leq 2\sum_{k\neq j}\frac{(k^2+j^2)(A_{kj}^2+B_{kj}^2+C_{kj}^2+D_{kj}^2)}{(k^2-j^2+\tau)^2} \\
&\leq K_2\|h(u)\|_2^2 \\
&\leq 4\pi^2K_2\|h\|_\infty^2
\end{aligned} \tag{19}
$$

Taking $K = \max\{K_1, 2\pi\sqrt{K_2}\}$ the lemma is proven. □

Let $D > 0$ be as in (3). Now (see (12))

$$
\begin{aligned}
|u(x,t)| &= |a\sin(x+t) + \bar{v} + z(x,t) + y(x,t)| \\
&\geq [(|c| - 2\|h\|_\infty)|\sin(x+t)| - (9 + K_1\tau)\|h\|_\infty]/\tau.
\end{aligned} \tag{20}
$$

Hence

$$
h(u(x,t)) = 0 \text{ if } \ |\sin(x+t)| \geq \frac{\tau D + (9+K_1\tau)\|h\|_\infty}{|c| - 2\|h\|_\infty}. \tag{21}
$$

Therefore there exists a positive constants c_0 and m such that if $|c| \geq c_0$ then

$$
m\{(x,t)\in\Omega; h(u(x,t)) \neq 0\} \leq \frac{m}{c}. \tag{22}
$$

Hence $\|h(u)\|_2 \leq m^{1/2}\|h\|_\infty c^{-1/2}$ for $|c| \geq c_0$. Replacing this in (18) we have

$$
|y(x,t)| \leq K\|h\|_\infty c^{-1/2}, \tag{23}
$$

for $|c| \geq c_0$. Also

$$
\begin{aligned}
\tau|\bar{v}| &= |\int_\Omega h(u(x,t))dxdt| \\
&\leq \|h\|_\infty m\{(x,t)\in\Omega; h(u(x,t))\neq 0\} \\
&\leq \frac{m\|h\|_\infty}{c}.
\end{aligned} \tag{24}
$$

Similarly (see (12))

$$
|\tau a - c| \leq m\|h\|_\infty c^{-1}. \tag{25}
$$

For $0 \leq r \leq s \leq 2\pi$, let $\chi_{[r,s]}$ be the 2π-periodic function such that $\chi_{[r,s]}(t) = 1$ if $t \in [r,s]$, and $\chi_{[r,s]}(t) = 0$ if $t \in [0,2\pi] - [r,s]$. Let $\phi(x,t) = \chi_{[r,s]}(x-t)$,

$\bar{z}_1(x,t) = z_1(x+t)$, and $\bar{z}_2(x,t) = z_2(x-t)$. Using that $\phi \in N$ and the mean value theorem for integrals we have

$$\begin{aligned} 0 &= \int_\Omega \phi((a\tau - c)\alpha + \tau(\bar{z}_1 + \bar{z}_2) + \bar{v} + h(u))dxdt \\ &= 2\pi(s-r)\tau z_2(s_2) + \int_\Omega \phi h(u)dxdt + 2\pi\bar{v}(s-r), \end{aligned} \tag{26}$$

where $s_2 \in (r,s)$. Since $|\int_\Omega \phi h(u)dxdt| \leq \|h\|_\infty (r-s)m/c$, we conclude

$$|z_2(r)| \leq M\|h\|_\infty/c, \tag{27}$$

with M independent of c. Similarly, letting $\psi(x,t) = \chi_{[r,s]}(x+t)$ and multiplying (1) by ψ,

$$\begin{aligned} 0 &= \int_\Omega \psi((a\tau - c)\alpha + \tau(\bar{z}_1 + \bar{z}_2) + \bar{v} + h(u))dxdt \\ &= 2\pi(s-r)((a\tau - c)\alpha(0,s_3) + \tau z_1(s_1)) + \tau\bar{v}2\pi(s-r) \\ &\quad + \int_\Omega \psi(h(u) - h(a\alpha + \bar{z}_1))dxdt + \int_\Omega \psi h(a\alpha + \bar{z}_1)dxdt, \end{aligned} \tag{28}$$

with $s_1, s_3 \in (r,s)$. Letting $s \to r$,

$$\begin{aligned} 0 &= 2\pi((a\tau - c)\alpha(0,r) + \tau z_1(r) + h((a\alpha + \bar{z}_1)(0,r)) + \bar{v}) \\ &\quad + \int_0^{2\pi} (h(y + \bar{v} + \bar{z}_1 + a\alpha + \bar{z}_2) - h(a\alpha + \bar{z}_1))(x, r-x)dx \end{aligned} \tag{29}$$

Hence (see (23), (24), (27))

$$\tau z_1(r) + h(a\alpha(0,r) + z_1(r)) = O(c^{-1/2}) \tag{30}$$

4. **Proof of Theorem 2.1.**

Proof. Without loss of generality we may assume that $c > 0$. Since for c large $a\alpha(0,\pi/2) + z_1(\pi/2) > D$ and $a\alpha(0,3\pi/2) + z_1(3\pi/2) < 0$, there exists t_1, t_2 such that $\pi/2 < t_1 < t_2 < 3\pi/2$, $a\alpha(0,t_1) + z_1(t_1) = D/2$, and $a\alpha(0,t_2) + z_1(t_2) = 0$. From (30)

$$\tau z_1(t_1) = -h(D/2) + O(c^{-1/2}). \tag{31}$$

Thus $a\alpha(0,t_1) = D/2 - z_1(t_1) = D/2 + (h(D/2)/\tau) + O(c^{-1/2}) < 0$. On the other hand, by (30), $\tau z_1(t_2) = -h(0) + O(c^{-1/2})$ which implies that $a\alpha(0,t_2) = -z_1(t_2) = O(c^{-1/2}) > O(c^{-1/2}) + (D/2 + h(D/2)/\tau)/2 > a\alpha(0,t_1)$, which contradicts that $t \to \alpha(0,t)$ defines a decreasing function on $[\pi/2, 3\pi/2]$.

□

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