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Alfonso Castro
Harvey Mudd College

Benjamin Preskill '09
Harvey Mudd College

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EXISTENCE OF SOLUTIONS FOR A SEMILINEAR WAVE EQUATION WITH NON-MONOTONE NONLINEARITY

Alfonso Castro and Benjamin Preskill

Department of Mathematics
Harvey Mudd College
Claremont, CA 91711, USA

Abstract. For double-periodic and Dirichlet-periodic boundary conditions, we prove the existence of solutions to a forced semilinear wave equation with asymptotically linear nonlinearity, no resonance, and non-monotone nonlinearity when the forcing term is not *flat* on characteristics. The solutions are in L^∞ when the forcing term is in L^∞ and continous when the forcing term is continuous. This is in contrast with the results in [4], where the non-enxistence of continuous solutions is established even when forcing term is of class C^∞ but is flat on a characteristic.

1. **Introduction and main result.** Motivated by the results in [2, 11, 8, 7, 5], we consider the existence of weak solutions, i.e. solutions in the sense of distributions, to the problem

$$\Box(u) \equiv u_{tt} - u_{xx} = p(x,t) - g(u) \quad x, t \in \mathbf{R}, \tag{1}$$

subject to either the double periodic condition

$$u(x,t) = u(x,t+2\pi) = u(x+2\pi,t) \qquad x,t \in \mathbf{R}, \tag{2}$$

or the Dirichlet periodic condition,

$$u(0,t) = u(\pi,t) = 0, \quad u(x,t) = u(x,t+2\pi) \qquad x,t \in \mathbf{R}. \tag{3}$$

For solutions to (1),(3) we assume u to be defined only on $[0,\pi] \times R$. We assume g to be differentiable and asymptotically linear but **need not be monotone.** More precisely we assume that

$$g(t) = \tau t + h(t) \quad \text{with} \quad \tau \in (0,\infty), \tag{4}$$

and that for some $\beta < 0$ and $A \in \mathbf{R}$

$$|h'(u)| \le |u|^\beta \quad \text{for} \quad |u| \ge A. \tag{5}$$

The spectrum of $\Box$ (D'Alembert's operator) subject to (2), respectively (3), is given by

$$\begin{aligned} \sigma_p(\Box) &= \{k^2 - j^2; k, j = 0, 1, \ldots\}, \text{ respectively} \\ \sigma_d(\Box) &= \{k^2 - j^2; k = 1, 2, \ldots,\ j = 0, 1, \ldots\}. \end{aligned} \tag{6}$$

In both cases all eigenvalues have finite multiplicity except for 0 which has infinite multiplicity.

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Imitating the arguments in [2] one sees that if g is monotone and $p \in L^2(\Omega)$, $\Omega \equiv (0, 2\pi) \times (0, 2\pi)$, the equation (1),(2) has a solution. Morover, if g and p are of class C^∞ and $|g'(u)| > \epsilon$ for some $\epsilon > 0$ and all $u \in \mathbf{R}$, then such a solution is of class C^∞. The results of [5] show that this is not the case when g is not monotone. In fact, in [5] it is proven that if g is not monotone the problem (1),(2) may not have continuous solutions regardless of the smoothness of g and p. Also, following the arguments in [11] and [8] one sees that for almost every $p \in L^2(\Omega)$ the equation (1),(2) has a *weak solution* but they provide no mechanism for determining the values of p for which (1),(2) has a solution.

In order to state our main result we introduce the concept of *flatness on characteristics*.

Definition 1.1. Let $a \in \{\pi, 2\pi\}$. We say that $\phi : [0, a] \times \mathbf{R} \to \mathbf{R}$ is not flat on characteristics if

$$m\{x \in [0, a]; \phi(x, r \pm x) = 0\} = 0 \quad \text{for all} \quad r \in \mathbf{R}. \tag{7}$$

Our main result is the following.

Theorem 1.2. *Let $\tau \notin \sigma_p(\Box)$, $p(x, t) = cq(x, t) \in L^\infty$, and φ the solution to*

$$\begin{cases} \varphi_{tt} - \varphi_{xx} + \tau\varphi = q(x, t) \\ \qquad\qquad \varphi(x, t) = \varphi(x + 2\pi, t) = \varphi(x, t + 2\pi) \end{cases} \tag{8}$$

If φ is not flat on characteristics then for $|c|$ sufficiently large the equation (1),(2) has a weak solution in L^∞ (see (12)). If q is continuous such a solution is continuous.

In section 5 we extend Theorem 1.2 to the boundary value problem (1),(3), generalizing the results of [7] (see Theorem 5.1.)

Central to the proof of Theorem 1.2 is the estimation of the projection into the kernel of $\Box$ subject to (2) of solutions to (1),(2). We achieve this by using (29), (30), (31), relations first derived in [5]. Theorem 1.2 is sharp in that if, for example, $q(x, t) = \sin(x+t)$ then (1),(2) cannot have continuous solutions (see [5]). Examples of functions satisfying (7) are plentiful; for instance $q(x, t) = \sin(x + t) + \sin(t - x)$ satisfies (7).

For studies on (1),(3) with g superlinear and monotone we refer the reader to [10]. For For other recent results on wave equations with non-monotone nonlinearities the reader is referred to [1]. Extensions of the results in [1] using techniques introduced in [5] are found in [6]. For a survey on semilinear wave equations see [9]

2. **Preliminaries and notations.** Let

$$\begin{aligned} \alpha_{k,j}(x, t) &= \sin(kx)\cos(jt), \quad \beta_{k,j}(x, t) = \sin(kx)\sin(jt), \\ \gamma_{k,j}(x, t) &= \cos(kx)\cos(jt), \quad \text{and} \quad \delta_{k,j}(x, t) = \cos(kx)\sin(jt). \end{aligned} \tag{9}$$

Let N denote the closed subspace of $L^2(\Omega)$ spanned by

$$\{\alpha_{k,k},\ \beta_{k,k}, \gamma_{k,k}, \delta_{k,k}; k = 0, 1, 2, \ldots\}. \tag{10}$$

That is, N is the null space of the wave operator $\Box$ subject to the boundary condition (2). We let H denote the *Sobolev* space of functions u that are 2π-periodic in both x and t, and such that u as well as its first order partial derivatives belong to $L^2(\Omega)$.

The norm in H is denoted by $\| \|_{1,2}$ and the norm in $L^2(\Omega)$ by $\| \|_2$. We let Y denote the subspace of H of functions y such that

$$\int_\Omega y(x,t)v(x,t)dxdt = 0 \text{ for all } \quad v \in N. \tag{11}$$

We say that $u = y + v \in Y \oplus N$ is a weak solution of (1),(2) if

$$\int_\Omega \{(y_t\hat{y}_t - y_x\hat{y}_x) - (g(u) - p)(\hat{y} + \hat{v})\}\, dxdt = 0, \tag{12}$$

for all $\hat{y}+\hat{v} \in Y \oplus N$. If g is linear, i.e. $h = 0$, then for every $p \in L^2(\Omega)$ the equation (1),(2) has a unique weak solution $y+v$, which we denote as $(\Box+\tau I)^{-1}(p)$. Moreover, there exist a real number k_1 such that if $p \in \{v \in L^2(\Omega); \int_\Omega vw = 0 \text{ for all } w \in N\}$ then

$$\|(\Box + \tau I)^{-1}(p)\|_{1,2} + \|(\Box + \tau I)^{-1}(p)\|_{C^{1/2}} \le k_1\|p\|_2, \tag{13}$$

where $C^{1/2}$ stands for the space of Hölder continuous functions with exponent $1/2$.

3. Analysis in the Kernel.

Let $u = v + y \in N \oplus Y$ be a weak solutions to (1),(2). In turn, $v(x,t) = \bar{v} + v_1(x+t) + v_2(t-x)$ with $\bar{v} \in \mathbf{R}$, v_1, v_2 2π-periodic functions with

$$\int_0^{2\pi} v_1(s)ds = \int_0^{2\pi} v_2(s)ds = 0. \tag{14}$$

Taking $\hat{y} = 0$ and $\hat{v} = 1$ we have (see (12))

$$\bar{v} = \frac{1}{4\pi^2\tau}\int_\Omega (p(x,t) - h(u(x,t)))dxdt \tag{15}$$

For $0 \le r \le s \le 2\pi$, let $\chi_{[r,s]}$ be the 2π-periodic function such that

$$\chi_{[r,s]}(t) = 1 \text{ if } t \in [r,s], \text{ and } \chi_{[r,s]}(t) = 0 \text{ if } t \in [0,2\pi] - [r,s]. \tag{16}$$

Let $\phi(x,t) = \chi_{[r,s]}(t-x)$, and

$$A_{r,s} \equiv A = \{(x,t) \in \Omega; t - x \in [r-2\pi, s-2\pi] \cup [r,s]\}. \tag{17}$$

Using that $\phi \in N$ and the mean value theorem for integrals we have

$$\begin{aligned}\int_A p(x,t)dxdt &= \int_\Omega \phi(x,t)p(x,t)dxdt \\ &= \int_\Omega \phi(x,t)(\tau v(x,t) + h(u(x,t)))dxdt \\ &= \int_A (\tau(\bar{v} + v_1(x+t) + v_2(t-x)) + h(u(x,t))dxdt \\ &= 2\pi\tau(s-r)(\bar{v} + v_2(s_2)) + \int_A h(u)dxdt,\end{aligned} \tag{18}$$

where $s_2 \in (r,s)$. By the Lebesgue differentiation theorem, dividing by $s-r$ in (18) and taking limit as $s \to r$ we have

$$\int_0^{2\pi} p(x,x+r)dx = 2\pi\tau(\bar{v} + v_2(r)) + \int_0^{2\pi} h(u(x,x+r))dx, \tag{19}$$

for almost every $r \in [0,2\pi]$. Letting $\psi(x,t) = \chi_{[r,s]}(x+t)$ and arguing as in (14)-(19), with A replaced by $\{(x,t) \in \Omega; t + x \in [r+2\pi, s+2\pi] \cup [r,s]\}$, we have

$$\int_0^{2\pi} p(x,r-x)dx = 2\pi\tau(\bar{v} + v_1(r)) + \int_0^{2\pi} h(u(x,r-x))dx, \tag{20}$$

for almost every $r \in [0, 2\pi]$.

Lemma 3.1. *If $\bar{v}, v_1, v_2$ satisfy (15), (19)-(20) then*

$$\tau v + Q(h(y+v)) = Q(p), \tag{21}$$

where Q denotes the orthogonal projection of $L^2(\Omega)$ onto N.

Proof. First we note that if $F(x,t)$ is 2π-periodic in t and integrable in Ω then

$$\begin{aligned} \int_\Omega F(x,t)dxdt &= \int_0^{2\pi} \int_x^{x+2\pi} F(x,t)dtdx \\ &= \int_0^{2\pi} \int_{-x}^{2\pi - x} F(x,t)dtdx. \end{aligned} \tag{22}$$

Let ϕ be as in (18). This, (22), and the fact that $\phi \in N$ imply

$$\begin{aligned} \int_\Omega \phi Q(p)dxdt &= \int_\Omega \phi p dxdt \\ &= \int_0^{2\pi} \int_0^{2\pi} \chi_{[r,s]}(t-x)p(x,t)dtdx \\ &= \int_0^{2\pi} \int_x^{x+2\pi} \chi_{[r,s]}(t-x)p(x,t)dtdx \\ &= \int_0^{2\pi} \int_0^{2\pi} \chi_{[r,s]}(\zeta)p(x,x+\zeta)d\zeta dx \\ &= \int_r^s \int_0^{2\pi} p(x,x+\zeta)dxd\zeta, \end{aligned} \tag{23}$$

where $\zeta = t - x$. This and (19) give

$$\begin{aligned} \int_\Omega \phi Q(p)dxdt &= \int_r^s \left[2\pi\tau(\bar{v} + v_2(\zeta)) + \int_0^{2\pi} h(u(x,x+\zeta))dx \right] d\zeta \\ &= \int_r^s \int_0^{2\pi} \left[\tau(\bar{v} + v_2(\zeta)) + h(u(x,x+\zeta)) \right] dxd\zeta \\ &= \int_0^{2\pi} \chi_{[r,s]}(\zeta) \int_0^{2\pi} \left[\tau(\bar{v} + v_2(\zeta)) + h(u(x,x+\zeta)) \right] dxd\zeta \\ &= \int_\Omega \chi_{[r,s]}(\zeta) \left[\tau(\bar{v} + v_1(2x+\zeta) + v_2(\zeta)) + h(u(x,x+\zeta)) \right] dxd\zeta \\ &= \int_0^{2\pi} \int_x^{x+2\pi} \chi_{[r,s]}(t-x)[\tau v(x,t) + h(u(x,t))]dxdt \\ &= \int_\Omega \phi \left[\tau v + Q(h(y+v)) \right] dxdt, \end{aligned}$$

Similarly, taking $\psi(x,t) = \chi_{[r,s]}(x+t)$, we see that

$$\int_\Omega \psi Q(p)dxdt = \int_\Omega \psi \left[\tau v + Q(h(y+v)) \right] dxdt. \tag{24}$$

Hence, if η is a linear combination of functions of the type $\chi_{[r,s]}(x+t)$ and $\chi_{[r,s]}(t-x)$,

$$\int_\Omega \eta Q(p)dxdt = \int_\Omega \eta \left[\tau v + Q(h(y+v)) \right] dxdt. \tag{25}$$

Since linear combinations of functions of the type $\chi_{[r,s]}(x+t)$ and $\chi_{[r,s]}(t-x)$, are dense in N, (25) hold for all η in N, which proves the lemma. □

For future reference we note that if $k \neq j$ and $f \in \{\alpha_{k,j}, \beta_{k,j}, \gamma_{k,j}, \delta_{k,j}\}$ then (see (9))

$$\int_0^{2\pi} f(x, r \pm x)dx = 0, \tag{26}$$

hence

$$\int_0^{2\pi} p(x, r \pm x)dx = \int_0^{2\pi} Q(p)(x, r \pm x)dx. \tag{27}$$

Now we write $v(x,t) = cQ(\varphi)(x,t) + z(x,t)$. Since $Q(\varphi) \in N$, we may write $Q(\varphi)(x,t) = \bar{\varphi} + \varphi_1(x+t) + \varphi_2(t-x)$, where φ_1, φ_2 are 2π-periodic functions of average equal to 0. Hence $z(x,t) = \bar{z} + z_1(x+t) + z_2(t-x)$, with $\bar{z} = \bar{v} - c\bar{\varphi}$, $z_1 = v_1 - c\varphi_1$, and $z_2 = v_2 - c\varphi_2$. Replacing this in (15), (19)-(20) we have

$$\begin{aligned} c\int_0^{2\pi} q(x, r-x)dx &= 2\pi\tau(\bar{z} + c\bar{\varphi} + z_1(r) + c\varphi_1(r)) \\ &\quad + \int_0^{2\pi} h((c\varphi + z + \zeta)(x, r-x)))dx, \end{aligned} \tag{28}$$

where $\zeta = y - \varphi + Q(\varphi)$. By the definition of φ, $\int_0^{2\pi} q(x, r-x)dx = \tau(\bar{\varphi} + \varphi_1(r))$. Thus have

$$z_1(r) = -\bar{z} - \frac{1}{2\pi\tau}\int_0^{2\pi} h((c\varphi + z + \zeta)(x, r-x)))dx, \tag{29}$$

where $P(\varphi) = \varphi - Q(\varphi)$. Similarly

$$z_2(r) = -\bar{z} - \frac{1}{2\pi\tau}\int_0^{2\pi} h((c\varphi + z + \zeta)(x, r+x)))dx. \tag{30}$$

Also

$$\bar{z} = -\frac{1}{4\pi^2\tau}\int_\Omega h((c\varphi + z + \zeta)(x,t))dxdt. \tag{31}$$

In what follows we will make extensive use of the following version of the contraction mapping principle *with parameters* (see [3]).

Theorem 3.2. *Let (X_1, d) be a complete metric space and (X_2, δ) a metric space. If $f; X_1 \times X_2 \to X_1$ is continuous and there exists $\gamma \in [0,1)$ such that*

$$d(f(x_1,y), f(x_2,y)) \le \gamma d(x_1,x_2) \text{ for all } x_1, x_2 \in X_1, y \in X_2, \tag{32}$$

then there exists a continuous function $\phi : Y \to X$ such that $f(\phi(y), y) = \phi(y)$. Moreover, If $f(x,y) = x$ then $x = \phi(y)$.

Lemma 3.3. *If φ is not flat on characteristics (see (7)), then there exists $c_0 > 0$ such that for $|c| \ge c_0$ and $\zeta \in Y$ with $\|\zeta\|_\infty \le c^\gamma$ the system (29), (30), (31) has a unique solution $z(\zeta)$. Moreover, the transformation*

$$\zeta \to z(\zeta)$$

is continuous with respect to the L^∞ norm and $\|z(\zeta)\|_\infty \le c^\gamma$.

Proof. By (5), there exists $A_1 \in \mathbf{R}$ such that

$$|h(s)| \leq A_1 + |s|^{\beta+1} \quad \text{for all} \quad s \in \mathbf{R}. \tag{33}$$

Let $\gamma \in (\max\{0, \beta+1\}, 1)$. Let X_1 be the metric spaces of function z of the form $z(x,t) = \bar{z} + z_1(x+t) + z_2(t-x)$ with $|\bar{z}| \leq c^\gamma/16$, and z_1, z_2 periodic measurable functions with $\|z_i\|_\infty \leq c^\gamma$, $i = 1, 2$, and metric given by

$$\begin{aligned} d(z,w) &\equiv d((\bar{z}, z_1, z_2), (\bar{w}, w_1, w_2)) \\ &= |\bar{z} - \bar{w}| + \|z_1 - w_1\|_\infty + \|z_2 - w_2\|_\infty. \end{aligned} \tag{34}$$

Let $X_2 = \{\zeta \in Y; \|\zeta\|_\infty \leq c^\gamma\}$ with $\delta(\zeta_1, \zeta_2) = \|\zeta_1 - \zeta_2\|_\infty$. We define $N_1(z,\zeta)(r)$ as the right hand side of (31), $N_2(z,y)(r)$ as the right hand side of the equation in (29), and $N_3(z,\zeta)(r)$ as the right hand side of the equation in (30). Also we denote

$$f(z,\zeta)(r) = (N_1(z,\zeta)(r), N_2(z,\zeta)(r), N_3(z,\zeta)(r)). \tag{35}$$

Let us see that, for $|c|$ sufficiently large, f satisfies the hypotheses of Theorem 3.2. The continuity of f follows from continuity of h. By (33), for $z \in X_1$ and $\zeta \in X_2$, for any $r \in [0, 2\pi]$, we have

$$\begin{aligned} |N_2(z,\zeta)(r)| &\leq \frac{|c|^\gamma}{16} + 2\pi A_1 + |c|^{\beta+1} \int_0^{2\pi} (|\varphi| + 2|c|^{\gamma-1})^{\beta+1} dx \\ &\leq |c|^\gamma, \end{aligned} \tag{36}$$

for $|c|$ large. Similarly, for $|c|$ large,

$$|N_3(z,\zeta)(r)| \leq |c|^\gamma \quad \text{for all} \quad r \in [0, 2\pi]. \tag{37}$$

Also

$$\begin{aligned} |N_1(z,\zeta)| &\leq \frac{A_1}{\tau} + |c|^{\beta+1} \int_\Omega (|\varphi| + 2|c|^{\gamma-1})^{\beta+1} dx \\ &\leq |c|^\gamma, \end{aligned} \tag{38}$$

for $|c|$ large. Thus f transforms $X_1 \times X_2$ into X_1.

Let us see that f is a contraction. Let $z, w \in X_1$ and $\zeta \in X_2$. For $r \in [0, 2\pi]$, let $D_r \equiv D = \{x \in [0, 2\pi]; |\varphi(x, r-x)| \geq 4|c|^{\gamma-1}\}$. By (7), there exists c_0 such that if $|c| > c_0$ then $m([0, 2\pi] - D) < 1/(10|h'|_\infty + 1)$. Also, for $x \in D$ there exists σ with $|\sigma| \geq |c|^\gamma$ such that

$$\begin{aligned} I_r(x) &\equiv |h((c\varphi + z + \zeta)(x, r-x)) - h((c\varphi + w + \zeta))(x, r-x)| \\ &= |h'(\sigma)||z(x, r-x) - w(x, r-x)| \\ &\leq 4^\beta c^{\beta\gamma} d(z,w). \end{aligned} \tag{39}$$

Hence

$$\begin{aligned} |N_2(z,\zeta)(r) - N_2(w,\zeta)(r)| &\leq \int_0^{2\pi} I_r(x) dx \\ &= \int_D I_r(x) dx + \int_{[0,2\pi]-D} I_r(x) dx \\ &= (|h'(\sigma)|(m([0,2\pi] - D) + 2\pi 4^\beta c^{\beta\gamma})\|z - w\|_\infty \\ &\leq ((1/10) + 2\pi 4^\beta c^{\beta\gamma}) d(z,w). \end{aligned} \tag{40}$$

Similarly

$$|N_3(z,\zeta)(r) - N_2(w,\zeta)(r)| \leq ((1/10) + 2\pi 4^\beta c^{\beta\gamma})\|z - w\|_\infty. \tag{41}$$

From (31) and (22) (see also (42))

$$\begin{aligned} |N_1(z,\zeta)(r) - N_1(w,\zeta)(r)| &\le \int_0^{2\pi}\int_0^{2\pi} I_r(x)dxdr \\ &\le 2\pi((1/10) + 2\pi 4^\beta c^{\beta\gamma})d(z,w). \end{aligned} \tag{42}$$

Now from (40), (41), (42) we see that there exists c_0 such that if $|c| \ge c_0$ then f is a contraction. Thus f satisfies the hypotheses of Theorem 3.2, and hence the lemma is proven. □

4. **Proof of Theorem 1.2.** Let $|c| \ge c_0$ (see Lemma 3.3) and Y_1 the orthogonal complement of N in $L^2(\Omega)$. Since $\tau \notin \sigma_p(\Box)$, the operator $(\Box + \tau I)^{-1}$ defines a continuous linear map between Y_1 and $H^{1,2}(\Omega)$. Moreover, there exists a real number K such that $\|(\Box+\tau I)^{-1}(\alpha)\|_\infty \le K\|\alpha\|_{L^2(\Omega)}$ for all $\alpha \in Y_1$. Let $X = \{\alpha \in Y_1; \|\alpha\|_{1,2} \le c^\gamma\}$. Endowing X with the L^∞ norm, we see that it is a closed convex subset of the Sobolev space H. Using the notation of Lemma 3.3 we see that

$$\zeta \to (\Box + \tau I)^{-1}((Q - I)(h(c\varphi + \zeta + z(\zeta))) \tag{43}$$

defines a compact transformation of X into itself. Thus by the Banach fixed point theorem there exists $\zeta \in X$ such that $\zeta = (\Box + \tau I)^{-1}(-Q(h(c\varphi + \zeta + z(\zeta))$. This and the definition of φ give

$$c(I - Q)\varphi + \zeta = (\Box + \tau I)^{-1}((Q - I)(h(c\varphi + \zeta + z(\zeta)) - cq)). \tag{44}$$

By Lemma 3.1,

$$\tau(Q(c\varphi) + z) + Q(h(c\varphi + z(\zeta) + \zeta)) = Q(cq). \tag{45}$$

Taking $y = c(I - Q)\varphi + \zeta$ and $v = Q(c\varphi) + z$ we see that $u = v + y$ is a solution to (1)-(2). If q is continuous we replace in the definition of X_1 measurability by continuity, keeping d as in (34). Now $u = c\varphi + z + \zeta$ is a continuous solution to (1),(2), which proves the theorem.

5. **The Dirichlet periodic case.** Now we turn our attention to the case (1),(3). We let $W = (0,\pi)\times(0,2\pi)$ and $\mathcal{N}$ the subspace of $L^2(W)$ spanned by the functions $\alpha_{k,k}, \beta_{k,k}$; $k = 1,2,\ldots$. The elements of $\mathcal{N}$ may be characterized as functions of the form $v(x,t) = v_1(x+t) - v_1(t-x)$ where v_1 is a 2π-periodic function in $L^2(0,2\pi)$. Without loss of generality one may assume that

$$\int_0^{2\pi} v_1(x)dx = 0. \tag{46}$$

We let $\mathcal{Y}$ denote the subspace of $H^1(W)$ spanned by $\alpha_{k,j}, \beta_{k,j}$; $k = 1,2,\ldots$, $j = 0,1,\ldots$, $k \ne j$. We say that $u = v + y \in \mathcal{N} \oplus \mathcal{Y}$ is a weak solution of (1),(3) if

$$\int_\Omega \{(y_t\hat{y}_t - y_x\hat{y}_x) - (g(u) - p)(\hat{y} + \hat{v})\}\, dxdt = 0, \tag{47}$$

for all $\hat{y} + \hat{v} \in \mathcal{N} \oplus \mathcal{Y}$. In this section we extend Theorem 1.2 as follows.

Theorem 5.1. *Let $\tau \in \mathbf{R} - \{k^2 - j^2; k, = 1,2\ldots j = 0,1,\ldots\}$. Let $p(x,t) = cq(x,t) \in L^\infty$ and φ the solution to*

$$\begin{cases} \varphi_{tt} - \varphi_{xx} + \tau\varphi = q(x,t) & x \in [0,\pi],\quad t \in \mathbf{R} \\ \varphi(x,t) = \varphi(x,t+2\pi) & x \in [0,\pi],\quad t \in \mathbf{R} \\ \varphi(0,t) = \varphi(\pi,t) = 0 & t \in \mathbf{R}. \end{cases} \tag{48}$$

If φ is not flat on characteristics (see (7)) then for $|c|$ sufficiently large the equation (1),(3) has a weak solution (see (47)). If q is continuous such a solution is continuous.

Let $u = v + y \in \mathcal{N} \oplus Y$ be a weak solutions to (1),(3). For $0 < r < s < 2\pi$ let $\phi(x,t) = \chi_{[r,s]}(x+t) - \chi_{[r,s]}(t-x)$ (see (16)). Hence $\phi \in \mathcal{N}$.

Let

$$
\begin{aligned}
B_{r,s,\pm} \equiv B_\pm = \{&(x,t) \in W; x \in [0,\pi], \\
&t \in \bigcup_{k=-1,0,1} [r + 2k\pi \pm x, s + 2k\pi \pm x]\}, \\
B_{r,s} \equiv &B = B_- \cup B_+.
\end{aligned}
\tag{49}
$$

By the Lebesgue differentiation theorem, if $f \in L^\infty(W)$ and f is 2π-periodic in the second variable then, for almost every $r \in [0,2\pi]$,

$$
\begin{aligned}
&\lim_{s\to r} \frac{1}{s-r} \int_{B_+} f(x,t)dxdt = \int_0^\pi f(x, r+x)dx \quad \text{and} \\
&\lim_{s\to r} \frac{1}{s-r} \int_{B_-} f(x,t)dxdt = \int_0^\pi f(x, r-x)dx.
\end{aligned}
\tag{50}
$$

Thus, by (46)

$$
\begin{aligned}
\int_{B_+} v_1(x+t)dxdt &= \int_0^\pi \int_r^s v_1(2x+z)dzdx \\
&= \int_r^s \int_0^\pi v_1(2x+z)dzdx \\
&= 0.
\end{aligned}
\tag{51}
$$

Similarly,

$$
\int_{B_-} v_1(x-t)dxdt = 0. \tag{52}
$$

Hence

$$
\begin{aligned}
\int_{B_-} p(x,t)dxdt - \int_{B_+} p(x,t)dxdt &= \int_\Omega \phi(x,t)p(x,t)dxdt \\
&= \int_\Omega \phi(x,t)(\tau v(x,t) + h(u(x,t)))dxdt \\
&= \int_B \phi(x,t)(\tau v(x,t) + h(u(x,t)))dxdt \\
&= \tau\left(\int_{B_-} v_1(x+t)dxdt + \int_{B_+} v_1(t-x)dxdt\right) \\
&\quad + \int_B \phi(x,t)h(u(x,t)))dxdt \\
&= \pi\tau(s-r)(v_1(s_1) + v_1(s_2)) \\
&\quad + \int_{B_-} h(u(x,t))dxdt - \int_{B_+} h(u(x,t))dxdt,
\end{aligned}
\tag{53}
$$

where $s_1, s_2 \in (r, s)$. Dividing by $s - r$ and taking limit as $s \to r$ we have (see (50))

$$\begin{aligned}\int_0^\pi p(x, r-x)dx - \int_0^\pi p(x, r+x)dx &= 2\pi\tau v_1(r) \\ + \int_0^\pi h(u(x, r-x))dx &- \int_0^\pi h(u(x, r+x))dx.\end{aligned} \tag{54}$$

Following the steps in the proof of Lemma 3.1 one has the following.

Lemma 5.2. *If v_1 satisfies (54) then*

$$\tau v + Q(h(y+v)) = Q(p), \tag{55}$$

where Q denotes the projection of $L^2(W)$ onto $\mathcal{N}$.

Since $Q(\varphi)(x,t) = \varphi_1(x+t) - \varphi_1(t-x)$ for some $\varphi_1 \in L^2(0, 2\pi)$, taking $h = 0$ in (54) yields

$$\int_0^\pi p(x, r-x)dx - \int_0^\pi p(x, r+x)dx = 2\pi\tau c\varphi_1(r). \tag{56}$$

Taking $z = Q(u - c\varphi)$, $\zeta = u - c\varphi - Q(u - c\varphi)$, and $z(x,t) = z_1(x+t) - z_1(t-x)$ we then have

$$\begin{aligned}2\pi\tau z_1(r) = &\int_0^\pi h((c\varphi + \zeta)(x, r+x) + z_1(2x+r) - z_1(r))dx \\ &- \int_0^\pi h((c\varphi + \zeta)(x, r-x) + z_1(r) - z_1(r-2x))dx\end{aligned} \tag{57}$$

Arguing as in Lemma 3.3 we also have the following.

Lemma 5.3. *If φ is not flat on characteristics (see (7)), then there exists $c_0 > 0$ such that for $|c| \geq c_0$ and $\zeta \in \mathcal{Y}$ with $\|\zeta\|_\infty \leq c^\gamma$ the equation (57) has a unique solution $z(\zeta)$. Moreover, the transformation*

$$\zeta \to z(\zeta)$$

is continuous.

Since

$$u = v + y = c\varphi + z + \zeta, \tag{58}$$

by Lemmas 5.2 and 5.3, in order to prove Theorem 5.1 it sufficient to see that there exists $\zeta \in \mathcal{Y}$ such that $\zeta = (\Box + \tau I)^{-1}(I - Q)(h(c\varphi + \zeta + z(\zeta))$. Such a ζ exists because $(\Box + \tau I)^{-1}(I - Q)$ maps compactly X_2 into itself, which proves Theorem 5.1.

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E-mail address: castro@math.hmc.edu
E-mail address: preskill@math.berkeley.edu