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Alfonso Castro
Harvey Mudd College

Jorge Cossio
Universidad Nacional de Colombia

Carlos Vélez
Universidad Nacional de Colombia

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EXISTENCE AND QUALITATIVE PROPERTIES OF SOLUTIONS FOR NONLINEAR DIRICHLET PROBLEMS

Alfonso Castro

Department of Mathematics
Harvey Mudd College, Claremont, CA 91711, USA

Jorge Cossio and Carlos Vélez

Escuela de Matemáticas
Universidad Nacional de Colombia, Sede Medellín
Apartado Aéreo 3840, Medellín, Colombia

Dedicated to Professor Jean Mawhin with admiration for this leadership role in the Nonlinear Functional Analysis community

Abstract. In this paper we study the existence of sign-changing solutions to semilinear elliptic problems in connection with their Morse indices. To this end, we first establish *a priori* bounds for one-sign solutions. Secondly, using abstract saddle point principles we find large augmented Morse index solutions. In this part, extensive use is made of critical groups, Morse index arguments, Lyapunov-Schmidt reduction, and Leray-Schauder degree. Finally, we provide conditions under which these solutions necessarily change sign and we comment about further qualitative properties.

1. **Introduction.** We consider the nonlinear Dirichlet problem

$$\begin{cases} \Delta u + f(u) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ is a smooth bounded domain, and $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1. We will also denote by $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ the sequence of eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω, and by $\{\varphi_k\}_{k\in\mathbb{N}}$ we denote a corresponding sequence of orthonormal eigenfunctions which is complete in the Sobolev space $H_0^1(\Omega)$. All our results hold in the case $N = 2$. However for the sake of brevity, we write all the statements for $N \geq 3$, where the formulae have a uniform notation (see, e. g. Theorem 2.1).

Existence of solutions for this kind of problems and some of their qualitative properties have been studied by many authors (see [2], [3], [7], [4], [5], [6], [8], [9], [10], [16], [19], [20], [21], [27], [29], [28], [30]). In this paper we mainly consider problem (1) when f satisfies either of the asymptotic linearity conditions

(f1) $f'(\infty) := \lim_{|t|\to\infty} f'(t) \in (\lambda_k, \lambda_{k+1})$ for some $k \geq 2$ (non-resonance), or
(f1') $f'(\infty) := \lim_{|t|\to\infty} f'(t) = \lambda_k$ for some $k \geq 2$ (resonance).

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We present results showing that there exist solutions with relatively large augmented Morse indices and prove that these solutions must change sign, at least when f' is relatively small on an interval around zero.

Using classical facts of the theory of elliptic PDE's, we establish an estimate whose simplified version reads as follows (the precise statement is presented in Section 2).

Theorem A. *Given $\epsilon > 0$, $A > 0$, and $D > 0$ there exists a positive constant $B := B(\epsilon, A, D, \Omega, N)$ such that if f satisfies*

(E1) $f(0) = 0$,
(E2) $|f'(t)| \leq D$ for all $t \in \mathbb{R}$,
(E3) $f'(t) \geq \lambda_1 + \epsilon$, for all $|t| > A$,
and u is either a positive or a negative solution of

$$\begin{cases} \Delta u + f(u) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

then u satisfies

$$\|u\|_{L^\infty(\Omega)} \leq B. \tag{3}$$

We point out that hypotheses (E2) and (E3) in Theorem A are much weaker than conditions (f1) or (f1'), since they allow f to be sublinear. On the other hand, here f cannot be superlinear as in [7] and [20].

In order to prove the existence of large augmented Morse index solutions to (1), we apply abstract results contained in [26] and [16]. To be more precise, let us introduce the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F(u) \right) dx,$$

where $F(\xi) = \int_0^\xi f(s)\, ds$. Under conditions (f1) or (f1'), $J \in C^2$ (see [29]) and, moreover,

$$DJ(u)\, v = \langle \nabla J(u), v \rangle = \int_\Omega (\nabla u \cdot \nabla v - f(u)\, v)\, dx, \qquad \forall u, v \in H_0^1(\Omega), \tag{4}$$

$$\left\langle D^2 J(u)\, v, w \right\rangle = \int_\Omega (\nabla v \cdot \nabla w - f'(u)\, vw)\, dx, \qquad \forall u, v, w \in H_0^1(\Omega). \tag{5}$$

It is well known that solutions to (1) agree with critical points of the functional J (see [23]). Since some of the techniques require so, from now on we assume the critical points of J to be isolated. If no such assumption were made, problem (1) would have infinitely many solutions.

For $c \in \mathbb{R}$, let us denote by J^c the set

$$J^c := \{u \in H | J(u) \leq c\}.$$

When $J \in C^1$ and u_0 is an isolated critical point of J, we define the critical groups of J at u_0 as

$$C_q(J, u_0) := H_q(J^c \cap U, J^c \cap U \setminus \{u_0\}),$$

where U is any neighborhood of u_0 that does not contain any other critical point of J, $J(u_0) = c$ and $H_q(X, Y)$ denotes the q-th relative singular homology group of the topological pair (X, Y) taking $\mathbb{R}$ as the coefficient group.

For $J \in C^2$ and u_0 a critical point of J we define the Morse index of J at u_0 as follows: if there is a nonnegative integer m such that there exists an m-dimensional subspace of H on which $D^2J(u_0)$ is negative-definite and m is maximal with respect to this property, we say that m is the *Morse index of J at u_0* and we denote it by $m(J, u_0)$, or $m(u_0)$ when there is no way of confusion. If such an m does not exist, we say the Morse index of J at u_0 is infinity. We define the *augmented Morse index of J at u_0* in a similar fashion, changing the expression "negative-definite" by "nonpositive-definite". In this case we use the notation $m_a(u_0, J)$, or $m_a(u_0)$ when there is no way of confusion. A critical point u_0 of J is said to be non-degenerate if $D^2J(u_0)$ is invertible.

Condition (f1) allows the application of the abstract results contained in [26] to the functional J. Actually, this is done in [14] and [15] under additional assumptions on the critical set of J. By applying a convenient version of Lazer-Solimini results from [26], here we prove the following proposition. Recently, we discovered that a more general result was proved by Chang, Li and Liu in [17] using similar arguments. Hence, we claim no originality for it.

Proposition B. *Let f satisfy (f1). Then:*

(a) There exists a solution u_0 of (1) such that $C_k(J, u_0) \neq \{0\}$. In particular, $m(u_0) \leq k \leq m_a(u_0)$.

(b) If, in addition, f satisfies

(f3) $f(0) = 0$,

(f4) $f'(0) < \lambda_1$,

then (1) has at least three nontrivial solutions u_+, u_- and u_0 of (1). Moreover, $u_+ > 0$ in Ω, $u_- < 0$ in Ω, and $C_k(J, u_0) \neq \{0\}$.

The reader is referred to [4] (Theorem 2.3) where the existence of sign-changing solutions u with $C_k(J, u) \neq \{0\}$ is obtained allowing $f'(0) > \lambda_1$. Unlike the methods in [4] that use critical point theory on partially ordered Hilbert spaces, our methods rely on a priori estimates of one-sign solutions (see Section 2 below). See also [12].

Now, since $f \in C^1$, condition (f1) (alternatively (f1')) implies the existence of $j \geq k+1$ so that $f'(t) \leq \omega < \lambda_j$ for all t. Then condition

(f2) there exist $\gamma > 0$ such that $f'(t) \leq \gamma < \lambda_{k+1}$ for all $t \in \mathbb{R}$,

is a kind of limit case of (f1) (or (f1')). Under conditions (f1)-(f2), starting with the work by A. Castro and A. Lazer in [16], the Lyapunov-Schmidt reduction method has been used in connection with problem (1). We use some of the arguments and results of [16] and [8] to obtain solutions of (1) whose augmented Morse indices are exactly k. More recently (see [27] and its references), the Lyapunov-Schmidt method has been applied in the resonant case (f1')-(f2), and here we use some of the results of [27] to also get the existence of solutions of (1) whose augmented Morse indices equal k. For doing this, we first prove that the Morse index and the augmented Morse index are invariant under the Lyapunov-Schmidt reduction method, which is essentially contained in [16] although not explicitly stated there.

By combining the a priori estimates of Theorem A and the existence of large augmented Morse index solutions of Proposition B, we prove the following.

Theorem C. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy (f1) and (f3). Let $\epsilon > 0$, $A > 0$, $D > 0$, and $B > 0$ as in Theorem A. If*

$$f'(t) < \lambda_k \quad \forall t \in [-B, B], \tag{6}$$

then there exists at least one sign-changing solution u_* of (1) such that

$$\|u_*\|_{L^\infty(\Omega)} > B.$$

In the limit case given by conditions (f1) and (f2), we have the following

Theorem D. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy (f1)-(f4). Let $\epsilon > 0$, $A > 0$, $D > 0$, and $B > 0$ as in Theorem A. If*

$$f'(t) < \lambda_k \quad \forall t \in [-B, B], \tag{7}$$

then there exist at least two sign-changing solutions u_* and v_* of (1). Moreover, one of them, let us say u_*, satisfies

$$\|u_*\|_{L^\infty(\Omega)} > B.$$

See also [4] (Corollary 2.4) for the existence of two sign-changing solutions. In the resonance case given by conditions (f1') and (f2), we have:

Theorem E. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy (f1')-(f4). Let $\epsilon > 0$, $A > 0$, $D > 0$, and $B > 0$ as in Theorem A. Assume, in addition, that f satisfies*

(f5) $F(t) - \frac{1}{2}\lambda_k t^2 \to \infty$ as $|t| \to \infty$.

If

$$f'(t) < \lambda_k \quad \forall t \in [-B, B] \tag{8}$$

then there exists at least one sign-changing solution u_* of (1) such that

$$\|u_*\|_{L^\infty(\Omega)} > B.$$

We remark that in [10] the existence of sign-changing solutions is proved, regardless of resonance, under different hypotheses and by means of a different argument. In [11] it is shown that the solution given by Theorem E is different from the solution given by [10] when $k \geq 3$.

The body of this paper is organized as follows. In Section 2 we state and prove the precise form of Theorem A and include Lemmas to be used in Section 4. In Section 3, we re-state an abstract Lazer-Solimini result in a convenient way, and we recall the Lyapunov-Schmidt reduction method. In Section 4 we prove Proposition B, Theorems C, D, and E, and we include some additional results concerning Morse index, sign-changing solutions, and symmetries.

2. A priori estimates.

Throughout this section we assume A, D and ϵ are positive constants and f satisfies the following hypotheses:

(E1) $f(0) = 0$,
(E2) $|f'(t)| \leq D$ for all $t \in \mathbb{R}$,
(E3) $f'(t) \geq \lambda_1 + \epsilon$, for all $|t| > A$.

We define

$$-m := \min_{t \geq 0} f(t), \tag{9}$$

$$-M := \min_{t \geq 0}\{f(t) - (\lambda_1 + \epsilon)t\}, \tag{10}$$

$$-K := \min_{t \geq 0}\{t|t|^{\frac{1}{N+1}} - f(t)\}. \tag{11}$$

We observe that hypotheses (E2) and (E3), particularly for $t \geq 0$, guarantee that these constants are well-defined. Before continuing, some comments are in order. First we note that (E2) implies $f' \in L^\infty(\mathbb{R})$. Secondly, $m \leq M$. Thirdly, as we shall see, we could have chosen a power of the form $1 + \frac{1}{N+\mu}$, $\mu \in (0, \infty)$, when defining $-K$. For the sake of simplicity we choose $\mu = 1$. Finally, we observe these constants are bounded in terms of A, D and ϵ.

In this section, a *positive* (respectively *negative*) *solution* of (1) is a non-zero function $u \in C^2(\overline{\Omega})$ which is non-negative (respectively non-positive) in Ω and satisfies both conditions in (1). Now we state the main result of this section.

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function satisfying (E1), (E2) and (E3).*
There exist positive constants $C_i(\Omega, N)$, $i = 1, ..., 5$, and $r > 0$, depending only on Ω, N and D, such that if u is a positive solution of (1) then

$$\|u\|_{L^\infty(\Omega)} \leq M^{1+\frac{1}{N+1}}\, r^{2-N} C_1(\Omega, N) + K\, r^{2-N} C_2(\Omega, N) + mC_3(\Omega, N) \tag{12}$$

and

$$\|u\|_{L^\infty(\Omega)} \leq MD\, r^{2-N} C_4(\Omega, N) + mC_5(\Omega, N). \tag{13}$$

Note: r is defined by (34) and (39).

Before proving these estimates, we recall some facts we extensively use along the proof. Given $\delta > 0$, denote by Ω_δ the set $\{x \in \Omega |\, d(x, \partial\Omega) < \delta\}$. For $x \in \partial\Omega$, let us denote by $\overrightarrow{n}(x)$ the inward unit normal to $\partial\Omega$ at x. Because of smoothness of Ω (see [22]), the following lemma about the existence of a tubular neighborhood of $\partial\Omega$ can be shown.

Lemma 2.1. *There exists $\delta_0 > 0$ such that*

(i) if $z \in \Omega_{\delta_0}$ there exists a unique $(x_z, t_z) \in \partial\Omega \times (0, \delta_0)$ such that $z = x_z + t_z \overrightarrow{n}(x_z)$. Moreover, $t_z = \|x_z - z\| = d(z, \partial\Omega)$.
(ii) If $x \in \partial\Omega$ then, for every $t \in (0, \delta_0)$, $x + t\overrightarrow{n}(x) \in \Omega$.

Secondly, according to the Sobolev embeddings, for $j, m \in \mathbb{N} \cup \{0\}$ and $p > 1$ the inclusion $W^{j+m,p}(\Omega) \hookrightarrow C^{j,\alpha}(\overline{\Omega})$ is continuous if $mp > N > (m-1)p$ and $0 < \alpha \leq m - \frac{N}{p}$ (see [1] for definitions and proofs).

Finally, we recall the *Agmon-Douglis-Nirenberg estimates* (see [23]):
Let $p \in (1, \infty)$. If $h \in L^p(\Omega)$ then

$$\begin{cases} -\Delta u &= h \text{ in } \Omega, \\ \quad\; u &\in W_0^{1,p}(\Omega), \end{cases} \tag{14}$$

has a unique solution $u_0 \in W^{2,p}(\Omega)$. Moreover, there exists a positive constant C (independent of u) such that

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|\Delta u\|_{L^p(\Omega)} \;\forall u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega).$$

Proof of Theorem 2.1. We complete this proof in three steps. Given a positive solution u of (1), in the first step we estimate the integrals $\int_\Omega u\varphi_1$, $\int_\Omega u$, $\int_\Omega u^{1+\frac{1}{N+1}}$, and $\int_\Omega f(u)$ in terms of K, m and M. In the second step, we demonstrate that if $\xi_u \in \Omega$ and $\|u\|_{L^\infty(\Omega)} = u(\xi_u)$, then the distance from ξ_u to $\partial\Omega$ is bounded below in terms of $\|f'\|_{L^\infty(\mathbb{R})}$. In the last step, we make use of Green's function and the previous estimates to prove the theorem.

STEP 1. Let u be a positive solution of (1). Multiplying the differential equation in (1) by the eigenfunction φ_1 and integrating by parts, we get

$$0 = \int_\Omega \varphi_1 \Delta u + \varphi_1 f(u) = \int_\Omega (f(u) - \lambda_1 u)\varphi_1. \tag{15}$$

From the definition of M we have

$$\int_\Omega u\varphi_1 \le \frac{M}{\epsilon}\int_\Omega \varphi_1. \tag{16}$$

Now, because of classical existence results and the Strong Maximum Principle (see [23]), there exists a unique solution $\psi > 0$ to

$$\begin{cases} -\Delta\psi = 1 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

As an application of Hopf's Lemma and comparison arguments between φ_1 and ψ in Ω, one can prove the existence of positive constants c_1 and c_2, depending only on Ω, such that

$$c_1\varphi_1 \le \psi \le c_2\varphi_1 \text{ in } \Omega \tag{18}$$

(we will consider a more general case below). Multiplying the differential equation in (17) by u and integrating by parts, we obtain

$$\int_\Omega u = \int_\Omega f(u)\psi. \tag{19}$$

As a consequence of (18) and (19), it follows that

$$\int_\Omega u \le \int_{f(u)\ge 0} f(u)\psi \le c_2\int_{f(u)\ge 0} f(u)\varphi_1 = c_2\left(\int_\Omega f(u)\varphi_1 - \int_{f(u)<0} f(u)\varphi_1\right).$$

From (15), (16) and the definitions of m and M,

$$\int_\Omega u \le Mc_2\left(\frac{\lambda_1}{\epsilon}+1\right)\int_\Omega \varphi_1. \tag{20}$$

Using (20), a combination of Hopf's Lemma, Sobolev embeddings and Agmon-Douglis-Nirenberg estimates, allow us to bound $\int_\Omega u^{1+\frac{1}{N+1}}$. To this end, let us consider the auxiliary problem

$$\begin{cases} -\Delta\omega = u^{\frac{1}{N+1}} & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

Because of Sobolev embeddings and Agmon-Douglis-Nirenberg estimates, there exists a unique weak solution $\omega \in C^1(\overline{\Omega})$ for this problem. Moreover $\omega \ge 0$ (see [23]). Again, Agmon-Douglis-Nirenberg estimates imply the existence of a constant $c_3 > 0$, depending only on Ω and N, such that

$$\|\omega\|_{W^{2,N+1}(\Omega)} \le c_3\|u^{\frac{1}{N+1}}\|_{L^{N+1}(\Omega)}. \tag{22}$$

Thus, from (20) and (22) it follows that

$$\|\omega\|_{W^{2,N+1}(\Omega)} \le M^{\frac{1}{N+1}}\left(c_2\left(\frac{\lambda_1}{\epsilon}+1\right)\int_\Omega \varphi_1\right)^{\frac{1}{N+1}} c_3. \tag{23}$$

From the continuity of the embedding $W^{2,N+1}(\Omega) \subset C^1(\overline{\Omega})$, there exists c_4, depending only on Ω and N, such that

$$\|\omega\|_{C^1(\overline{\Omega})} \le M^{\frac{1}{N+1}}\left(c_2\left(\frac{\lambda_1}{\epsilon}+1\right)\int_\Omega \varphi_1\right)^{\frac{1}{N+1}} c_3c_4. \tag{24}$$

Now we intend to compare ω and φ_1 in Ω. We make use of Hopf's Lemma and the existence of a tubular neighborhood of $\partial\Omega$.

Lemma 2.2. *There exists a positive constant c, depending only on Ω and M, such that*

$$\omega \leq c\varphi_1 \text{ in } \Omega. \tag{25}$$

Proof of Lemma 2.2. Let $m_1 := \min_{\partial\Omega} |\nabla\varphi_1|$. Observe that Hopf's Lemma implies $m_1 > 0$. Now let us consider δ_0 as in Lemma 2.1. Let us denote by $\overrightarrow{n}(x)$ the inward unit normal to $\partial\Omega$ at x.

Claim 1: There exists δ such that $0 < \delta < \delta_0$ and

(a) For every $z \in \Omega_\delta$, $|\nabla\varphi_1(z)| \geq \frac{1}{2}m_1$,
(b) For every $x \in \partial\Omega$ and every $t \in (0, \delta)$,

$$\frac{\nabla\varphi_1(x + t\overrightarrow{n}(x))}{|\nabla\varphi_1(x + t\overrightarrow{n}(x))|} \cdot \overrightarrow{n}(x) \geq \frac{1}{2}.$$

This claim comes from the uniform continuity of the function $z \mapsto |\nabla\varphi_1(z)|$ on $\overline{\Omega}$, the uniform continuity of the function $(x,t) \mapsto \frac{\nabla\varphi_1(x+t\overrightarrow{n}(x))}{|\nabla\varphi_1(x+t\overrightarrow{n}(x))|} \cdot \overrightarrow{n}(x)$ on $\partial\Omega \times [0, \delta]$ and the fact that $\frac{\nabla\varphi_1(x)}{|\nabla\varphi_1(x)|} \cdot \overrightarrow{n}(x) = 1$ for all $x \in \partial\Omega$.

Now, take $m_2 := \min_{\Omega\setminus\Omega_\delta} \varphi_1 > 0$. We observe that m_1, δ, and m_2 depend only on Ω. Being motivated by (24), let us pick $c > 0$ so that

$$c \min\{\frac{1}{4}m_1, m_2\} \geq M^{\frac{1}{N+1}} \left(c_2\left(\frac{\lambda_1}{\epsilon} + 1\right)\int_\Omega \varphi_1\right)^{\frac{1}{N+1}} c_3c_4.$$

Claim 2: If $z \in \Omega \setminus \Omega_\delta$ then $\omega(z) \leq c\varphi_1(z)$.

Indeed, if $z \in \Omega \setminus \Omega_\delta$,

$$\omega(z) \leq \|\omega\|_{C^1(\overline{\Omega})} \leq M^{\frac{1}{N+1}} \left(c_2\left(\frac{\lambda_1}{\epsilon} + 1\right)\int_\Omega \varphi_1\right)^{\frac{1}{N+1}} c_3c_4 \leq cm_2 \leq c\varphi_1(z).$$

In order to complete the proof of (25), we demonstrate the following

Claim 3: If $z \in \Omega_\delta$ then $\omega(z) \leq c\varphi_1(z)$.

To prove this claim, let us write $z = x + t\overrightarrow{n}(x)$, for some $x \in \partial\Omega$ and $t \in (0, \delta)$, which is given by Lemma 2.1. Since $\omega \in H_0^1(\Omega) \cap C^1(\overline{\Omega})$, $\omega(x) = 0$. Hence,

$$\omega(z) = \int_0^t \nabla\omega(x + s\overrightarrow{n}(x)) \cdot \overrightarrow{n}(x)\, ds \leq \int_0^t |\nabla\omega(x + s\overrightarrow{n}(x))|\, ds.$$

On the other hand, from the choice of c and Claim 1, for every $s \in [0, t]$ it follows that

$$|\nabla\omega(x + s\overrightarrow{n}(x))| \leq \|\omega\|_{C^1(\overline{\Omega})} \leq \frac{1}{4}cm_1$$
$$\leq \frac{1}{2}c|\nabla\varphi_1(x + s\overrightarrow{n}(x))| \leq c\nabla\varphi_1(x + s\overrightarrow{n}(x)) \cdot \overrightarrow{n}(x).$$

Consequently,

$$\omega(z) \leq \int_0^t |\nabla\omega(x + s\overrightarrow{n}(x))|\, ds \leq \int_0^t c\nabla\varphi_1(x + s\overrightarrow{n}(x)) \cdot \overrightarrow{n}(x)\, ds = c\varphi_1(z).$$

This is just (25). Finally, observe that c can be taken so that

$$c = c_5 M^{\frac{1}{N+1}}, \tag{26}$$

where c_5 depends only on Ω and N. □

Now we proceed to estimate $\int_\Omega u^{1+\frac{1}{N+1}}$. Multiplying by u the differential equation in (21), integrating by parts and using (25), we have

$$\int_\Omega u^{1+\frac{1}{N+1}} = \int_\Omega f(u)\omega \le c \int_{f(u)\ge 0} f(u)\varphi_1.$$

Arguing as we did to get (20), from (15) and (16) it follows

$$\int_\Omega u^{1+\frac{1}{N+1}} \le Mc\left(\frac{\lambda_1}{\epsilon}+1\right)\int_\Omega \varphi_1. \tag{27}$$

Therefore, from (27) and the definition of K it follows that if $\widehat{\Omega} \subset \Omega$,

$$\int_{\widehat{\Omega}} f(u) \le \int_\Omega u^{1+\frac{1}{N+1}} + K|\Omega| \le Mc\left(\frac{\lambda_1}{\epsilon}+1\right)\int_\Omega \varphi_1 + K|\Omega|. \tag{28}$$

As a variant of the previous estimate, we observe that because of the Mean Value Theorem and (E1), $f(t) \le Dt$ for all $t \ge 0$. Keeping into account this fact, as well as (20) we get

$$\int_{\widehat{\Omega}} f(u) \le MDc_2\left(\frac{\lambda_1}{\epsilon}+1\right)\int_\Omega \varphi_1, \tag{29}$$

where $\widehat{\Omega}$ is any measurable subset of Ω.

STEP 2.

Lemma 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy (E1), (E2) and (E3). Then, there exists $r_0 > 0$ depending only on Ω, N and $\|f'\|_{L^\infty(\mathbb{R})}$ such that if u is a positive solution of (1), then:*

$$\forall \xi_u \in \Omega : \|u\|_{L^\infty(\Omega)} = u(\xi_u) \Rightarrow d(\xi_u, \partial\Omega) \ge r_0. \tag{30}$$

Proof of Lemma 2.3. Since Ω is smooth and bounded, there exists $\delta_0 > 0$, that depends only on Ω, satisfying (i) and (ii) in Lemma 2.1. Let u a positive solution of (1). Let $\xi_u \in \Omega$ such that $\|u\|_{L^\infty(\Omega)} = u(\xi_u)$. In order to prove the lemma, it suffices to consider the case in which $\xi_u \in \Omega_{\delta_0}$ (otherwise $d(\xi_u, \partial\Omega) \ge \delta_0$ and the result follows).

Let us write $\xi_u = x + t\overrightarrow{n}(x)$, with $x \in \partial\Omega$ and $t \in (0, \delta_0)$. As a consequence of (ii) in Lemma 2.1, function $s \mapsto u(x + st\overrightarrow{n}(x))$ is well-defined and it is differentiable in $[0,1]$. Because of the Mean Value Theorem, there exists $s \in (0,1)$ such that

$$u(\xi_u) = (\nabla u(x + st\overrightarrow{n}(x)) \cdot t\overrightarrow{n}(x).$$

Since ξ_u is a critical point of u, $\nabla u(x + t\overrightarrow{n}(x)) = \nabla u(\xi_u) = 0$. Thus,

$$u(\xi_u) = (\nabla u(x + st\overrightarrow{n}(x)) - \nabla u(x + t\overrightarrow{n}(x))) \cdot t\overrightarrow{n}(x). \tag{31}$$

On the other hand, from Sobolev imbeddings and Agmon-Douglis-Nirenberg estimates, there exist positive constants c_6 and c_7, depending only on Ω and N, such that

$$\|u\|_{C^{1,\frac{1}{2(N+1)}}(\overline{\Omega})} \le c_6\|u\|_{W^{2,N+1}(\Omega)} \le c_7\|f(u)\|_{L^{N+1}(\Omega)}. \tag{32}$$

Applying the Mean Value Teorem to $f(u)$, because of (E1),

$$\|f(u)\|_{L^{N+1}(\Omega)} \le c_8\|f'\|_{L^\infty(\mathbb{R})}\|u\|_{L^\infty(\Omega)}, \tag{33}$$

for some positive constant c_8 that depends only on Ω.

From (31), (32) and (33), it follows that

$$\|u\|_{L^\infty(\Omega)} = u(\xi_u) \le t^{1+\frac{1}{2(N+1)}}\|u\|_{C^{1,\frac{1}{2(N+1)}}(\overline{\Omega})}$$

$$\leq c_7 c_8 t^{1+\frac{1}{2(N+1)}} \|f'\|_{L^\infty(\mathbb{R})} \|u\|_{L^\infty(\Omega)}.$$

Because of the previous and the note after Lemma 2.1, we have

$$(d(\xi_u, \partial\Omega))^{1+\frac{1}{2(N+1)}} = |\xi_u - x|^{1+\frac{1}{2(N+1)}} = t^{1+\frac{1}{2(N+1)}}$$

$$\geq \frac{1}{c_7 c_8 \|f'\|_{L^\infty(\mathbb{R})}} =: r_0^{1+\frac{1}{2(N+1)}} \tag{34}$$

and the proof is complete. □

Remark: In the case Ω is, in addition, *convex*, there exists r_0 that depends only on Ω, such that if u is a positive solution of (1), there exists $\xi_u \in \Omega$ such that $\|u\|_{L^\infty(\Omega)} = u(\xi_u)$ and $d(\xi_u, \partial\Omega) \geq r_0$ (see [21] and [20]).

STEP 3. Now we complete the proof of Theorem 2.1. Using Green's function for u in Ω (see [23]), we get

$$u(\xi_u) = \int_\Omega \left(\frac{C(N)}{\|x - \xi_u\|^{N-2}} - v(\xi_u, x) \right) f(u(x)) dx, \tag{35}$$

where $v(\xi_u, .)$ is a positive harmonic function in Ω and $C(N) > 0$. Observe that

$$\int_\Omega \left(\frac{C(N)}{\|x - \xi_u\|^{N-2}} - v(\xi_u, x) \right) dx \geq 0$$

(from applying Green's function to the solution of (17)). Hence

$$\int_\Omega v(\xi_u, x) dx \leq \int_\Omega \frac{C(N)}{\|x - \xi_u\|^{N-2}} dx \leq \int_{B_R(\xi_u)} \frac{C(N)}{\|x - \xi_u\|^{N-2}} dx =: C(N, \Omega),$$

where $R = 2diam(\Omega)$. Therefore, from the definition of m

$$\int_\Omega -v(\xi_u, x) f(u(x)) dx \leq C(N, \Omega) m. \tag{36}$$

Now we estimate $u(\xi_u)$. Let $r \in (0, r_0)$. Because of (30), $B_r(\xi_u) \subset \Omega$. Hence, from (35) and (36),

$$u(\xi_u) \leq \int_{B_r(\xi_u)} \frac{C(N) f(u(x))}{\|x - \xi_u\|^{N-2}} dx + \int_{\Omega \setminus B_r(\xi_u)} \frac{C(N) f(u(x))}{\|x - \xi_u\|^{N-2}} dx + C(N, \Omega) m. \tag{37}$$

Now we bound the integrals on the right hand side of (37). First, we observe that by virtue of the Mean Value Theorem, (E1) and the definition of D, it follows

$$\int_{B_r(\xi_u)} \frac{C(N) f(u(x))}{\|x - \xi_u\|^{N-2}} dx \leq C_1(N) D\, r^2 u(\xi_u), \tag{38}$$

where $C_1(N) > 0$ is a constant that depends on N. Let us take $r > 0$ in $(0, r_0)$ so that

$$0 < r^2 < \frac{1}{2C_1(N) D}. \tag{39}$$

This choice of r depends only on Ω, N and D. Moreover, from (37), (38) and (39) we have

$$\frac{1}{2} u(\xi_u) \leq \int_{\Omega \setminus B_r(\xi_u)} \frac{C(N) f(u(x))}{\|x - \xi_u\|^{N-2}} dx + C(N, \Omega) m. \tag{40}$$

Regarding the integral on the right hand side of (40), due to (28),

$$\int_{\Omega \setminus B_r(\xi_u)} \frac{f(u(x))}{\|x - \xi_u\|^{N-2}} dx \leq M c\, r^{2-N} \left(\frac{\lambda_1}{\epsilon} + 1 \right) \int_\Omega \varphi_1 + K\, r^{2-N} |\Omega|. \tag{41}$$

From (40) and (41) there exist positive constants $C_1(\Omega,N)$, $C_2(\Omega,N)$ and $C_3(\Omega,N)$, that depend on Ω and N, such that

$$\|u\|_{L^\infty(\Omega)} \le Mc\,r^{2-N}C_1(\Omega,N) + K\,r^{2-N}C_2(\Omega,N) + mC_3(\Omega,N). \tag{42}$$

As a variant of the previous estimate, using (29) as well as (40) we obtain

$$\|u\|_{L^\infty(\Omega)} \le MD\,r^{2-N}C_4(\Omega,N) + mC_5(\Omega,N), \tag{43}$$

for some positive constants $C_4(\Omega,N)$ and $C_5(\Omega,N)$. From (26), (42) and (43) our result follows. □

Similarly we can obtain estimates for the $L^\infty(\Omega)$-norm of negative solutions of (1).

Corollary 2.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a C^1-function that satisfies (E1), (E2) and (E3). There exist positive constants $C_i(\Omega,N)$, $i=1,\dots,5$, and $r>0$, depending only on Ω, N and D, such that if u is a negative solution of (1) then*

$$\|u\|_{L^\infty(\Omega)} \le \widetilde{M}^{1+\frac{1}{N+1}}\,r^{2-N}C_1(\Omega,N) + \widetilde{K}\,r^{2-N}C_2(\Omega,N) + \widetilde{m}C_3(\Omega,N) \tag{44}$$

and

$$\|u\|_{L^\infty(\Omega)} \le \widetilde{M}D\,r^{2-N}C_4(\Omega,N) + \widetilde{m}C_5(\Omega,N), \tag{45}$$

where the constants $\widetilde{m}$, $\widetilde{M}$, and $\widetilde{K}$ are defined by

$$-\widetilde{m} := \min_{t\ge 0}\{-f(-t)\}, \tag{46}$$

$$-\widetilde{M} := \min_{t\ge 0}\{-f(-t) - (\lambda_1+\epsilon)t\}, \tag{47}$$

$$-\widetilde{K} := \min_{t\ge 0}\{t|t|^{\frac{1}{N+1}} + f(-t)\}. \tag{48}$$

Note: r is defined by (34) and (39).

Throughout the remaining part of this paper we extensively make use of the following inequalities coming from the variational characterization of $\{\lambda_j\}_j$: given $k\in\mathbb{N}$,

$$\|x\|^2_{H^1_0(\Omega)} \le \lambda_k \int_\Omega x^2 \;\; \forall x\in X := span\{\varphi_1,\varphi_2,\dots,\varphi_k\} \subset H^1_0(\Omega), \tag{49}$$

and

$$\|y\|^2_{H^1_0(\Omega)} \ge \lambda_{k+1} \int_\Omega y^2 \;\; \forall y\in Y := X^\perp = \overline{\{\varphi_{k+1},\varphi_{k+2},\dots\}} \subset H^1_0(\Omega). \tag{50}$$

We remark that conditions (E2) and (E3) in the following proposition are purely technical and can be changed by some condition that ensures $J\in C^2$.

Proposition 2.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a C^1-function satisfying (E2) and (E3). Let us assume u is a solution of (1) and there exists $j\in\mathbb{N}_0$ such that*

$$f'(t) < \lambda_{j+1} \text{ for all } t\in u(\Omega).$$

Then $m_a(J,u)\le j$.

Proof. From (5),

$$\left\langle D^2 J(u)\, v, v\right\rangle = \int_\Omega \left(\nabla v \cdot \nabla v - f'(u)\, v^2\right) dx.$$

Thus, if $Y := \overline{\langle \varphi_j, \varphi_{j+1}, \ldots\rangle} \subset H_0^1(\Omega)$ and $v \in Y \setminus \{0\}$, from the hypothesis and the inequality (50), it follows that $\left\langle D^2 J(u)\, v, v\right\rangle > 0$.

If $j = 0$ the result is already proven. If $j > 1$ we argue by contradiction. Suppose there is a subspace $W \subset H_0^1(\Omega)$ such that $\dim W =: p > j-1$ y $\left\langle D^2 J(u)\, w, w\right\rangle \leq 0$ for all $w \in W \setminus \{0\}$. Let $\{w_1, \ldots, w_p\}$ be a basis for W. Let us define $X := span\{\varphi_1, \varphi_2, \ldots, \varphi_{j-1}\}$. Then, because of the decomposition $H_0^1(\Omega) = X \oplus Y$, we have $w_1 = x_1 + y_1, \ldots, w_p = x_p + y_p$, for some $x_1, \ldots x_p \in X$ and $y_1, \ldots, y_p \in Y$. Given that $p > j-1$, there exist $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$, not all zero, so that $\alpha_1 x_1 + \ldots + \alpha_p x_p = 0$. Hence, $w := \alpha_1 w_1 + \ldots + \alpha_p w_p \in W \cap Y \setminus \{0\}$. But this is a contradiction and the result follows. □

To complete this section we state a proposition which will be useful to prove the existence of sign-changing solutions in Section 4. Let us introduce the following notations

$$B_+ := M^{1+\frac{1}{N+1}}\, r^{2-N} C_1(\Omega, N) + K\, r^{2-N} C_2(\Omega, N) + m C_3(\Omega, N),$$
$$b_+ := M D\, r^{2-N} C_4(\Omega, N) + m C_5(\Omega, N),$$

where the right-hand sides are those appearing in (12) and (13). Similarly,

$$B_- := \widetilde{M}^{1+\frac{1}{N+1}}\, r^{2-N} C_1(\Omega, N) + \widetilde{K}\, r^{2-N} C_2(\Omega, N) + \widetilde{m} C_3(\Omega, N),$$
$$b_- := \widetilde{M} D\, r^{2-N} C_4(\Omega, N) + \widetilde{m} C_5(\Omega, N),$$

where the right-hand sides are those appearing in (44) and (45).

Proposition 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function satisfying (E1), (E2) and (E3). In addition, suppose f satisfies*

$$f'(t) < \lambda_k \quad \forall t \in [-B_-, B_+]. \tag{51}$$

Then, every solution u of (1) whose augmented Morse index is greater than $k-1$ is a sign-changing solution and satisfies

$$\|u\|_{L^\infty(\Omega)} > \min\{B_-, B_+\}.$$

The same conclusion follows if we change B_+ by b_+ and/or B_- by b_- in (51).

Proof. Apply Theorem 2.1, Corollary 2.1, and Proposition 2.1. □

3. **Abstract results.** In this section H denotes a real Hilbert space with inner product $\langle \cdot, \cdot\rangle$ and X and Y will denote closed subspaces of H such that $\dim X =: k < \infty$ and $H = X \oplus Y$. Given a functional $J : H \to \mathbb{R}$, we recall that J is said to satisfy *the Palais-Smale condition*, referred to as (PS), if given a sequence $\{u_n\}_n$ in H such that $DJ(u_n) \to 0$ and $\{J(u_n)\}_n$ is bounded, $\{u_n\}_n$ contains a convergent subsequence.

We recall the Saddle Point Theorem of P. Rabinowitz (see, for example, [29]).

Theorem 3.1. (Saddle Point Theorem) *Let $J \in C^1(H, \mathbb{R})$ be a functional satisfying the (PS) condition and the following hypotheses*

(S1) $\inf\{J(y) : y \in Y\} =: d > -\infty$,

(S2) $J(x) \to -\infty$ *as* $\|x\| \to \infty$ *with* $x \in X$.

Then J has at least one critical point.

A. Lazer and S. Solimini in [26] show that, under additional conditions, at least one of the critical points of J has Morse index equal to $\dim X$. More precisely, they prove the following lemma.

Lemma 3.1. (Lazer-Solimini) *Let $J \in C^2(H, \mathbb{R})$ be a functional satisfying (PS) condition and hypotheses (S1)-(S2). Assume J has only a finite number of critical points, all of which are nondegenerate. Then there must exist at least one critical point whose Morse index equals $k = \dim X$.*

A careful reading of the proof presented in [26] makes clear that this lemma can be reformulated in terms of critical groups instead of Morse index, eliminating the nondegeneracy hypothesis of critical points. In fact, the proof needs not be modified at all. Besides, J has just to be assumed of class C^1 in this reformulation. We also point out that the following version of Lazer-Solimini's result, which will be very useful for our purposes in Section 4; it is a consequence of infinite dimensional Morse theory (see, for example, [13], chapter II). Hence, we omit its proof.

Lemma 3.2. *Let $J \in C^1(H, \mathbb{R})$ be a functional that satisfies (PS) condition, and hypotheses (S1)-(S2). Suppose J has only a finite number of critical points. Then there exists a critical point u_0 of J such that $C_k(J, u_0) \neq \{0\}$. If, in addition, $J \in C^2(H, \mathbb{R})$ then the critical point u_0 is such that $m(u_0, J) \leq k \leq m_a(u_0, J)$.*

Now we recall a version of the Lyapunov-Schmidt reduction method. We refer the reader to [16] and [8] for details. We recall that if $u_0 \in H$ is a critical point of a functional $J \in C^1$ and $c = J(u_0)$, then u_0 is said to be a *critical point of mountain pass type* of J if there exists a neighborhood U of u_0 such that, for every neighborhood $V \subset U$ de u_0, the set

$$V \cap \{u \in H |\, J(u) < c\}$$

is neither empty nor path-connected (see [24] and [13]).

Lemma 3.3. *Let $J : H \to \mathbb{R}$ be a function of the class $C^2(H, \mathbb{R})$. Suppose there exists $c > 0$ such that*

$$\langle D^2 J(u)y, y\rangle \geq c\|y\|_H^2; \qquad \forall u \in H \quad \forall y \in Y. \tag{52}$$

Then:

(i) There exists a function $\psi : X \to Y$, of the class C^1, such that

$$J(x + \psi(x)) = \min_{y \in Y} J(x + y).$$

Moreover, given $x \in X$, $\psi(x)$ is the unique element of Y such that

$$\langle \nabla J(x + \psi(x)), y\rangle = 0 \quad \forall y \in Y. \tag{53}$$

(ii) The functional $\hat{J} : X \to \mathbb{R}$, defined by $\hat{J}(x) := J(x + \psi(x))$ for $x \in X$, is of class C^2. Moreover,

$$D\hat{J}(x)h = \left\langle \nabla \hat{J}(x), h \right\rangle = \langle \nabla J(x + \psi(x)), h\rangle \quad \forall x, h \in X. \tag{54}$$

(iii) Given $x \in X$, x is a critical point of $\hat{J}$ if and only if $u = x + \psi(x)$ is a critical point of J.

(iv) If $u_0 = x_0 + \psi(x_0)$ is a critical point of mountain pass type of J then x_0 is a critical point of mountain pass type of $\hat{J}$.

(v) If $x_0 \in X$ is an isolated critical point of $\hat{J}$ then the local Leray-Schauder degree is preserved under reduction, i.e.

$$d_{loc}(\nabla \hat{J}, x_0) = d_{loc}(\nabla J, u_0).$$

(vi) If $x_0 \in X$ is a critical point of $\hat{J}$ then the Morse index is invariant under reduction, i.e.

$$m(u_0, J) = m(x_0, \hat{J})$$

and

$$m_a(u_0, J) = m_a(x_0, \hat{J}).$$

(vii) If $u_0 = x_0 + \psi(x_0) \in H$ is a nondegenerate critical point of J then x_0 is a nondegenerate critical point of $\hat{J}$.

4. **Sign-changing solutions, Morse index and further qualitative properties.** Along this section, f will be assumed to satisfy one of the asymptotic linearity conditions

(f1) $f'(\infty) := \lim_{|t|\to\infty} f'(t) \in (\lambda_k, \lambda_{k+1})$ for some $k \geq 2$, or

(f1') $f'(\infty) := \lim_{|t|\to\infty} f'(t) = \lambda_k$ for some $k \geq 2$,

unless otherwise stated. Any of these assumptions automatically guarantee that f satisfies (E2) and (E3) as in Section 1.

Our starting result is an application of Lemma 3.2 and some previous results regarding critical groups of critical points of mountain pass type (see [13]). The first part gives a solution to (1) whose augmented Morse index is bounded below. This fact will be useful to get sign-changing solutions. For the most part the proof of the following proposition is well-known (see [17]). We include it here for the sake of completeness.

Proposition B. *Let f satisfy (f1). Then:*

(a) There exists a solution u_0 of (1) such that $C_k(J, u_0) \neq \{0\}$. In particular, $m(u_0) \leq k \leq m_a(u_0)$.

(b) If, in addition, f satisfies

(f3) $f(0) = 0$,

(f4) $f'(0) < \lambda_1$,

there exist at least three nontrivial solutions u_+, u_- and u_0 of (1). Moreover, $u_+ > 0$ in Ω, $u_- < 0$ in Ω, and $C_k(J, u_0) \neq \{0\}$.

Proof. a) Due to condition (f1), it is well-known that the functional J, defined in the introduction, is of the class $C^2(H_0^1(\Omega), \mathbb{R})$, satisfies condition (PS) and its set of critical points is bounded in $H_0^1(\Omega)$ (see [29], [18], [15]). In order to apply Lemma 3.2, we must verify J satisfies (S1), (S2) and that the set of critical points of J be finite. This last condition is simply a consequence of the fact that the critical set of J is bounded, (PS) condition and the assumption of isolation of critical points of J. Let $X := span\{\varphi_1, \varphi_2, \dots, \varphi_k\}$ and $Y := X^\perp = \overline{\{\varphi_{k+1}, \varphi_{k+2}, \dots\}}^{H_0^1(\Omega)}$.

Claim 1: J satisfies (S1).

Indeed, let $y \in Y$. because of (f1), there exist $a_1 < \lambda_{k+1}$ and $a_2 \in \mathbb{R}$ such that

$$F(s) \leq \frac{a_1}{2} s^2 + a_2 \quad \forall s \in \mathbb{R}.$$

Thus,

$$J(y) \geq \frac{1}{2}\|y\|^2_{H_0^1(\Omega)} - \frac{a_1}{2}\int_\Omega y^2 \; - a_2|\Omega|.$$

From (50),

$$J(y) \geq \frac{1}{2}\left(1 - \frac{a_1}{\lambda_{k+1}}\right) \|y\|^2_{H^1_0(\Omega)} - a_2\,|\Omega|.$$

Since $a_1 < \lambda_{k+1}$, J satisfies (S1).

Claim 2: J satisfies condition (S2).

To prove this, let $x \in X$. Again, (f1) implies the existence of $a_3 > \lambda_k$ and $a_4 \in \mathbb{R}$ such that

$$F(s) \geq \frac{a_3}{2}s^2 + a_4 \quad \forall s \in \mathbb{R}.$$

Hence,

$$J(x) = \frac{1}{2}\|x\|^2_{H^1_0(\Omega)} - \int_\Omega F(x) \;\leq \frac{1}{2}\|x\|^2_{H^1_0(\Omega)} - \frac{a_3}{2}\int_\Omega x^2 - a_4|\Omega|.$$

From (49),

$$J(x) \leq \frac{1}{2}\left(1 - \frac{a_3}{\lambda_k}\right) \|x\|^2_{H^1_0(\Omega)} - a_4\,|\Omega|.$$

As $a_3 > \lambda_k$, J satisfies (S2).

Consequently, part a) follows directly from Lemma 3.2.

b) Let u_0 be the solution coming from part a). The existence of one-signed solutions u_+ and u_- of (1) is a well-known application of the Mountain Pass Theorem.

By virtue of a result by H. Hofer (see [24] and [13]),

$$C_q(J, u_+) = C_q(J, u_-) = \delta_{q,1}\mathbb{R}.$$

Since $k \geq 2$ and $C_k(J, u_0) \neq \{0\}$, $u_+ \neq u_0$ and $u_- \neq u_0$. Finally, from hypotheses (f3), (f4) and Poincar's inequality, a direct computations shows 0 is an isolated local minimum of J. Consequently, $C_q(J, 0) = \delta_{q,0}\mathbb{R}$ and $u_0 \neq 0$, which proves part b). □

Remark: We observe that Proposition B improves those results presented in [14] and [15] thanks to the reformulation of Lemma 3.1 in terms of critical groups as in Lemma 3.2, which gives finer information of u_0. On the other hand, if nondegeneracy is assumed, the existence of at least four nontrivial solutions is obtained, as pointed out in [15].

As an application of the estimates of Section 2 and the previous proposition, we have the following result about existence of sign-changing solutions.

Theorem C. *Let f satisfy (f1) and (f3). Let $\epsilon > 0$, $A > 0$, $D > 0$, and $B > 0$ as in Theorem* A. *Suppose, in addition, that*

$$f'(t) < \lambda_k \;\; \forall t \in [-B, B]. \tag{55}$$

Then, there exists at least one sign-changing solution u_* of (1) such that

$$\|u_*\|_{L^\infty(\Omega)} > B.$$

Proof. Because of (f1), part (a) in Proposition B implies the existence of a solution u_* of (1) whose augmented Morse index is at least k. From Proposition 2.2 the result follows. □

In the remaining part of this section we consider the limit case of condition (f1) mentioned in the Introduction. From now on, we assume that f satisfies, besides (f1), condition (f2) there exist $\gamma > 0$ such that $f'(t) \leq \gamma < \lambda_{k+1}$ for all $t \in \mathbb{R}$,

where k is as in (f1). As we mentioned above, since $f \in C^1$, condition (f1) forces the existence of a $j \geq k+1$ so that $f'(t) \leq \omega < \lambda_j$ for all t. So (f1)-(f2) is simply the case $j = k+1$.

In [16], A. Castro and A. Lazer considered the case in which the range of f' crosses exactly one eigenvalue. In [8], f' is supposed to cross several eigenvalues, gaining the existence of more solutions. Let us recall a result on multiplicity of solutions to (1) proved in [8].

Theorem 4.1. ([8]) *If f satisfies (f1)-(f4), then, problem (1) has at least five solutions. Moreover, exactly one of the following cases holds:*

(a) k is even and problem (1) has two sign-changing solutions.
(b) k is even and problem (1) has six solutions, three of which have the same sign.
(c) k is odd and problem (1) has two sign-changing solutions.
(d) k is odd and problem (1) has three solutions of the same sign.

Let us denote by $u_i, i = 1, \cdots, 5$ the solutions of (1) given by Theorem 4.1. The solution $u_1 \equiv 0$ is an isolated local minimum of J. Using the Mountain Pass Theorem (see [29]) A. Castro and J. Cossio proved the existence of a positive solution u_2 and a negative solution u_3. The solution u_4 comes from an application of the Lyapunov-Schmidt reduction method (see Lemma 3.3), and the solution u_5 is obtained by using Leray-Schauder degree arguments.

The following proposition is, somehow, a big remark after a careful reading of the proof of Theorem 4.1 in [8] and partially complements it by means of the use of Mountain Pass Theorem and Morse index invariance under the Lyapunov-Schmidt reduction method. On the other hand, it opens some interesting questions.

Proposition 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy (f1)-(f4) where $k \geq 4$ is even. Suppose that the critical points of J are nondegenerate. Then:*

(a) $m(J, u_4) = k$.
(b) $m(J, u_5)$ is even and $m(J, u_5) \leq k$.
(c) If $m(J, u_5) = k$, there exist two additional solutions u_6 and u_7 of (1). Moreover $m(J, u_6) = k-1$.
(d) If u_4 has one sign, there also exist two additional solutions u_6 and u_7 of (1).

Proof. As in the proof of Theorem 4.1, $m_a(J, u_4) = k$. Hence, (a) follows from the nondegeneracy assumption. Let us prove (b). The inequality $m(J, u_5) \leq k$ follows from hypothesis (f2) and Proposition 2.1.

In the case u_4 changes sign, a review of the proof of Theorem 4.1 makes it clear that u_5 satisfies $d_{loc}(\nabla J, u_5) = (-1)^k = 1$. On the other hand, the nondegeneracy assumption implies $d_{loc}(\nabla J, u_5) = (-1)^{m(J,u_5)}$ (see [25]). Then, $m(J, u_5)$ is even in this case. To complete the proof of (b) we consider the case in which u_4 has one sign. As in the proof of Theorem 4.1, there exists a one sign solution u_6. Moreover, the existence property of the Leray-Schauder degree and the nondegeneracy assumption imply the existence of solutions u_5 and u_7 such that $d_{loc}(\nabla J, u_5) = 1 = d_{loc}(\nabla J, u_7)$ (see [25]). Again, $m(J, u_5)$ is even and the proof of (b) is complete. Also, as a by-product, we have proved (d).

Finally we prove (c). Suppose $m(J, u_5) = k$. Writing u_5 as $u_5 = x_5 + \psi(x_5)$, Lemma 3.3 guarantees $m(\hat{J}, x_5) = k$, so x_5 is a local maximum of $\hat{J}$. Thus, x_4 and x_5 are points of local minima of $-\hat{J}$. Now we apply the Mountain Pass Theorem to

$-\hat{J}$. Directly from its definition, $\hat{J}$ satisfies (PS) condition since J does it so. The Mountain Pass Theorem (see [24], [29]) implies the existence of a critical point x_6 of mountain pass type for $-\hat{J}$. Since it is nondegenerate, $m(-\hat{J}, x_6) = 1$ (see [24]). Hence, $m(\hat{J}, x_6) = k - 1 \geq 3$. Again, because of Lemma 3.3, $u_6 = x_6 + \psi(x_6)$ is a critical point of J whose Morse index is $k - 1$.

Now we use a degree counting to obtain u_7. Let S a sub-region of $B_R(0)$ such that $\overline{S} \cap K = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Then, using $d_{loc}(\nabla J, u_5) = (-1)^k = 1$ and $d_{loc}(\nabla J, u_6) = (-1)^{k-1} = -1$, we have

$$\begin{aligned}(-1)^k &= d(\nabla J, B_R(0), 0)\\ &= d(\nabla J, S, 0) + d(\nabla J, B_R(0) \setminus \overline{S}, 0)\\ &= 1 - 1 - 1 + (-1)^k + (-1)^k + (-1)^{k-1} + d(\nabla J, B_R(0) \setminus \overline{S}, 0).\end{aligned}$$

Consequently, $d(\nabla J, B_R(0) \setminus \overline{S}, 0) = 1 \neq 0$ and existence property of the Leray-Schauder degree implies that of u_7. We have proved (c) and the result follows. $\square$

Remarks:

1. A number of questions naturally arise from Proposition 4.1. To mention just a couple: what can be said about (b)-(d) when k is odd? and, when k is still even, how to make true the assumption in part (c)?

2. Parts (c) and (d) in Proposition 4.1 are not true when $k = 2$. Suppose f satisfies (f1)-(f4),

(f5) $tf''(t) > 0$, for $t \neq 0$,

and (51) with $k = 2$. In this particular case, problem (1) has exactly five solutions, all of which are nondegenerate. This is a consequence of some of the results contained in [10] and Proposition 2.1. A similar result is contained in [16] when the range of f' crosses one eigenvalue.

3. Another natural thing to ask is whether u_4 changes sign or not. We give a partial answer in the following result. It is analogous to Theorem C, but now in the setting of the limit case given by hypothesis (f2). Its proof has three key ingredients: the characterization of u_4 and u_5 in the proof of Theorem 4.1, the invariance of augmented Morse index under the Lyapunov-Schmidt reduction method in the form of Lemma 3.3, and the estimates of Section 2.

Theorem D. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy (f1)-(f4). Let $\epsilon > 0$, $A > 0$, $D > 0$, and $B > 0$ as in Theorem A. If, in addition,*

$$f'(t) < \lambda_k, \quad \forall t \in [-B, B], \tag{56}$$

then (1) has two sign-changing solutions, u_* and v_*. Moreover, at least one of them, let us say u_*, satisfies

$$\|u_*\|_{L^\infty(\Omega)} > B.$$

Proof. Hypotheses (f1)-(f4) allow the use of those arguments leading to Theorem 4.1. We use the same notations. First, we observe that Lemma 3.3 implies $m_a(J, u_4) = m_a(\hat{J}, x_4)$. Since x_4 is a local maximum of a functional defined on a k-dimensional space, $m_a(J, u_4) = k$. Proposition 2.2 implies $u_4 := u_*$ changes sign. As in the proof of Theorem 4.1, solution $v_* := u_5$ also changes sign when $u_* = u_4$ does it so. The result follows. $\square$

In the resonance setting, we use the results of [27], the estimates of Section 2 and Lemma 3.3, to prove the following theorem also stated in the introduction.

Theorem E. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy (f1')-(f4). Suppose, in addition, f satisfies*
(f5) $\frac{1}{2}\lambda_k t^2 - F(t) \to -\infty$ *as* $|t| \to \infty$,
and

$$f'(t) < \lambda_k, \quad \forall t \in [-B, B]. \tag{57}$$

Then, there exist a sign-changing solution u_* of (1) such that

$$\|u_*\|_{L^\infty(\Omega)} > B.$$

Proof. Because of the results of [27], problem (1) also has at least five solutions under assumptions (f1')-(f5), as in the non-resonant case of [8]. One of those solutions, let us say u_*, is built up as $u_* = x_* + \psi(x_*)$, where $\psi : X \longrightarrow Y$ is as in Lemma 3.3, and $x_* \in X$ is such that $\hat{J}(x_*) = \max_X \hat{J}$. Consequently, by virtue of the invariance of the Morse index under Lyapunov-Schmidt reduction method, $m_a(J, u_*) = m_a(\hat{J}, x_*) = k$. Then, from Proposition 2.2 the proof follows. ☐

Remark:
If Ω is a ball or an annulus centered at 0 and $f \in C^2(\mathbb{R})$ satisfies the hypotheses in Theorem C for $k \leq N$, then, $m(J, u_*) \leq k$ and this solution changes sign. On the other hand, from the results of A. Aftalion and F. Pacella in [2], every radial sign-changing solution u to (1) satisfies $m(J, u) \geq N + 1$. Thus:

Corollary 4.1. *Let Ω be a ball or an annulus centered at 0. Let $f \in C^2$ satisfy (f1), (f3) and (51) with $k \leq N$. Then, solution u_* in Theorem C is not radially symmetric.*

Under hypothesis (f2), every solution of (1) has Morse index less than or equal to k. So, the analogue of the previous corollary in reduction setting is:

Corollary 4.2. *Let Ω be a ball or an annulus centered at 0. let $f \in C^2$ satisfy (f1)-(f4) and (57) with $k \leq N$. Then, solutions u_* and v_* in Theorem D are not radially symmetric.*

Of course, an analogous statement holds in the resonant case considered above.

REFERENCES

[1] R. A. Adams, "Sobolev Spaces," New York, AcademicPress, 1975.

[2] A. Aftalion and F. Pacella, *Qualitative propertiesof nodal solutions of semilinear elliptic equations in radiallysymmetric domains*, C. R. Math. Acad. Sci. Paris, **339** (2004), 339–344.

[3] T. Bartsch, Z. Liu and T. Weth, *Sign-changing solutions ofsuperlinear Schrödinger equations*, Comm. Partial Diff. Eq., **29** (2004), 25–42.

[4] T. Bartsch, K. C. Chang and Z. Q. Wang, *On the Morse indices of sign changing solutions of nonlinear ellipticproblems*, Math. Z., **233** (2000), 655–677.

[5] T. Bartsch and Z. Q. Wang, *On the existence of sign changing solutions for semilinear Dirichlet problems*, Topol. Methods Nonlinear Anal., **7** (1996), 115–131.

[6] T. Bartsch and T. Weth, *A note on additional properties ofsign-changing solutions to superlinear elliptic equations*, Topol. Methods Nonlinear Anal., **22** (2003), 1–14.

[7] H. Brezis and R. E. L. Turner, *On a class of superlinearelliptic problems*, Comm. in Partial Diff. Equations, **2** (1977), 601–614.

[8] A. Castro and J. Cossio, *Multiple solutions for a nonlinear Dirichlet problem*, SIAM J. Math. Anal., **25** (1994), 1554–1561.

[9] A. Castro, J. Cossio, and J. M. Neuberger, *A sign changing solution for a superlinear Dirichlet problem*, Rocky Mountain J. M., **27** (1997), 1041–1053.

[10] A. Castro, J. Cossio and J. M. Neuberger, *A minmax principle,index of the critical point, and existence of sign-changing solutions to elliptic boundary value problems*, Electronic Journal Diff. Eqns., **1998** (1998), 1–18.

[11] A. Castro, J. Cossio and C. Vélez, *Existence of seven solutions for an asymptotically linear Dirichlet problem*, Annali di Mat. Pura et Appl., to appear.

[12] A. Castro, P. Drabek and J. M. Neuberger, *Asign-changing solution for a superlinear Dirichlet problem*, 101–107 (electronic), Electron. J. Diff. Eqns., Conf. 10, Southwest Texas State Univ., San Marcos, TX, 2003.

[13] K. C. Chang, "Infinite Dimensional Morse Theory and MultipleSolution Problems," Birkhäuser, Boston, 1993.

[14] J. Cossio and S. Herrón, *Nontrivial Solutions for asemilinear Dirichlet problem with nonlinearity crossing multipleeigenvalues*, Journal of Dynamics and Differential Equations, **16** (2004), 795–803.

[15] J. Cossio, S. Herrón and C. Vélez, *Existence of solutions for an asymptotically linearDirichlet problem via Lazer-Solimini results*, Nonlinear Anal., **71** (2009), 66–71.

[16] A. Castro and A. C. Lazer, *Critical point theory and the number of solutions of a nonlinear Dirichlet problem*, Ann. Mat. Pura Appl., **70** (1979), 113–137.

[17] K. Chang, S. Li and J. Liu, *Remarks on multiple solutions for asymptotically linear elliptic boundary value problems*, Topol. Methods in Nonlinear Anal., **3** (1994), 179–187.

[18] J. Cossio and C. Vélez, *Soluciones no triviales para un problema de Dirichlet asintóticamente lineal*, Rev. Colombiana Mat., **37** (2003), 25–36.

[19] E. N. Dancer, *Counterexamples to some conjectures on the number of solutions of nonlinear equations*, Mathematische Annalen, **272** (1985), 421–440.

[20] D. De Figueiredo, P. L. Lions and R. D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures et Appl., **61** (1982), 41–63.

[21] B. Gidas, W. M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., **68** (1979), 209–243.

[22] V. Guillemin and A. Pollack, "Differential Topology," New York, NY, Prentice-Hall, 1974.

[23] D. Gilbart and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer Verlag, Berlin 1977.

[24] H. Hofer, *The topological degree at a critical point of mountain pass type*, Proc. Sympos. Pure Math., **45** (1986), 501–509.

[25] S. Kesavan, "Nonlinear Functional Analysis (A First Course)," Text and readings in mathematics 28. Hindustan Book Agency, India, 2004.

[26] A. C. Lazer and S. Solimini, *Nontrivial solutions of operator equations and Morse indices of critical points of min-max type*, Nonlinear Anal., **12** (1988), 761–775.

[27] S. Liu, *Remarks on multiple solutions for elliptic resonant problems*, J. Math. Anal. Appl., **336** (2007), 498–505.

[28] Z. Liu, F. A. van Heerden, and Z. Q. Wang, *Nodal type bound states of Schrödinger equations via invariant sets and minimax methods*, Journal of Differential Equations, **214** (2005), 358–390.

[29] P. H. Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations," Regional Conference Series in Mathematics, number 65, AMS, Providence, R.I., 1986.

[30] X. F. Yang, *Nodal sets and Morse indices of solutions of super-linear elliptic PDEs*, Journal of Functional Analysis, **160** (1998), 223–253.

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E-mail address: castro@math.hmc.edu
E-mail address: jcossio@unal.edu.co
E-mail address: cauvelez@unal.edu.co