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Introduction

Locating the roots of nonlinear functions has been an active area of study for centuries. Among the myriad of solution approaches which have been developed for general nonlinear functions, one of the better known and often used methods is Newton's method (see, for example, [2]). In fact, some currently available and widely used software packages base their polynomial root-finding algorithms upon Newton's method (*e.g.*, CMLIB routines RPZERO or CPZERO [1]).

Recall that Newton's method for finding a root of a nonlinear function, $f(x)$, is an iterative process in which a sequence of approximations to the root is generated based on tangent line approximations to the function $f(x)$. Newton's formula for the $(N+1)^{st}$ iterate in the sequence is given by

$$x_{N+1} = x_N - \frac{f(x_N)}{f'(x_N)}. \tag{1}$$

The goals of this paper are twofold: first to reveal the interestingly dichotomous behavior of Newton's Method when applied to a quadratic versus a cubic polynomial, and second to highlight a fascinating and surprising feature of Newton's formula when applied to cubic polynomials, in particular. We will describe a way in which some a priori knowledge will allow us to use only *one* step of formula (1) to locate a root of a cubic polynomial, and more generally, to locate a point of intersection between a cubic polynomial and any other polynomial of degree three or less.

For the interested reader, a related study can be found in [6], in which the author explores graphical approaches to locating roots of polynomials of degree four or less. In [3], a geometric interpretation of the solution of a quartic is investigated. In [4], a method is discussed which obtains an expression for two roots of a cubic in terms of a third root. And in [5], a very nice overview of both historical and recent algorithmic approaches to solving for polynomial roots is provided, along with an extensive bibliography.

Newton's Mishaps

When Newton's method converges it converges rapidly (see, for example, [2]) for discussion of convergence properties). However, as any undergraduate numerical analysis student knows, there are cases in which Newton's method will diverge.

One illustrative example of divergence involves using Newton's formula on a quadratic function. Suppose, for example, we apply Newton's method to $q(x) = x^2 - 2x - 8$ (Figure 1).

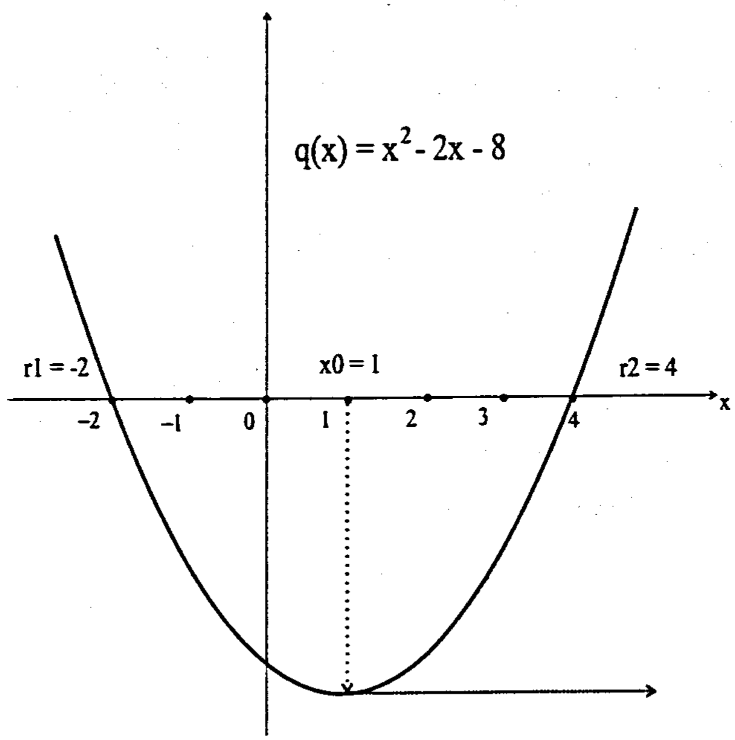

Figure 1: Newton diverging on a quadratic polynomial.

In this case we know that the roots are $r_1 = -2$ and $r_2 = 4$. If we happen to choose our initial guess to be the average of the two roots ($x_0 = 1$), then the tangent line passing through $(1, q(1))$ has zero slope, *i.e.*, $q'(1) = 0$. By equation (1), then, x_1 is not defined, and Newton's method diverges.

It is, in fact, straightforward to show that when x_0 is the average of the two roots of any quadratic polynomial $q(x)$, $q(x_0)$ will always be a local maximum or minimum. Specifically, we introduce:

Observation 1: Suppose $q(x)$ is a quadratic polynomial of the form

$$q(x) = a_2x^2 + a_1x + a_0,$$

where a_2, a_1 and a_0 are real coefficients, and $a_2 \neq 0$. Suppose r_1 and r_2 are the roots of $q(x)$. Then if

$$x_0 = (r_1 + r_2)/2,$$

$q(x_0)$ is a local maximum or minimum.

Why is Observation 1 True? Standard calculus techniques can be employed to provide a convincing argument that Observation 1 is true. We suggest this as an appropriate exercise for the student.

It follows that the choice of the midpoint of two roots of a quadratic as the initial guess in Newton's method will always cause the method on the quadratic to diverge.

Newton's Victories

Interestingly, if we now apply Newton's method to searching for the roots of a cubic polynomial $p(x)$, as opposed to a quadratic polynomial, the effect of choosing initial guess x_0 equal to the average of two distinct roots is now quite the opposite. Not only is $p'(x_0)$ well defined, but the real surprise is that exactly *one* step of Newton's method, in this case, will immediately produce the third root of the cubic polynomial $p(x)$. In particular we have:

Observation 2: Suppose $p(x)$ is a cubic polynomial of the form

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0,$$

where a_3, a_2, a_1 and a_0 are real coefficients, and $a_3 \neq 0$. Suppose r_1, r_2 and r_3 are the roots of $p(x)$, and $r_1 \neq r_2$. Then exactly *one* step of Newton's method applied to $p(x)$ with initial guess $x_0 = (r_1 + r_2)/2$ produces the third root, $x_1 = r_3$.

Why is Observation 2 True? Factor $p(x)$:

$$p(x) = a_3(x - r_1)(x - r_2)(x - r_3).$$

Now let

$$q(x) = a_3(x - r_1)(x - r_2).$$

Then we can write:

$$p(x) = q(x)(x - r_3). \tag{2}$$

Take the first derivative of $p(x)$:

$$p'(x) = q'(x)(x - r_3) + q(x). \tag{3}$$

Set

$$x_0 = (r_1 + r_2)/2. \tag{4}$$

Apply one step of Newton's method to $p(x)$ with initial guess x_0. We then have

$$\begin{aligned}
x_1 &= x_0 - p(x_0)/p'(x_0) && \text{from (1)}\\
&= x_0 - q(x_0)(x_0 - r_3)/\left(q'(x_0)(x - r_3) + q(x_0)\right) && \text{from (2) and (3)}\\
&= x_0 - q(x_0)(x_0 - r_3)/q(x_0) && q'(x_0) = 0 \text{ by (4) and Observation 1}\\
&= r_3 && \text{by algebraic simplification.}
\end{aligned}$$

We emphasize that Observation 2 holds whenever x_0 is chosen to be the average of *any* two distinct roots out of three. In the special case that all roots are real and distinct, the process can be pictured geometrically. In Figure 2 we illustrate such a case, and how each of the choices for x_0 ($x_0 = (r_1 + r_2)/2$, $x_0 = (r_2 + r_3)/2$, $x_0 = (r_1 + r_3)/2$) reveals the third root (r_3, r_1, or r_2, respectively) in precisely one Newton step.

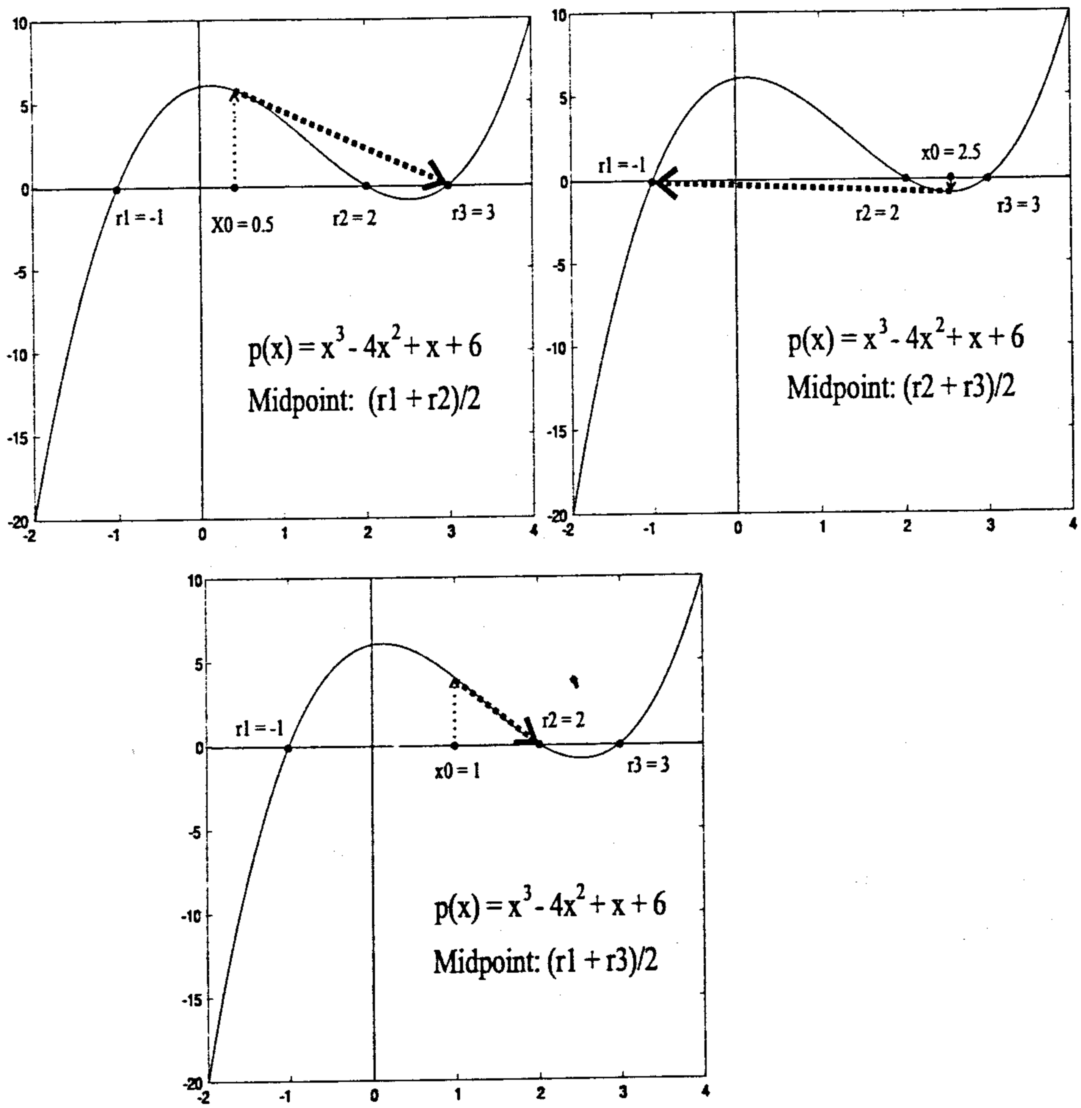

Figure 2: Finding the third root in one Newton step.

But Wait – There's More!

This interesting property of Newton's formula can be employed to find more than just polynomial roots. We can also use Newton's formula to discover points of intersection between two polynomials. In particular, we have

Observation 3: Suppose $p(x)$ is a cubic polynomial of the form

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0,$$

where a_3, a_2, a_1 and a_0 are real coefficients, and $a_3 \neq 0$. Suppose $w(x)$ is a polynomial of degree three or less, of the form

$$w(x) = b_3x^3 + b_2x^2 + b_1x + b_0,$$

where b_3, b_2, b_1 and b_0 are real coefficients, and $b_3 \neq a_3$. Then

1. There are always three points of intersection, r_1, r_2 and r_3 between $p(x)$ and $w(x)$ (that is, $p(r_i) = w(r_i)$ for $i = 1, 2, 3$). The points of intersection need not be distinct.
2. If $r_1 \neq r_2$ are two distinct points of intersection between $p(x)$ and $w(x)$, then exactly one step of Newton's method applied to $(p(x) - w(x))$ with initial guess $x_0 = (r_1 + r_2)/2$ produces the third point of intersection, $x_1 = r_3$.

Why is Observation 3 True? Let

$$g(x) \coloneqq p(x) - w(x). \tag{5}$$

Since $p(x)$ is a cubic polynomial, and $w(x)$ is at most a cubic whose leading term does not match that of $p(x)$, then $g(x)$ is a third degree polynomial with real coefficients.

Part 1: Since $g(x)$ is a cubic, $g(x)$ has three roots, r_1, r_2 and r_3 (*i.e.*, $g(r_i) = 0$ for $i = 1, 2, 3$). By definition (5), the three roots of $g(x)$ are the three points of intersection between $p(x)$ and $w(x)$. This proves part 1.

Part 2: The two distinct points of intersection r_1 and r_2 between $p(x)$ and $w(x)$ are also two distinct roots of cubic polynomial $g(x)$, by (5). By Observation 2, one step of Newton's method applied to $g(x)$ with initial guess $x_0 = (r_1 + r_2)/2$ produces the third root of $g(x)$, $x_1 = r_3$. By definition, the third root of $g(x)$, r_3, is also the third point of intersection between $p(x)$ and $w(x)$. Part 2 has been proved.

Let us look at an example. Suppose we have cubic

$$p(x) = x^3 - 4x^2 + 3x + 1,$$

and quadratic

$$q(x) = x^2 + x - 7.$$

Suppose we already know that

$$p(2) = q(2) = -1$$
$$p(4) = q(4) = 13$$

It is not immediately obvious that the third point of intersection can be easily found. However, using Observation 3, we can very quickly calculate the third point of intersection in one application of Newton's formula. Setting

$$g(x) = p(x) - q(x) = x^3 - 5x^2 + 2x + 8,$$

and choosing $x_0 = 3$, we have:

$$x_1 = 3 - g(3)/g'(3) = -1.$$

We confirm that $x_1 = -1$ is in fact a third point of intersection between $p(x)$ and $q(x)$:

$$p(-1) = q(-1) = -7.$$

Conclusion

These interesting relationships between Newton's method and cubic polynomials are so structurally elegant and clear to picture, that they could be incorporated into any undergraduate lecture relating to Newton's method of root-finding. This also opens the door to further speculation: Can these observations be generalized, and in which direction? Although our observations stand on their own as far as being amusing and a bit surprising, it is likely not the end of the story, but in fact a piece in a much larger, and as yet undiscovered, puzzle.

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