1-1-1984

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Recommended Citation

A. Castro and R. Shivaji. "Uniqueness of positive solutions for a class of elliptic boundary value problems," Proc. of the Royal Soc. of Edinburgh, 98A (1984), pp. 267-269.

Proceedings of the Royal Society of Edinburgh, **98A**, 267–269, 1984

Uniqueness of positive solutions for a class of elliptic boundary value problems

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(MS received 22 December 1983)

Synopsis

Uniqueness of non-negative solutions conjectured in an earlier paper by Shivaji is proved. Our methods are independent of those of that paper, where the problem was considered only in a ball. Further, our results apply to a wider class of nonlinearities.

1. Introduction

Here we study the uniqueness of non-negative solutions to the boundary value problem

$$\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{for} \quad x \in \Omega, \\ u &= 0 \quad \text{for} \quad x \in \partial\Omega, \end{aligned} \tag{1.1$_\lambda$}$$

where $\lambda > 0$ and Ω is a bounded region in R^n with smooth boundary.

We assume that $f(0) > 0$ and that for some positive constant C and some $d > 1{\cdot}5$,

$$0 \leqq f'(u) \leqq C(1+u)^{-d} \tag{1.2}$$

for all $u \geqq 0$. Our main result is:

THEOREM 1.1. *There exists λ_0 such that for $\lambda \geqq \lambda_0$, the problem $(1.1)_\lambda$ has a unique non-negative solution.*

Our proof depends on the maximum principle for subharmonic functions and trace inequalities (see [**1**]). Theorem 1.1 proves a conjecture raised in [**4**] where the result was proved when Ω is a ball and $d = 2$.

2. Proof of Theorem 1.1

As is well documented (see [**4**]), for each $\lambda > 0$, the problem $(1.1)_\lambda$ has a minimal and a maximal positive solution. Let them be u_λ and v_λ respectively. Let $w_\lambda = v_\lambda - u_\lambda \geqq 0$. Hence, by the mean value theorem, there exist continuous functions $\theta_\lambda : \Omega \to R$ such that $u_\lambda \leqq \theta_\lambda \leqq v_\lambda$ and

$$\begin{aligned} -\Delta w_\lambda &= \lambda f'(\theta_\lambda) w_\lambda \quad \text{for} \quad x \in \Omega, \\ w_\lambda &= 0 \quad \text{for} \quad x \in \partial\Omega. \end{aligned} \tag{2.1$_\lambda$}$$

Let $z : \Omega \to R$ denote the solution of $-\Delta z = 1$ in $\Omega, z = 0$ on $\partial\Omega$. By the maximum principle, it follows that $\lambda f(0)z$ is a subsolution to $(1.1)_\lambda$ and λMz is a supersolution to $(1.1)_\lambda$, where $M=\sup\{f(t);\ t\in R\}$. Since, by Hopf's maximum principle, z is positive in Ω and the inward normal derivative of z at $\partial\Omega$ is positive, there exists constants $c_1>0$ and $c_2>0$ such that

$$\lambda sc_1 \leqq \lambda z(x)f(0) \leqq u_\lambda(x) \leqq v_\lambda(x) \leqq \lambda Mz(x) \leqq \lambda c_2 s, \tag{2.2}$$

where s is the distance from x to $\partial\Omega$.

Suppose Theorem 1.1 is not true. Hence, there exists a sequence $\lambda_n \to \infty$ such that $w_{\lambda_n} \neq 0$. Let $z_n = w_{\lambda_n}/|w_{\lambda_n}|_{2,2}$, where $|\ |_{2,2}$ denotes the norm in $H^{2,2}(\Omega)$ (see [**1**]). Let $\alpha>0$ be such that if $x, y\in\partial\Omega$, ξ and η are the unit outward normals at x and y respectively, then $\{x+s\xi;\ 0\leqq s\leqq\alpha\}\cap\{y+s\eta;\ 0\leqq s\leqq\alpha\}=\emptyset$ (see [**5**, p. 69]). For $0<\beta\leqq\alpha$, we set

$$\Omega_\beta = \Omega - \{x+s\xi;\ x\in\partial\Omega,\ 0\leqq s\leqq\beta,\ \xi \text{ as above}\}. \tag{2.3}$$

Since the boundary of the Ω_β's is parallel to the boundary of Ω, by the Trace Theorem (see [**1**]), there exists a constant c_3 such that

$$|z_n|_\beta \leqq c_3, \tag{2.4}$$

where $|\ |_\beta$ denotes the norm in $H^{3/2,2}(\partial\Omega_\beta)$.

Now we establish an estimate for the L^2-norm of the z_n's on $D(\beta)=\Omega-\Omega_\beta$ for $0<\beta\leqq\alpha$. Let c_4 be a uniform bound for the Jacobian of the transformation $(x_1,\ldots,x_n)\to(y, s)$ with $y\in\partial\Omega$ and s the component in the unit inward normal direction at y. Hence

$$\int_{D(\beta)} z_n^2\, dx_1 \ldots dx_n \leqq c_4\int_0^\beta \int z_n^2(y, s)\, dy\, ds, \tag{2.5}$$

where y ranges in $\partial\Omega$. By the fundamental theorem of calculus and (2.5), we have

$$\begin{aligned}\int_{D(\beta)} z_n^2\, dx &\leqq c_4\int_0^\beta\int_{\partial\Omega}\left(\int_0^s (\partial z_n/\partial\xi(y,t))\, dt\right)^2 dy\, ds\\ &\leqq c_4\int_0^\beta s\int_0^s\int_{\partial\Omega}(\partial z_n/\partial\xi)^2\, dy\, dt\, ds\\ &\leqq c_3^2 c_4\beta^3/3.\end{aligned} \tag{2.6}$$

Let $\varepsilon<\delta<0$ and $\lambda_n\equiv\lambda$ be large enough so that $\lambda^\delta<\alpha$. By (1.2) and (2.2), we see that there exists a positive constant c_5 such that $|f'(\theta_\lambda(y+s\xi))|\leqq c_5(\lambda s)^{-d}$ for all $y\in\partial\Omega$, $0\leqq s\leqq\alpha$. This and (2.6) give

$$\begin{aligned}\int_Z |f'(\theta_\lambda)z_n|^2\, dx &\leqq c_5\lambda^{-2d(1+\varepsilon)}\int_{D(\lambda^\delta)} z_n^2\, dx\\ &\leqq c_3^2 c_4 c_5 \lambda^{-2d(1+\varepsilon)+3\delta},\end{aligned} \tag{2.7}$$

where $Z\equiv D(\lambda^\delta)-D(\lambda^\varepsilon)$. Also from (1.2) and (2.2), we have

$$\begin{aligned}\int_{\Omega_{\lambda^\delta}} |f'(\theta_\lambda)z_n|^2\, dx &\leqq c_5\lambda^{-2d(1+\delta)}\int_\Omega z_n^2\, dx\\ &\leqq c_5\lambda^{-2d(1+\delta)},\end{aligned} \tag{2.8}$$

where we have used the fact that $|z_n|_{2,2}=1$. Hence, on combining (2.6), (2.7) and (2.8), we see that there is a constant $M>0$ such that

$$\int_\Omega |f'(\theta_\lambda)z_n|^2\,dx \leqq M(\lambda^{3\varepsilon}+\lambda^{-2d(1+\varepsilon)+3\delta}+\lambda^{-2d(1+\delta)}). \tag{2.9}$$

By choosing $\varepsilon=-(4d^2+6d)/(4d^2+6d+9)$, $\delta=-4d^2/(4d^2+6d+9)$, we have

$$|f'(\theta_\lambda)z_n|_{L^2(\Omega)} \leqq (3M)^{\frac{1}{2}}\lambda^{3\varepsilon/2}. \tag{2.10}$$

Thus, by *a priori* estimates for elliptic equations (see [**2**]), we have $|z_n|_{2,2}\leqq M_1\lambda^{1+(3\varepsilon/2)}$. By our choice of ε and since $d>\frac{3}{4}$ (see 1.2), we have $1+3\varepsilon/2<0$ which contradicts the fact that $|z_n|_{2,2}=1$ for all n, and the theorem is proved.

It is easy to see that Theorem 1.1 can be extended to more general situations such as: (i) $-\Delta$ can be replaced by any elliptic operator satisfying the maximum principle, and (ii) $f(u)$ can be replaced by $f(x,u)$ with an adequate change in (1.2).

3. Applications

The problem

$$\begin{aligned} -\Delta u &= \lambda \exp\{\alpha u/[\alpha+u]\} && \text{for} \quad x\in\Omega, \\ u &= 0 && \text{for} \quad x\in\partial\Omega, \end{aligned} \tag{3.1$_\lambda$}$$

with $\alpha>0$, arises in the theory of combustion. It was proved in [**3**] that there is a range of λ for which $(3.1)_\lambda$ has at least three non-negative solutions for large values of α. Our results prove that this range is bounded.

References

1 R. A. Adams. *Sobolev Spaces* (New York: Academic Press, 1975).

2 S. Agmon, A. Douglis and L. Nirenberg. Estimates near the boundary of solutions of elliptic partial differential equations satisfying general boundary conditions, I. *Comm. Pure Appl. Math.* **12** (1959), 623–727.

3 K. J. Brown, M. M. Ibrahim and R. Shivaji. S-shaped bifurcation curves. *Nonlinear Anal.* **5** (1981), 475–486.

4 R. Shivaji. Uniqueness results for a class of positone problems. *Nonlinear Anal.* **7** (1983), 223–230.

5 V. Guillemin and A. Pollack. *Differential Topology* (New Jersey: Prentice Hall, 1974).

(*Issued* 25 *October* 1984)