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Man's Cards and God's Dice: A Conceptual Analysis of Probability for the Advanced Student

Elie Feder
City University of New York, Kingsborough Community College

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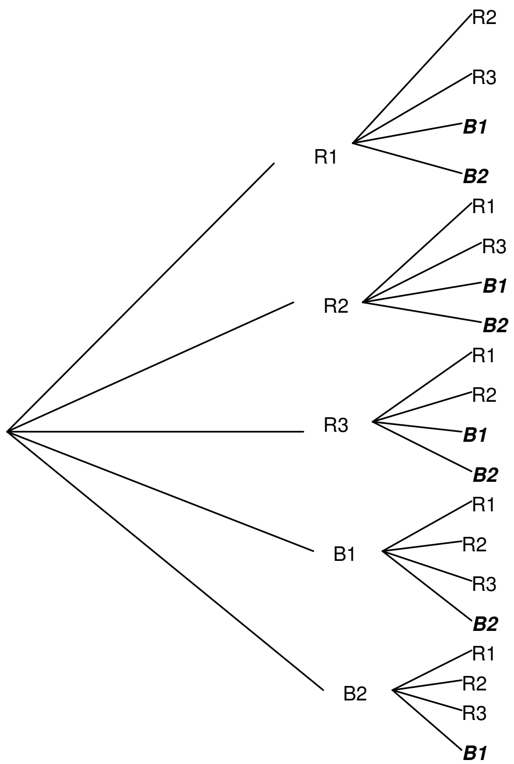
A Conceptual Analysis of Probability for the Advanced Student

Elie Feder
efeder@kbcc.cuny.edu
Kingsborough Community College

Probability is a field of mathematics which truly has no equal. Its underlying conceptual basis is of a qualitatively differentiated nature from that of all other branches of mathematics. I was prompted to realize this lesson by a probing question from one of my advanced students. The answer to her question reveals a new world of insight into the foundations of probability theory. This insight provides the basis for understanding the famous Einstein-Bohr debate regarding the assumptions of quantum mechanics.

My journey began when teaching "mathematics for liberal arts majors". An advanced student was puzzled by an exercise in probability, and concluded "This is why probability doesn't make any sense." The exercise was as follows: You are presented with a pile of cards containing 3 red cards and 2 black cards. If you pick one card, the probability of getting a black is 2/5. Let's say you pick two cards. What is the probability of getting a black card in the second pick? The answer is again 2/5. The student asked me "But does it not depend upon what you picked in the first pick? If you chose black, then the probability is now ¼. If you chose red, the probability is now ¾. How could you just ignore the first pick?" At first, I had two responses, both of which did not satisfy her. Firstly, I reasoned "You would be correct if we knew what happened on the first pick. But since we do not know, we cannot factor that in. How would you suggest that we factor in an unknown result?" She answered, "I don't know, but how could you just ignore the fact that there was a first pick?" Realizing that this approach was not going to satisfy her, I went on to a second, more concrete approach. I illustrated the sample space with a tree diagram, as shown below. (next page)



The three red cards are denoted R1, R2, and R3 while the two black cards are denoted B1 and B2. Notice that 8 of the 20 possibilities have black as the second card. These are in italics. Thus, the probability of picking black on the second pick is 8/20 = 2/5.

We see that 20 different possibilities emerge, 8 of which have black as the second card. This yields a probability for black on the second pick of 8/20 = 2/5. She agreed that this was the correct answer, but was still bothered why it should be so. In frustration she said "this type of problem is why I never understood probability". Apparently, none of her previous teachers had been able to assist her. Instead of just dismissing her, I decided to analyze her problem and to try to resolve her difficulty. Firstly, I felt that this is my duty as a teacher. Secondly, I realized that I may learn something about probability in the process. This is usually the way things work; good teaching promotes good learning.

When approaching the problem, my first realization was that there is a theorem which allows us to calculate the answer to this problem theoretically, without the use of a cumbersome tree diagram. This theorem, which is essential in the study of Bayesian probability, is as follows:

> <u>Theorem</u>: Let S be a sample space. If B_1, $B_2,\ldots,B_n$ are mutually exclusive events such that $S = B_1 \cup B_2 \cup \cdots \cup B_n$ and $P(B_i) > 0$ for all $i=1,2,\ldots n$. Then for any event A of S, we have $P(A)= P(A|B_1)\cdot P(B_1) + P(A|B_2)\cdot P(B_2) + \cdots + P(A|B_n)\cdot P(B_n)$.

To apply this theorem to our problem, let B_1= the event of picking a red card on the first pick, and B_2 = the event of picking a black card on the first pick. We want to compute P(A), where A = the event of picking a black card on the second pick. We have

$$\begin{aligned} P(A) &= P(A|B_1)\cdot P(B_1) + P(A|B_2)\cdot P(B_2) \\ &= (2/4)\cdot(3/5) + (1/4)\cdot(2/5) \\ &= 6/20 + 2/20 \\ &= 8/20 \\ &= 2/5. \end{aligned}$$

This result is, of course, in line with our answer above. However, this theorem did not lie within the scope of the course. Since I wanted to satisfy the student's desire to understand probability, I did not think that quoting a theorem and illustrating its application would suffice. She was looking for the underlying sense of the solution, not merely the mathematical tool used to arrive at it. I, therefore, decided to think about the problem logically, without making recourse to this theorem.

When thinking about the issue, I realized that her question goes to the heart of probability. Her question could be strengthened as follows: If I know what happened on the first pick (say, black was chosen) then I will say the probability of black on the second pick is ¼. If somebody else doesn't know what happened on the first pick, they will say the probability is 2/5. Who is correct? Am I mathematically wrong because I didn't know what happened on the first pick? It would seem that is not the case. But, is there no *right* answer? Where do you find a mathematical problem where the answer depends upon the knowledge of the person to whom the problem is presented? For instance, sin30 = ½ whether or not I know what a 30-60-90 triangle looks like. I may not know the answer, but it is a mathematical "truth". In probability, however, it seems that there is no such "truth". This demands an explanation. What is the precise uniqueness of probability which my student sensed, but could not define?

Allow us to investigate the underlying nature of probability by means of a simple example. Let us assume we spin a fair die. The probability that it will land on 1 is 1/6. Let us investigate further. Assume that we knew the precise position of the die at the moment it was released, and we knew the precise amount of spin that the person put onto the die, and we knew the precise speed of the wind currents present at the time,… Of course we cannot possibly determine all of these factors, but if we could then we would say that the die will definitely land on a 1 (let us leave the modern results of quantum physics on the side, for the time being). There would be no room for probability. What then is the place for probability, since these conditions are, after all, fixed despite our lack of knowledge of them? It would seem that the answer is that probability is based upon the fact that we do not know all of these factors; it is based upon our lack of knowledge. Thus in studying probability, as opposed to in other areas of science and mathematics, we are taking our lack of knowledge as a given and asking for the best prediction based upon what we do know. Since we lack knowledge of all the causative factors in spinning the die, we say that the probability of a 1 is 1/6. Therefore, probability can best be described as the mathematics of incomplete, or lack of knowledge.

Given this explanation, we can now solve our problem with the cards. How can we say that the probability of a black card on the second pick is 2/5 when the second pick is dependent upon the first pick? The answer is that this is analogous to saying that the probability of a die resulting in 1 is 1/6 despite the fact that the result is dependent upon many factors. The reason why we could say this is because since we don't have knowledge of all those factors, we ignore them and determine the probability based upon what we do know. The same is true with our cards. Since we don't know the result of the first pick, we must ignore it and analyze the possibilities for the second pick. Since there are five possibilities, two of which are black, we can say the probability of black is 2/5. This also explains how one who knows the result of the first pick will get a different probability for the second pick than one who lacks this knowledge. Who is correct? We can explain as follows: every probability question has an assumed, unspoken introduction: “Given that we know such and such, what is the probability of…” Thus, if we know what happened on the first pick, then we are being asked a different question then if we did not know about the result of the first pick. If we had to answer who is correct, we would say “to which question?” The two people are being asked two different questions and they are both giving a correct answer to their respective question. This is only possible because of the unique nature of probability. In trigonometry we are looking for the reality of the sine of a given angle. This has nothing to do with the knowledge of the observer. His limitations do not affect the true answer. This is the case in all areas of math and science. In probability however, the truth is that there is no such thing as a reality to probability. The premise of understanding probability is recognizing that it is based upon an observer who has intrinsic limitations in his ability to isolate all of the factors involved in determining the outcome of a given experiment.

Based upon this understanding of probability, we can provide a significant backdrop for the study of conditional probability. Since we see that any probability question is based upon what we know and what we do not know, we can frame questions assuming certain pieces of knowledge. We can ask, what is the probability of event A occurring, given that we know that event B occurred. These are the types of problems which come up in Bayesian probability. We have shown, however, that a similar type of analysis goes into every question in probability. For instance, when we spin a die, we have no knowledge of any of the contributing factors and therefore consider it as if all of the possibilities are equally likely. Similarly, we must always consider which factors we know and which are beyond our knowledge. It is only after we have this clarified, that we can proceed towards a solution.

With this insight into the underlying nature of probability, we can perhaps shed light upon a famous debate which took place in the beginning of the twentieth century. It was between the great scientists Albert Einstein and Neils Bohr regarding the results of quantum physics. Experiments indicated that if one would send a photon through a slit onto a photographic plate, then its path was indeterminate. Where does the particle go? The answer of quantum physics was that the particles path will be determined by a probability function. It will go here with probability x, there with probability y, and so on. This was a new type of scientific explanation. Previously, science would answer questions by providing a definite effect for a given cause. Here, science was saying that the effect is a probability, nothing definite. Einstein could not accept this. He responded with his famous quip, “God doesn’t play dice with the universe. “ Bohr responded back, “Don’t tell God what to do.” On the surface it is difficult to understand Einstein’s point. Bohr seems decidedly correct. Does Einstein have an intimate knowledge of “God” that he could say how He runs the universe? One mark of Albert Einstein’s character was his great humility and he obviously would not make such a claim. However, I believe that we are in a position to gain insight into Einstein’s position. He was not claiming to have a deep knowledge of God, but of probability. He understood that the nature of probability was a science based upon an observer’s lack of knowledge of causative factors in a given experiment. This being the case, “from God’s eyes” there could be no such thing as probability. When we are trying to identify the true causative factors of a given experiment, there’s no room for probability, but for exact scientific principles. What quantum was suggesting is that God plays dice with the universe- that there is new type of probability which is woven into the very fabric of the universe. This is a new application of probability which was never before envisioned. Einstein could not accept this. Bohr, however insisted that despite the difficulty of such a proposition, the evidence pointed to its truth and God could certainly create a world which is in fact based upon an intrinsic probability. Modern day scientists favor the results of quantum physics, but I believe that based upon our conceptual understanding of probability, we can have a deeper understanding of the contention of the great Albert Einstein.

After analyzing my student's question, I responded to her and to the rest of the class with the above explanation. I conveyed to the class the underlying nature of probability and its uniqueness. My previously puzzled student accepted the answer and her frustration with probability came to an end. It is encouraging how the most basic questions can lead us to uncover the fundamentals upon which mathematics is built. I believe that this insight into the underlying nature of probability must be kept in mind whenever we study this unique subject. It will "probably" lead us to a deeper understanding of many other examples in probability.