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Maria Mannone

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cARTegory Theory: Framing Aesthetics of Mathematics

Maria Mannone[1]

Palermo, ITALY
manno012@umn.edu

Synopsis

Mathematics can help investigate hidden patterns and structures in music and visual arts. Also, math in and of itself possesses an intrinsic beauty. We can explore such a specific beauty through the comparison of objects and processes in math with objects and processes in the arts. Recent experimental studies investigate the aesthetics of mathematical proofs compared to those of music. We can contextualize these studies within the framework of category theory applied to the arts (cARTegory theory), thanks to the helpfulness of categories for the analysis of transformations and transformations of transformations. This approach can be effective for the pedagogy of mathematics, mathematical music theory, and STEAM.

Keywords: *proof; elegance; crossmodal correspondences; gestural similarity; categories*

1. Introduction: The Math of Art, the Art of Math

If we leaf through Byrne's edition of the Euclid's *Elements*, beautifully illustrated, we first think of some Mondrian artworks [3]. Even if the result may appear as a collection of abstract paintings, Byrne's intention was merely pedagogical: he aimed to make a visual aid for learning, an enjoyable tool to diminish the effort and shorten the time spent by students to study the *Elements*.

[1] Maria Mannone is an Italian theoretical physicist and a composer. She achieved her Ph.D. in Composition at the University of Minnesota.

Art can be helpful in teaching mathematics through examples that include visuals, music, and story-telling. A famous story helps us imagine being lifted to higher geometric dimensions, as in the case of *Flatland*, a story that also appears as a social satire through an enjoyable geometric metaphor [1].

Mathematics can use resources from figurative and abstract visual art: we could also start teaching mathematical concepts from a collection of contemporary Fine Arts works. Even some *conceptual* and *abstract* art can *embody* mathematical concepts [35], possibly independently from the artist's intention.

Art can also help students get rid of their fear of mathematics. This is helpful in the practice of teaching. During a psychology-of-mathematics experiment, researchers asked prospective math teachers and elementary school students to represent mathematics through a metaphor, an animal, or a monster [25]. The answers embodied both students' and prospective teachers' fears. After training, the outcome was less frightening, and probably, for the subjects involved in the experiment, future teaching will be more effective.

From elementary education to university classes, mathematics is sometimes loved and sometimes feared. According to some scholars, there is a need for a new way to convey mathematical content, beyond the formalism and the rigor [32]. According to [29], sometimes a (temporary) lack of rigor can be helpful if we are dealing with a cutting-edge concept, method, or frontier of research. Arts, and even humor, can be a way to convey passion about the field and surprise for mathematical methods, results, and way of thinking, as illustrated by the stand-up mathematician Matt Parker during his public lectures.

The aspects of entertainment, enjoyment, comparison of arts and nature can stimulate students' learning, and help them see the world in a new light. A classic book about *Symmetry* takes the ground from beautiful shapes and patterns in nature and the arts [40]. In this book, the author, Hermann Weyl, recalls the image of the Triskelion and Monreale mosaics in Sicily.[2] These sources inspired Escher, the celebrated artist who spent several years in Southern Italy, and whose woodcuts illustrate several math books and have given inspiration to generations of mathematical artists.

[2] As pointed out by Weyl, mathematicians were familiar with the Triskelion because it was the symbol of the journal *Rendiconti del Circolo Matematico di Palermo.*

Connections between art and mathematics are not only limited to *objects*; they also include the *transformations* of some visual or musical material. Transformational processes in the context of the arts are discussed by the composer Salvatore Sciarrino [36]. Even if the mathematical reference is limited to simple operations such as addition or multiplication, we can investigate and formalize these examples in the framework of categories. Category theory is a branch of mathematics born to formalize generic transformations and transformations between transformations. It has also been applied to music, including to the description of musical gestures [27, 9, 8].

The concept of transformation can be framed in mathematical discourse, as well as in discourse on a theory of arts. The connecting ideas are also suitable for applications in mathematical pedagogy: the math within arts and the art within math allow us to teach math starting from the arts. The pedagogy of mathematics can be related to education in mathematical music theory. In this field, one of the main challenges is accessibility [41]: in fact, the popularization of topics — and their connection with other areas — can be a helpful tool to overcome difficulties generated by the need of specialized knowledge and high skills to access math-musical interdisciplinary research [30, 41]. With a well-organized popularization strategy, a broader audience can get acquainted with both math and music. The same applies to the more general pedagogy of STEAM, where the abstract power of categories can help connecting objects and processes from different research areas.

Moreover, the employment of games, by using adequate tools, can also facilitate math learning and, at the same time, it can constitute a source and an opportunity for enjoyment. The well-known Rubik's cube was invented as a teaching tool to visualize and introduce group theory. A musical application of Rubik's cube, the CubeHarmonic,[3] can be used as a teaching tool in the context of mathematical music theory, joining, in a tangible device, group theory and chord permutations [23].

In mathematical research, aesthetic enjoyment and investigation play a relevant role. According to Poincaré, aesthetics of mathematical ideas may

[3] The CubeHarmonic was first thought of by Maria Mannone in 2013 during her studies of mathematics and music in Paris, and first published in a math-music textbook [28]. The CubeHarmonic is currently under development as a physical device and a real musical instrument, in collaboration with the Tohoku University in Japan [24].

lead to more productive results [33, 37]. Elegance and aesthetic qualities in mathematics are often emphasized [7]. In the same way, the importance of aesthetics in the "making" of mathematics is highlighted [6]. Aesthetic judgments may be relevant also in the search for new proofs of existing theorems [12], preferring *the most beautiful and simplest* proofs [26], and appreciating the aesthetic pleasure of a *perfect* solution [34]. As it happens for the arts, the style of mathematical proofs, too, can depend on the historical period and on the specific personalities of scientists. For example, we find reciprocal influences between Archimedes' mathematical work and the Hellenistic art style [31].

In this article I consider a recent study [10, 11] on the aesthetic psychology of mathematics within the framework of category theory applied to the arts. I use the framework of current research on cARTegory theory [19, 20], that is, the application of basic concepts of category theory to the arts, to include theoretical concepts and to connect elements of music and visual arts. Some examples of application involve the categorical description of a spherical inflorescence [20] and of the pangolin's armor sonification.[4] See Figure 1 for the pangolin and Figure 2 for the flowers. According to this approach, which constitutes a development of the mathematical theory of musical gestures, the gesture is seen as a common generator of both images and sounds and helps us define *gestural similarity*.[5] Recent research on mathematical theory of musical gestures also involves knots, networks, and tensor categories [22], because, as said above, mathematics can describe elements of the arts, and, vice versa, the arts can help describe characteristics and methods of math.

[4] The pangolin is a particular mammal with keratin scale armor.

[5] Let us summarize what *gestural similarity* is [19]. In music, a crescendo can be seen as a transformation between two loudness levels, and an accelerando transforms a slower crescendo into a faster crescendo. Two musical gestures are defined as *similar* if the performer's movements have visual analogies in their respective spaces — e.g., a hitting of the piano in a forte, a strong hitting in timpani playing, an intense air emission for trumpet — that induce similar transformations in their respective sound spectra — e.g., a 'forte' loudness/timbre effect for piano, timpani, and trumpet. A visual shape and a musical fragment are similar if their gestural generators are similar — e.g., a detached movement in piano playing and a detached movement of a pencil making points on a paper — and their effect present crossmodal correspondences — e.g., a collection of staccato notes and isolated dots. Two gestures can be similar concerning some components, and not for others. We can also define a *degree* of similarity, for example as a value comprised within $[0, 1]$.

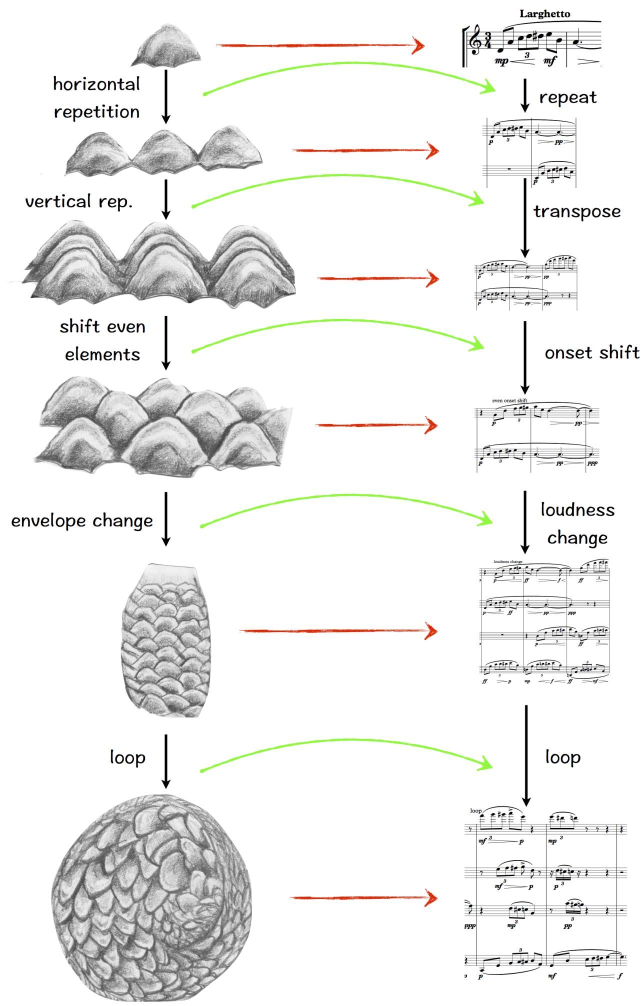

Figure 1: The pangolin is a mammal that, when it is scared, curls up into an armored ball. In the image, the figure of the closed pangolin is reconstructed starting from a single scale and following a sequence of transformations. Each black arrow in the left half of the image corresponds to a morphism in the category "scales" or, better, "visual fragments". In the right half of the image, there is a sequence of musical fragments and musical transformations, which belong to the category of musical fragments. Each musical fragment is derived from visuals. The sequence of black (musical) arrows represents a sequence of musical transformations, which are the musical transposition of visual transformations discussed before. Red arrows transform scales and their combinations into musical fragments, and green arrows transform visual morphisms into musical morphisms. Thus, red and green arrows describe the action of a functor, that is, the *sonification functor* [23]. Drawings and music by M. Mannone.

Category theory, thanks to its abstraction power, seems particularly suitable to describe and compare artistic methods. Let us focus on music: while the mathematical theory of music aims to mathematically describe forms, structures, and practice of music, we may use music, as well as visual arts, to capture some essential mechanisms in mathematics. Also, we can recursively use mathematics to describe the artistic aspect that is inside mathematics itself. This would lead us to both the "Math of Art" and the "Art of Math"!

This article is organized in the following way. In Section 2, I summarize the main points of recent research on mathematics and aesthetics [10, 11]. In this framework I introduce the categorical contextualization that is the focus of the article, referring to some elements of Gestalt analysis of music and visual arts. Section 3 concludes the paper with a view toward further research, both from the mathematical as well as musical point of view.

2. This proof . . . sounds good!

A *category* is given by objects and morphisms between them [17]. In diagrams, objects are represented by points and morphisms by arrows. Morphisms verify the associative and identity properties. Morphisms between two categories connect objects of the first category with objects of the second category and morphisms of the first category with morphisms of the second category. Morphisms between categories are called *functors*. If we have two functors F and G, a natural transformation is defined from F to G; see Figure 2 for a natural — and naturalistic — depiction.[6] Categories may help connect mathematics with other disciplines, and to compare their methods, styles, maybe also aesthetics. But . . . is it possible to *quantitatively* approach aesthetics in mathematics?

In [10, 11], Samuel Johnson and Stefan Steinerberger tried to determine if there might be some quantitative explanation behind statements such as "this theorem is elegant" *et similia*. The authors first picked four mathematical proofs, easy enough to be understood and also appreciated by non-

[6] In category theory, a *natural transformation* allows the comparison between the action of a functor and the action of another functor. As an intuitive example, we can look at the different shapes of the inflorescences in Figure 2. The natural transformation α transforms the action of S into the action of H: $\alpha : S \to H$; in the flower example, it transforms the spherical shape into the ellipsoidal shape.

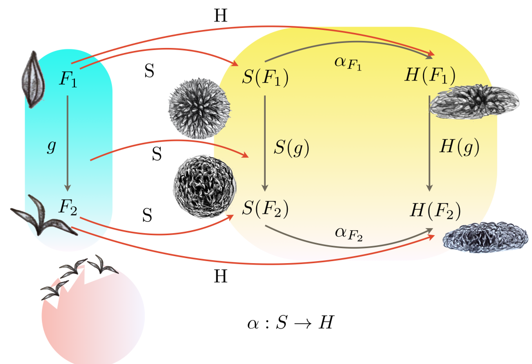

Figure 2: A floral representation of functors and natural transformations [20]. This is inspired by the spherical inflorescences of the Southern globe thistle, *Echinops Ritro*. Functor S brings single (simplified) flowers (objects of the category *single flowers*) into spheres of flowers as in the product flower × sphere (objects of the category *spheres of flowers*). Functor H creates an imaginary variation, an ellipsoidal inflorescence. Natural transformation α accounts for the shape difference between spherical and ellipsoidal inflorescences. Drawings by M. Mannone.

mathematicians. Then, they asked the participants to establish a relationship and assign a grade of similarity between each mathematical proof and the beginning of a short musical piece selected among four choices, and moreover between each mathematical proof and a painting selected among four choices. Experimental material included piano pieces of different genre, and landscape images. Landscapes are not traditionally compared with mathematics, and this set to zero any bias for the experiment. The *similarity degree* values ranged from a minimum of 0 to a maximum of 10.

Data analysis revealed significant correlations between specific proof styles and music styles, and between proof styles and painting styles, with stronger correlations between proofs and musical pieces. Experimental subjects were professional mathematicians, math students, and non-specialists. The authors found analogies between the answers of mathematicians and math students; however, also the results given by laymen-data analysis revealed significant correspondences between proof-styles and artwork-styles.

Using abstract artwork might lead to an interesting extension of the experiments above, as well as the comparison with "atomic elements" — in the sense of "essential elements" — taken from visuals and musical pieces, such as an isolated chord sequence, or unexpected articulation change.

The research by Johnson and Steinerberger [10, 11] aims to find similarities between the structure and style of a mathematical proof, a musical piece, and a painting. This process of comparison can be analyzed in light of the categorical formalism I have been working on in recent works [20, 22]. Let us consider Figure 3.

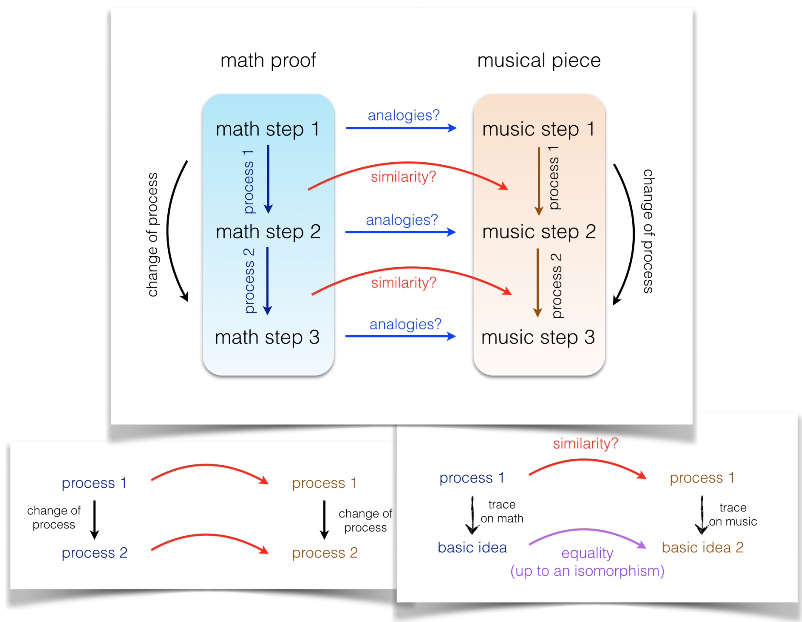

Figure 3: A categorical depiction of steps and transformations in a mathematical proof, steps and transformations within a musical piece, and their connection as well as possible structural similarities.

On the left, we have a simple three-step mathematical proof. On the right, we have a three-step musical fragment. In the mathematical proof, each step is connected with the following one through transformations called *process 1* and *process 2*, respectively. If these two transformations are identical, then the arrow *change of process* will be an identity. On the musical side, the steps might correspond to musical sequences, musical chords, some musical elements that are transformed the one into the other through a musical *process 1* and *process 2*. Also in this case, if these two transformations are identical, the arrow *change of process* will be an identity.

Because associative and identity properties are verified,[7] mathematical steps and processes belong to the category *math proof*, while musical fragments and their transformations belong to the category *musical piece*. It is worth mentioning that we are not considering an entire musical piece, but parts of it, though. Blue and red arrows represent the action of an imaginary functor of "structure translation" from math processes to musical processes. Blue arrows represent the translation of objects, while the red arrows the translation of processes. While translating processes, we can investigate about their *similarity* of mechanisms. We can compare the same mathematical proof with musical sequences, ideally defining different functors, and the differences between these functors can be formalized by natural transformations.

Data collection and analysis conducted by Johnson and Steinerberger can be framed in terms of natural transformations. The authors evaluated the similarity degree between mathematical and musical processes. We could make an equivalent diagram by using 'abstract' visual transformations, or some transformations extracted from an existing painting. Mathematical objects and musical objects, connected by blue arrows, may or may not present analogies. However, this is not fundamental for our research, because we are here more focused on transformational processes.

In the lower part of Figure 3, we see the arrows "processes" as points, and, on the bottom right, we have a metaphorical exemplification of the *partial trace* concept from quantum physics [2]. Let us consider a state that describes the configuration of a system and its environment. If we perform the operation of trace with respect to the environment, we are left with the system only, and vice versa. Let us suppose that we can perform an operation equivalent to the trace operation on the processes of Figure 3. If we "trace" a mathematical process on math, we get the basic idea of the process; at the same time, if we "trace" a musical process on music, we get the basic idea of that process, too.[8]

[7] Combining mathematical steps we get mathematical steps and mathematical objects, thus objects and morphisms are still in the same category, and the associative property is verified. The identity process is defined; the same is true for musical transformations.

[8] We can see such a "trace operation" as an extraction of essential information, as a filtering. If our object of study is a process, a "trace" of it gives the essential information about the given process: this is a "filtered process", obtained via some "filtering process". In this way, the comparison between different processes can be reduced to an analysis and comparison of their filtered versions.

If we have a high degree of similarity between the considered mathematical process and the considered musical process, we would expect to have an equality of their basic ideas. Or, more realistically, an equality up to an isomorphism.[9] By "tracing" on the specific medium — visual arts, music, or a mathematical proof — we can find *essential elements* of sequences of transformations, and we can compare them, finding their similarities. This would constitute a theoretical framework to contextualize data analysis in [10, 11]. In [11], the authors explicitly talk about *similarity* with music. A musical piece can be seen as a process, as an entity developing in time. However, an image is not usually seen as an entity in time. If we "scan" an image, we explore it through time, thus we add a temporal dimension. More broadly, think here of the "making" of an image, with the creation of a whole starting from one element, as a collection of steps to paper-and-pencil drawing an image. Once we have such a building process, we can translate it into music.[10]

In Figure 4, the mechanism described in Figure 3 is applied to a simple case. On the left is a categorical representation of the proof scheme for the sum of infinite geometric series through half-cutting a square. This is the first example provided in [11]. On the right is a sequence of musical duration values, from a whole note to a quarter note. On the left, we are in the category of mathematical objects; on the right, we are in the category of music. The similarity of processes is evident, with a musical equivalent of "taking the half" applied to musical duration. When we consider processes as points, and we perform a trace-like operation with respect to math or music respectively, we lose all but the main, essential idea of "taking half". We could make a corresponding analysis with visual arts: for example, a representation of the complete human body that is little by little reduced to the representation of isolated parts, or the repetition of the same shape whose size is taken progressively smaller by half.

This can be connected with the concept of musical gesture in the following manner: a shorter note requires a shorter permanence of the hand on the keyboard or an earlier interruption of the air stream for a wind player.

[9] The expression "up to an isomorphism" is quite used in category theory, where the notion of equality is weakened in favor of a more flexible, and general, concept.

[10] See the musical rendition of the pangolin, Figure 1.

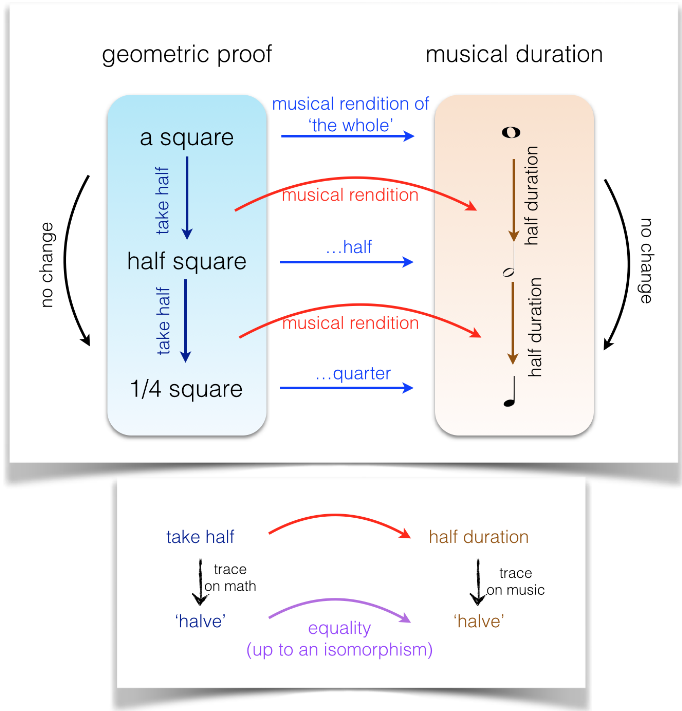

Figure 4: An application of the abstract diagrams shown in Figure 3.

In mathematics, a geometrical transformation might be seen as the result of a movement, an action. This is a concept already present in Gilles Châtelet's work [4]. *Taking half of a square* makes us thinking of the action of half-folding a paper, or cutting it into two equal parts.

The analogies between the beauty that is inside a mathematical structure and within a piece of art stimulate more deep questions about cognition, in particular about Gestalt (meaning "shape, figure, form" in German). Gestalt psychology is a theory of cognition aiming to explain how we can perceive a whole as more than the sum of its parts. (See [13] for one of the foundational texts.) There are several studies in the field of shape perception of a musical piece or of a visual artwork, and in particular, about musical versions of Gestalt laws [16]. Translations from music to visual arts (visualization) and vice versa (sonification) can be framed in a categorical context (see Figure 1) and be the object of a joint mathematical-cognitive research [21, 22].

Such a formalization would especially regard the similarity between initial and final objects, and the possibility to keep an "essential Gestalt" during such a translational process. We can wonder if an artwork could be "Gestalt-invariant", i.e., if its shape/main aesthetic content could remain unchanged during a medium translation. This implies that the main shape and aesthetic content of an artwork could be independent of the specific media chosen: a piano piece, a fresco painting, or a black and white picture. We can make music out of a snail shell, a mathematical conchoid: the music reproduces the "getting closer" of the shell's spirals, through shortening of silence and approaching of the pitch.[11] In this sense, we are keeping the *main idea* of the conchoid from the domain of visuals to the domain of sounds, through an opportune mapping of its mathematical definition via a 3-component parametric equation [18, 28]. We can explain the eventual analogies between mathematical steps and visual steps through crossmodal correspondences [38] and the identification of audiovisual objects [14], as well as gestural similarity [19]. We can build networks of similar images and sounds and investigate their connections [21].

If we shift our attention from visual/auditory objects to visual/auditory transformations, in categorical terms we are focusing not on points but on arrows. Investigating similarities between processes means examining transformations of transformations. In this way, we can analyze the style of a mathematical proof. It means finding how some given mathematical objects are modified from the beginning to the end of a proof, and how the arrows differ, e.g., how a process between two steps differs from a process between other two steps. Because transformational processes in arts are relevant for the definition of style — let us think of the theme treatment in a Beethoven's symphony or in an Italian opera aria — we can investigate the Gestalt not only of objects but also of morphisms.

We can wonder if the style of a mathematical proof can be labeled with some sort of *Mathematical Gestalt*. If this is true, we can use basic concepts of category theory to describe Gestalt translations between artistic media and to investigate the Gestalt of a proof.

[11] We can have different musical renditions of the same snail-shell according to the choice of mapping or of the composition style, however, an effective musical rendition of the same snail's shell should evoke the described effect of progressively approaching of points.

In mathematics, the concept of the envelope is well-defined. This might be related to Gestalt: we see some elements, but we also perceive the overall shape they belong to, or they create. In the Gestalt examples, the contour can be recreated by the mind: in the Kanizsa triangle, there is no real white triangle, but our mind builds it obeying to the law of the "good shape". In visual arts, we can have a collection of small images that, taken together, constitute a new image: this is the case of Arcimboldo portraits, for example. In music, we can have micro-meaningful unities that, when overall considered, constitute a clearly-identifiable structure. In the mathematical theory of musical gestures, the concept of *gesture of gestures* is an equivalent of this [27], and it has been extended to visual arts and higher categories [22]. In a mathematical proof, we can distinguish between the detail of every single step, of every single transformation/process between two steps, and the overall structure constituted by the whole sequence of transformations. This might be seen as an equivalent of the concept of the envelope, and it may be helpful to identify some stylistic features within more structured and complex proofs. A higher-level Gestalt, that is, some hypergestural structure in a mathematical proof-style, easily envisaged in higher-categories, may be an object of future research.

3. Conclusion and further research

This article started from a recent study in the field of aesthetics of mathematics, attempting to give a quantitative meaning to qualitative statements such as "this is an elegant mathematical proof" [11]. The authors therein quantitatively compared mathematicians', students', and laymen's judgements about similarity between simple mathematical proofs, musical pieces, and paintings [10]. Here I contextualized this research within a categorical framework that had been applied to musical performance theory [19, 22], and to music/visual arts comparisons [22, 21]. In fact, the *stylistic* similarities between mathematical proofs and artistic materials can be envisaged not in the "matter" — mathematical objects are usually not directly related with musical themes, or landscape features — but instead in their transformational processes. In this sense, we can approach the structure of a musical piece or of a painting through the lenses of mathematical processes, as suggested by Salvatore

Sciarrino[12] [36]. The formalism that naturally allows us to compare transformations, and thus to investigate transformations of transformations, is category theory. To illustrate this, I began first by framing a general structure (Figure 3), and then offering a very simple application (Figure 4), in order to contextualize a comparison between a generic mathematical proof and a sequence of generic musical elements within the diagrammatic thinking. My goal throughout has been to initiate a deeper discussion between categorical thinking and cognition.

Future research may involve more advanced categorical concepts through intuitive musical examples. A possible starting point for new studies is the categorical notion of subobject classifier and truth values [17]. In the case of sets, we say that the object a belongs to the subgroup A' of set A by attributing a truth value 1 (true) or 0 (false). In music: *Is this musical gesture a subobject of "staccato"?* A characteristic function gives the answer. When we deal with degrees of similarities, our truth set extends from $\{0, 1\}$ to the whole interval $[0, 1]$. We can ask *how far is an object* a from the objects of A'? and we can define a distance, and *how much does the object* a belongs to A'?, and we can define a fuzzy inclusion. We need classifiers and also degrees of inclusion, that are easily defined in a machine learning framework. This would constitute a development of pioneering research in category theory applied to machine learning [5].

Further theoretical developments would involve the use of fractals with their self-similarity, which can be expressed through loop arrows in a categorical framework, as well as with the formalism of attractors and fixed points [15, 39].

Other developments may involve composition and cognition experiments. We can compose a musical piece based on a mathematical proof. Provocatively, we may wonder about some "Beethoven's Fifth Theorem" or "Fermat's Last Symphony". We can take a "good" or "beautiful" proof, and we can make a musical piece out of it, by using the sequence of transformations as a list of composition techniques and instructions. Then, we can ask non-specialists what do they think of it. We can ask a group of composers to write musical

[12] Sciarrino's musical works are well-known worldwide, but unfortunately this is not true for his book *Le Figure della Musica*, that has never been translated into English, and it was not reprinted.

pieces based upon specific proof styles. Then, we can quantitatively measure the (expected) similarities between pieces derived from the same theorem, and we can ask listeners, selected among professional mathematicians, math students, and laymen, to match proofs/musical pieces and to assign them a degree of similarity, e.g., a real number between 0 and 1, or a natural number between 0 and 10 as in former experiments. Finally, the obtained results and the developed research methods can be used as teaching materials for mathematics and music classes.

Of course, hidden mathematical beauty cannot be completely unrevealed through simple experiments or measures. However, this work can help even more appreciate the potentiality of math to describe art, and also the richness of art hidden in mathematics.

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