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Cover Page Footnote

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From Solvability to Formal Decidability: Revisiting Hilbert's "Non-Ignorabimus"

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Abstract

The topic of this article is Hilbert's axiom of solvability, that is, his conviction of the solvability of every mathematical problem by means of a finite number of operations. The question of solvability is commonly identified with the decision problem. Given this identification, there is not the slightest doubt that Hilbert's conviction was falsified by Gödel's proof and by the negative results for the decision problem. On the other hand, Gödel's theorems do offer a solution, albeit a negative one, in the form of an impossibility proof. In this sense, Hilbert's optimism may still be justified. Here I argue that Gödel's theorems opened the door to proof theory and to the remarkably successful development of generalized as well as relativized realizations of Hilbert's programs. Thus, the fall of absolute certainty came hand in hand with the rise of partially secure and reliable foundations of mathematical knowledge. Not all was lost and much was gained.

Keywords: *David Hilbert, Ignorabimus, impossibility proof, solvability, decision problem, decidability*

1. Introduction

For a long time, David Hilbert's contribution to the philosophy of mathematics was reduced to his alleged role as advocate of naive formalism, often viewed as a single-minded and ultimately inadequate response to the well-known foundational crisis in mathematics early in the twentieth century.

More than that, Hilbert's philosophy of mathematics was stigmatized as a pure formalistic approach which was nothing more than the manipulation of meaningless symbols akin to a game.¹ This naive picture has been revised by a significant number of excellent studies on Hilbert's foundational programs, and this newer, more comprehensive account has been corroborated by previously unpublished material. Nevertheless, many traces of Hilbert's approaches in philosophy of mathematics are still unnoticed, neglected, and undeveloped. This article aims to deepen our understanding of Hilbert's philosophy of mathematics by focusing on the meaning and role of Hilbert's axiom of solvability, that is, his conviction that every mathematical problem can be solved by means of a finite number of operations.

Hilbert articulated his conviction against the background of the *ignorabimus* ("we will be ignorant") dispute, which is explored in detail in Section 3.² The *ignorabimus* was more than a "foolish phrase" [50, page 963]. It was a serious controversy about the limits of scientific knowledge, with remarkable consequences for Hilbert's metamathematical program. Why remarkable? The formal definition of the decision problem as it is used today in logic and computer science was developed from the vague and informal idea of the solvability of every scientific question in the context of the *ignorabimus* controversy. The aim of this paper is to reconsider Hilbert's axiom of solvability within this historical context.

¹ Hilbert's former student Hermann Weyl has substantially helped to promote the appropriation of Hilbert as a strict formalist when he joined the Brouwerian camp in 1920/21 [85, page 136]. Just for a brief period, Weyl strongly defended intuitionism [84]. More important for his later development was Weyl's readings of Husserl's phenomenology and Fichte's "constructivism" [75].

² In recent decades several books and articles appeared about the *ignorabimus* controversy. A fundamental and substantive work on the theme was carried out by Vidoni, first published in Italian [81] and translated into German three years later [82]. Volume 3 *Der Ignorabimus-Streit* of the German series *Naturwissenschaft, Philosophie und Weltanschauung im 19. Jahrhundert* [4] entails several contributions from different disciplines and perspectives, ranging from physics and biology to medicine and psychology, from the historical roots in the philosophy of Enlightenment to its actuality in the philosophy of mind. The 2016 book *Limits of Knowledge: The Nineteenth-Century Epistemological Debate and Beyond* [1] focuses on the *ignorabimus* from a more analytical and epistemological perspective. The *ignorabimus* dispute in the philosophy and history of mathematics was never exhaustively investigated. So far as I know there are only two articles dealing with the *ignorabimus* regarding the history and philosophy of mathematics [61, 76].

2. Hilbert’s Axiom of Solvability Contextualized

At the Second International Congress of Mathematicians, which was held in Paris in the summer of 1900, David Hilbert presented ten unsolved mathematical problems. Among them were Cantor’s continuum hypothesis (the first problem) and the proof of the consistency, or compatibility, of the arithmetical axioms (the second problem).³ Hilbert’s list became a standing challenge for mathematicians.

“The modernistic flavour of the problems”, Ivor Grattan-Guinness remarked, “lay not only in their unresolved status but also in the high status given to axiomatization in solving or even forming several of them” [39, page 753]. It was in this context, in his 1900 talk (and in the ensuing papers), right before proposing specific problems, that Hilbert made the claim that all well-posed mathematical problems were solvable:

The conviction of the solvability of every mathematical problem is a powerful incentive to our work. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure thinking. For in mathematics there is no *ignorabimus*! [47, page 445].⁴

In 1930, when the biennial meeting of the *German Society of Natural Scientists and Physicians* was held in Königsberg, Hilbert repeated his maxim that for mathematics there was no *ignorabimus*. And again, in his talk “Logic and the Knowledge of Nature”, Hilbert extended his optimism:

³ Because of time constraints Hilbert presented only ten problems at the Paris conference of the International Congress of Mathematicians. A list of twenty-three problems was published in *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* [46]. Hilbert originally included a twenty-fourth problem on his list concerning the simplicity of proofs. This fact was only recently discovered by Rüdiger Thiele from a study of Hilbert’s handwritten notes at the University Library in Göttingen [77, 78]. Numerous further reprints and translations of Hilbert’s famous talk appeared, for example, in French and Russian. For further details see, among others, [39, 40, 89].

⁴ “Diese Ueberzeugung [sic] von der Lösbarkeit eines jeden mathematischen Problems ist uns ein kräftiger Ansporn während der Arbeit; wir hören in uns den steten Zuruf: Da ist das Problem, suche die Lösung. Du kannst sie durch reines Denken finden; denn in der Mathematik giebt [sic] es kein Ignorabimus!” [46, page 262].

For the mathematician there is no *ignorabimus*, nor, in my opinion, for any part of natural sciences. [...] Instead of the foolish *ignorabimus*, our answer is on the contrary: We must know; we will know [32, page 1165].⁵

Hilbert's confidence in the solvability (or at least the proof of the impossibility of a solution) of every mathematical problem follows a long tradition that assigns a privileged epistemological status to mathematical knowledge. In modern philosophy this view is best exemplified by Immanuel Kant, whose *Critique of Pure Reason* opened with this concise statement:

Human reason has the peculiar fate in one species of its cognitions that it is burdened with questions which it cannot dismiss, since they are given to it as problems by the nature of reason itself, but which it also cannot answer, since they transcend every capacity of human reason [41, page 99].⁶

Certain questions, like why there is something rather than nothing, lie beyond our limited capacity for understanding. Following Aristotle, Kant called these questions metaphysical, for within the realm of science there were no limits to knowledge. Paul Bernays, who can be called the “real architect” of proof theory [31, page 132], similarly compared Hilbert's program with Kant's critique: “The task falling to metamathematics vis-à-vis mathematics is analogous to the one which Kant ascribed to the critique of reason vis-à-vis the system of philosophy.”⁷

⁵ “Für den Mathematiker gibt es kein Ignorabimus, und meiner Meinung nach auch für die Naturwissenschaft überhaupt nicht. [...] Statt des törichten Ignorabimus heiße im Gegenteil unsere Lösung: Wir müssen wissen. Wir werden wissen” [50, page 963].

⁶ “Die menschliche Vernunft hat das besondere Schicksal in einer Gattung ihrer Erkenntnisse: daß sie durch Fragen belästigt wird, die sie nicht abweisen kann; denn sie sind ihr durch die Natur der Vernunft selbst aufgegeben, die sie aber auch nicht beantworten kann; denn sie übersteigen alles Vermögen der menschlichen Vernunft” [55, A 1].

⁷ The quotation is from the English translation by Paolo Mancosu, revised by Steve Awodey, Bernd Buldt, Dirk Schlimm, and Wilfried Sieg. The translation is available online via the Bernays Project and can be viewed at <http://www.phil.cmu.edu/projects/bernays/Pdf/philmath.pdf>, last accessed on January 22, 2019. See also [59, pages 234–265]. In German: “Die Aufgabe, welche der Metamathematik gegenüber dem System der Mathematik zufällt, ist analog der, welche Kant der Vernunftkritik gegenüber dem System der Philosophie zugewiesen hat” [8, page 57].

Hilbert’s relationship to Kant has its roots in his close connection to the neo-Kantian philosopher Leonard Nelson, the founding and leading representative of the school named for Jacob Friedrich Fries (1773–1843).⁸ Paul Bernays was one of the co-founders of the society.⁹ Nevertheless, the comparison between Hilbert and Kant should be treated with caution: In Hilbert’s time it was common to adorn oneself with Kantian quotations, and philosophers inspired by Kant were not only engaged in interpreting his words, but also in rejecting his claims. On the other hand, Hilbert’s references to Kant cannot be denied.

The link between Hilbert and Kant makes most sense when viewed through the lens of the *ignorabimus* controversy. Hilbert was certainly familiar with Emil du Bois-Reymond’s legendary talks and the works of his younger brother Paul.¹⁰

3. The Ignorabimus Controversy

In 1872 Emil du Bois-Reymond, an influential physiologist at the University of Berlin and father of modern electrophysiology, closed his address “On the Limits of the Knowledge of Nature” with the pronouncement:

The Bernays Project aims to prepare and publish English translations of Bernays’ papers on the philosophy of mathematics. The originals were written in either German or French. Bernays played a pivotal role in the discussion of foundations of mathematics in the twentieth century. As David Hilbert’s assistant from 1917 into the 1930s, Bernays was actively involved in the development of the finitist conception of mathematics as the basis upon which the consistency of all of mathematics was to be proved [90].

⁸ Volker Peckhaus has documented and analyzed Hilbert’s long standing interest in the philosophy of Leonard Nelson in several articles and contributions [66, 67, 68, 69]. Significant aspects of Nelson’s career are also documented in [72, 44].

⁹ Jakob Friedrich Fries, a corresponding member of the Königlich-Preußische Akademie der Wissenschaften (Royal Prussian Academy of Sciences), played an important role in reinterpreting Kant’s philosophy in light of recent developments in mathematical and physical sciences, especially with respect to the calculus and probability theory [54].

¹⁰ Several letters from Paul du Bois-Reymond to David Hilbert, Felix Klein, Carl Neumann, Moritz Cantor, and others are preserved in the “Handschriftenabteilung” of the Göttingen University Library. Leonard Nelson was well acquainted with Emil und Paul du Bois-Reymond because Nelson’s mother, Elisabeth Nelson, was related to the du Bois-Reymond-family [44].

As regards the riddle of the nature of matter and force and how they are able to think, we must resign ourselves once and for all to the far more difficult verdict: “Ignorabimus.”¹¹

Du Bois-Reymond maintained he could prove — “similar to the work of a mathematician who has proved the unsolvability of a problem” — that natural science could not answer every question and will never be able to do so.¹² He gave three specific examples: the nature of matter and force, the genesis of motion, and the origin of consciousness.

Du Bois-Reymond’s lecture triggered a storm of papers and an overwhelming flood of criticism.¹³ In response he reinforced his position in a speech before the Prussian Academy of Sciences outlining seven riddles, namely: the origin of matter and force, the ultimate nature of matter and force, the origin of motion, the origin of life, the apparently teleological arrangements of nature, the origin of simple sensations and consciousness, the origin of intelligent thought and language, and the question of free will.

¹¹ “Gegenüber dem Räthsel aber, was Materie und Kraft seien, und wie sie zu denken vermögen, muss er ein für allemal zu dem viel schwerer abzugebenden Wahrspruch sich entschliessen: Ignorabimus” [24, page 464]. Du Bois-Reymond’s talk was translated by Joseph Fitzgerald in *The Popular Science Monthly* [25]. So far as I know there is no other English translation of this lecture nor of any of du Bois-Reymond’s other talks. Fitzgerald’s translation does not fulfill the criteria for a good, critical and modern translation. The quoted passages in English are my own translations. Some of them are difficult to translate. For example, “Rätsel” could be translated as “riddle” – “puzzle” – “enigma” – “conundrum” – “marvel”. The German word “Wahrspruch” could be translated as: “doctrine” – “confession” – “verdict”. At this point it is worth saying that common translations of the phrase “wie sie [matter and force] zu denken vermögen” are misleading: “how they can be thought about”, or “how they are to be conceived” should be corrected as “whether they [matter and force] can think”, or “how they [matter and force] are able to think”, or “their ability to think”. The “riddle” facing du Bois-Reymond was the mind-body-problem. There is also a French translation of du Bois-Reymond’s talk [26]. It appeared in *Revue scientifique de la France et de l’étranger* with the title “Les bornes de la philosophie naturelle”. – “Bornes” is the French translation for German “Grenzen”, or English “bounds”, often used synonymously with “limits”, or “frontiers”.

¹² Du Bois-Reymond drew the analogy to the working mathematician in his talk “Goethe und kein Ende”. To quote the passage in German: “wie denn Mathematik eine Aufgabe für bewältigt hält, deren Unlösbarkeit sie bewies” [27, page 164].

¹³ For more details, see Gabriel Finkelstein’s book [33], which traces the development of du Bois-Reymond’s ideas as well as their impact on Germany and Europe.

Concerning the first, the second, and the fifth riddles, he altered his verdict to: “Ignoramus et Dubitemus”.¹⁴ Du Bois-Reymond tried to underpin his position by referring to the impossibility of squaring the circle and achieving perpetual motion [23, page 15].

The same examples can be found in Hilbert’s “Mathematical Problems” (1900), where Hilbert discussed them as follows:

In later mathematics, the question as to the impossibility of certain solutions plays a preeminent part, and we perceive in this way that old and difficult problems, such as the proof of the axiom of parallels, the squaring of the circle, or the solution of equations of the fifth degree by radicals have finally found fully satisfactory and rigorous solutions, although in another sense than that originally intended [47, page 444].¹⁵

For Hilbert an impossibility proof, also called a negative proof, demonstrates that a particular problem cannot be solved (in principle). For example, perpetual motion machines violate either the first or second law of thermodynamics, and circles cannot be squared (as Ferdinand von Lindemann demonstrated in 1882), because the number π is transcendental and only algebraic numbers can be constructed with compass and straightedge. Additionally, Galois had shown that equations of the fifth degree are not solvable by radicals.

¹⁴ “We do not know and we doubt.” At first glance, du Bois-Reymond’s liberal agnosticism implied a gesture of modesty and a plea for tolerance toward different worldviews and toward religion and science. In fact, his argumentation was neither a matter of armchair philosophy nor a purely academic matter, but the object and weapon of politics. He emphasized the integrity of all natural sciences and rejected both the anti-scientific cultural tendencies prevailing among many intellectuals and anti-modernist trends within science, like spiritualism, occultism, and a cult of miracles dulling the mind of the masses. It was no accident that the *ignorabimus* debate arose hand in hand with the famous religious conflict of the new Reich, the “Kulturkampf” against German Catholics [6, 33, 57].

¹⁵ “In der neueren Mathematik spielt die Frage nach der Unmöglichkeit gewisser Lösungen eine hervorragende Rolle und wir nehmen so gewahr, daß alte schwierige Probleme wie der Beweis des Parallelenaxioms, die Quadratur des Kreises oder die Auflösung der Gleichungen 5ten Grades durch Wurzelziehen, wenn auch in anderem als dem ursprünglich gemeinten Sinne, dennoch eine völlig befriedigende und strenge Lösung gefunden haben” [46, 261].

Perhaps the most famous example of a negative / impossibility result Hilbert mentioned involves Euclid's axioms of geometry. From antiquity many attempts had been made to show that Euclid's parallel postulate was a logical consequence of the other axioms. However, early in the nineteenth century Nikolai Ivanovich Lobachevski, János Bolyai, Carl Friedrich Gauss, and Eugenio Beltrami had all come to the conclusion that Euclid's fifth axiom could not be derived from the other four.

All these mathematical problems had been solved. Why, then, were they said to be proofs of the impossible? For example, if the parallel postulate is set aside, Euclidean geometry no longer holds. Taking the impossibility as a challenge offers a way out of the conundrum. As Josiah Royce once remarked in his introduction to Henri Poincaré's *Foundations of Science*:

the circle-squarers and the inventors of devices for perpetual motion are today discredited, not because of any unorthodoxy of their general philosophy of nature, but because their views regarding special facts and processes stand in conflict with certain equally special results of science which themselves admit of very various general theoretical interpretations. [...] The ordinary inventors of the perpetual motion machines still stand in conflict with accepted generalizations; but nobody knows as yet what the final form of the theory of energy will be, nor can any one say precisely what place the phenomena of the radioactive bodies will occupy in that theory [70, page 11].

Emil du Bois-Reymond's *ignorabimus* elicited an enormous response, not only in the German-speaking world, but internationally as well. In his polemic *Free Science and Free Teaching*, Ernst Haeckel accused du Bois-Reymond of a "crusade against scientific freedom" [42, page 78]. Ernst Mach, to quote another example, called du Bois-Reymond's "riddles" meaningless "pseudo-problems" [58, page 14], a notion that was adopted by members of the Vienna Circle, who declared in their manifesto of 1929 that the scientific conception of the world does not admit insoluble riddles [80, page 15]. In this they echoed a famous proposition of Wittgenstein's *Tractatus*: "The riddle does not exist. If a question can be put at all, then it can also be answered" [88, 6.5].

Similarly, Mach's British colleague Karl Pearson dismissed the *ignorabimus* with the words: "the Brothers du Bois-Reymond's first and third Problems and their cry of *Ignorabimus* became meaningless. Matter and force and 'action at a distance' are witch-and-blue-milk problems" [65, page 302].

Today, the younger brother of Emil du Bois-Reymond, David Paul Gustav du Bois-Reymond (1831–1889), is seldom mentioned in the context of the *ignorabimus* controversy. Paul was a well-known mathematician of his time who made significant contributions to real analysis, differential equations, mathematical physics, and the foundations of mathematics.¹⁶ Two works written by Paul du Bois-Reymond deserve particular mention: *General Function Theory* [29], and: *On the Foundations of Knowledge in the Exact Sciences* [30]. The former book is written in the form of a dialogue between two imaginary mathematicians, Idealist and Empiricist (one might also say, Platonist and Constructivist), considering two philosophical approaches to mathematics. The topic of their dispute is the concept and idea of infinity in mathematics (to be more precise, in real analysis), among them questions regarding the limits for bounded, monotonically increasing sequences.

The Idealist claims that the actual infinity really exists and contains within itself an infinite magnitude (quantity). The infinite is viewed as a completed totality. The Empiricist denies the existence of the actual infinity. It cannot be proven, he says, let alone be conceived [29, page 110]. The Empiricist only accepts potential infinity, i. e. a non-terminating, endless process that never actually reaches infinity. Take, for example, the natural numbers: it is always possible to add 1 to the previous number to get an even larger number. Any given number one takes is finite, whereas the series of numbers may be extended indefinitely, for there is always a next number, without bound.

Further, the Empiricist rejects nonconstructive existence proofs that lead to the repudiation of classical logical principles, the most prominent being the *tertium non datur* ("no third is given" — the law of the excluded middle).

¹⁶ It is also worth mentioning that Paul du Bois-Reymond introduced the diagonalization technique in an 1875 article on approximation by infinitesimals [28] long before Georg Cantor presented his first diagonal proof to the public [10]. For more on the relationship between Georg Cantor and Paul du Bois-Reymond see [11, pages 71f.]. For further reading see [34, 43].

Consequently, the Empiricist does not share the Idealist's conception of mathematical existence. According to the principle of non-contradiction, any proposition which is formally inconsistent or which entails a contradiction is impossible (and so is necessarily false), provided that the *tertium non datur* holds. Thus, the consistency of a given system is a sufficient criterion for the existence of mathematical objects and their truths. The Empiricist stands in opposition, insisting that the existence of mathematical objects presupposes their conceivability and constructability.¹⁷ For the Idealist, such a restriction is incomprehensible, being little other than a theological doctrine:

Science, which is to say the embodiment of all aspirations of the human thirst for knowledge, rejects only the obviously wrong. Unrestricted research comes to rest only in the face of the unambiguous truth and abandons only demonstrably absurd problems, like perpetual motion. Science should not condemn the urge to cross the borders of the presently imaginable until necessary limits are drawn for human research. Are there any limits? Does our drive to expand ever reach its bounds?¹⁸

The Empiricist counters: “You blame me for dogmatism, preaching your own world-view at the same time”, which sounds like a Catholic confession, namely:

¹⁷ Much later, Luitzen E. J. Brouwer followed the same line of reasoning. He rejected, firstly, nonconstructive existence proofs, which have been classically used by mathematicians since Euclid, and, secondly, Cantor's actual infinities because of the paradoxes of classes found by Georg Cantor, Cesare Burali-Forti and Bertrand Russell. For Hilbert, Brouwer's rejection of the law of the excluded middle was an unacceptable restriction of mathematics, especially on the newfound mathematical freedom Cantor's set theory offered. Hilbert feared the return of Leopold Kronecker's constraints on mathematics and a trace of the *ignorabimus*.

¹⁸ “Die Wissenschaft aber, d. i. der Inbegriff aller Strebungen des menschlichen Erkenntnistriebes verwirft nur das offenbar Falsche. Die freie Forschung beruhigt sich erst vor der unzweideutigen Wahrheit, und giebt nur nachweislich widersinnige Probleme auf, wie das Perpetuum mobile. Was den Idealismus erzeugt, der Drang, die Grenzen des zur Zeit Vorstellbaren zu überschreiten, kann von der Wissenschaft nicht verurtheilt werden, so lange nicht der menschlichen Forschung ihre nothwendigen Grenzen gezogen sind. Giebt es deren, wird unser Expansionstrieb je seine Schranken erreichen?” [29, pages 149f.]. The English translation is my own.

As an Idealist I believe in the reality of my ideals. I believe in the reality of my ideas, going to the outermost limits of my thought, although they are truly not conceivable, like the infinite. I believe in the infinite and the actual existence of exact geometrical forms.¹⁹

The passages quoted are good examples of fine irony. One is tempted to read them as an allusion to the dispute between Georg Cantor, a devout Lutheran whose explicit Christian beliefs shaped his philosophy of mathematics and natural science [16], and Leopold Kronecker, an early proponent of constructivism and forerunner of Luitzen E. J. Brouwer, who only accepted mathematical objects that could be constructed finitely from the intuitively given set of natural numbers. At this point, Paul du Bois-Reymond emphasized that impossibility proofs allow a definite yes or no answer. By contrast, the debate between Idealist and Empiricist is merely a matter of personal preference, or temperament, like one's favourite colour. Thus, no final decision between Idealist and Empiricist will ever be reached [30, page 97].²⁰

Naive Cantorian set theory has been discredited by several antinomies that riddled it, that is, “with the discovery of pairs of mutually inconsistent propositions each of which could be justified with the same degree of certainty. Today, they are called logical antinomies (to distinguish them from semantical antinomies in which the concepts of meaning and reference are involved). Their source was the imprecise notion of a set used by Cantor” [62, page 581].

¹⁹ “Du wirfst mir Dogmatismus vor, und predigst doch zugleich deine eigene Weltanschauung [...] Als Idealist glaube ich an die Wirklichkeit meiner Ideale. Ich glaube an die Wirklichkeit meiner bis an die äussersten Marken meines Denkens verfolgten Ideen, wenn sie auch in Wahrheit, wie das Unendliche, unvorstellbar sind. Ich glaube an das Unendliche und an das wirkliche Vorhandensein genauer geometrischer Gebilde” [29, page 68].

²⁰ Although Paul du Bois-Reymond was one of the first mathematicians of his time who took Cantor's theory seriously [11, page 61], Cantor did not appreciate du Bois-Reymond's endeavours. It is well-known that Cantor fulminated against Paul du Bois-Reymond's reintroduction of infinitesimals and denounced his works as outdated. Cantor had such a deep aversion to infinitesimals that he branded them as “Cholera-bacillus of mathematics” in a letter to the Italian mathematician Giulino Vivanti, December 13, 1893 [16, page 233]. The *ignorabimus* call and its “dangerous” coalition with mechanistic, deterministic, and materialistic world-views was a thorn in Cantor's side, a grist for the mill of agnosticism which would help to undermine religious authority. Cantor's quarrel with mechanistic, deterministic and materialistic tendencies is discussed in [20, page 233].

There is a broad consensus that Cantor discovered the first set-theoretical paradox in 1899 or between 1895 and 1897 [2, page 34]. This discovery was the beginning of a series of such antinomies that forced mathematicians to revise their idea of what constituted a set in order to avoid these difficulties and achieve consistency: the Burali-Forti antinomy of the greatest ordinal (it was also known to Cantor), Cantor's antinomy of the set of all sets and Russell's antinomy of the irreflexive classes (called today Zermelo-Russell antinomy). However, it was not Russell, but Ernst Zermelo who had "pointed out to Hilbert a disturbing antinomy in set theory — the same antinomy which Bertrand Russell pointed out to Gottlob Frege as Frege was ready to send to the printer his definitive work on the foundations of arithmetic" [72, page 98]. As a consequence, Hilbert pleaded in his lecture at the third International Congress of Mathematicians at Heidelberg in the late summer of 1904 that the concept of proof itself should be made an object of mathematical investigations [48]. Hilbert called this endeavour "metamathematics".²¹

4. Hilbert's Way to Proof Theory

Hilbert's influential textbook *Foundations of Geometry* [45] reduced the proof of the consistency of geometrical axioms to the consistency of arithmetic. However, relative consistency proofs assert nothing about the absolute consistency of any system; they merely relate the consistency of one system to that of another. In other words, such proofs do not rule out the possibility that all of the theories involved are inconsistent.

²¹ Matthias Wille, among others, pointed out that "the term 'metamathematics' was extensively used and achieved the status of a catchphrase in a public and in part polemic discourse" [87, page 338] long before Hilbert. Metamathematics, in the sense of Emil and Paul du Bois-Reymond, was a meta-discipline, reflecting on the general epistemological conditions of mathematical reasoning and the ontological status of mathematical objects and relations. Paul argued that there are unsolvable, that is, undecidable questions of mathematics insofar as they are not mathematical problems at all; they are problems of metamathematics [30, pages 16–22]. In this respect, the practice of metamathematics can be compared to the task of the philosopher who criticizes the possibilities and limits of reason through reason itself. Within this context it would be a mistake to identify metamathematics with Hilbert's proof theory, although Hilbert used both terms synonymously.

As Hilbert's ultimate goal was to prove the consistency of mathematics as a whole, the foundation of analysis itself required axiomatization and an absolute consistency proof that would not depend on another system. To be more precise, Hilbert's program required a logical formalization of all of mathematics in axiomatic form, together with a proof by finitary methods that this axiomatization was consistent. Hilbert's main goal was to show that proofs that use ideal elements (such as the notion of an actual infinity) always yield correct results.

Hilbert had emphasized the importance of such a direct consistency proof for arithmetic, or analysis, in his 1900 talk on the open problems in mathematics. In his Heidelberg talk he had presented a sketch of such a proof [48]. Hilbert's program hinged on proving that the axioms "are not contradictory, that is, that a finite number of logical steps based upon them can never lead to contradictory results" [47, page 447].

According to the standard view, this program consisted of two steps. The first task was to formalize mathematics in a rigorous and complete way, that is, to reconstitute infinitary mathematics as a formal system (containing classical logic, infinite set theory, arithmetic of natural numbers, and analysis). One would need to specify axioms that would in turn serve as premises in the proofs and rules for deducing a formula from others. The axioms should be formulated in a formal language and the logic should be made explicit, through an effective deductive system. In this way theorems of mathematics would become those formulas that had formal proofs based on a given set of axioms and given rules of inference. The second step of Hilbert's program was to give a proof of the consistency and conservativeness of mathematics. Such a proof would need to be carried out by finitary methods.

Over the past few decades, historians of mathematics have argued persuasively against most aspects of this standard view as anachronistic. As Stewart Shapiro puts it in his review on Wilfried Sieg's book *Hilbert's Programs and Beyond* [74], there is "no such thing as the Hilbert Program. Hilbert's views, along with those of his main students and collaborators, notably Paul Bernays, evolved considerably over a period of some forty years, sometimes in reaction to results within mathematics, and sometimes due to interactions with colleagues, collaborators, and opponents" [73].

Nevertheless, there was a significant chronological split between the period from 1899 to 1917 and from 1917 to 1931/33. The year 1917 marks the beginning of many new and interesting developments in Hilbert's thought. This led to metamathematics (in Hilbert's sense) between 1917 and 1920. The period from 1920 to 1922 constitutes the transition period from logic to proof theory. The period from 1922 to 1925 is mainly devoted to the development of finitist proof theory. The period between 1925 and 1931 sees attempts to formulate an elementary finiteness theorem. In the early twenties Hilbert had arrived at a refined understanding of the formalization of mathematics and he had perceived a greater awareness of the considerable technical difficulties of his project and the need for a deeper and more careful probing into the logical structure of mathematical proofs and theories. In a series of courses from 1917–1921, assisted by Paul Bernays and Heinrich Behmann, Hilbert made significant new contributions to formal logic, especially to first-order logic, and modern proof theory (c.f. [74, 90, 91]). Traces of his outlook can be found in a September 1917 address he delivered to the Swiss Mathematical Society entitled “Axiomatic Thought” [49], his first published contribution to mathematical foundations since 1905:

The chief requirement of the theory of axioms must go farther, namely, to show that within every field of knowledge contradictions based on the underlying axiom-system are absolutely impossible [32, page 1112].²²

Hilbert's and Ackermann's textbook *Principles of Theoretical Logic* [51] provided an important summary of the work on logic done in Göttingen in the 1920s. This publication is important not only for the influence it had in terms of publicising the methods and results developed by Hilbert, his collaborators, and students. The book contains a number of improvements, concerning the semantics of the predicate calculus and the decision problem (c.f. [60, page 380]) that are based very closely on Bernays' notes for lectures given by Hilbert in Winter 1917/1918, Summer 1920, and Winter 1921/22.

²² “[D]ie prinzipielle Forderung der Axiomenlehre muß vielmehr weitergehen, nämlich dahin zu erkennen, daß jedesmal innerhalb eines Wissensgebietes aufgrund des aufgestellten Axiomensystems Widersprüche überhaupt unmöglich sind” [49, page 148].

5. The Decision Problem in the Light of Gödel's Theorems

Hilbert was convinced that every mathematical problem has a solution, and therefore, as he famously stated, “in mathematics there is no *ignorabimus*”. And as mentioned before, he expressed his belief on several occasions. In “Axiomatic Thought” he listed the problem of the “decidability of a mathematical question in a finite number of operations” [49, page 1113] as one of the fundamental problems for the axiomatic method. At the 1928 International Congress of Mathematicians in Bologna Hilbert refined the question to whether there is a generalized procedure of calculation that can decide whether any given logical expression can be derived from the axioms in finitely many operations. And in the 20s, he developed the specific question of an “Entscheidungsproblem” (“decision problem”) applied to Diophantine equations (within Hilbert’s tenth problem) into the more general question about a method of decision for any mathematical formula. Attempts to generalize this question were closely related to Hilbert’s foundational program of mathematics. “This proposal incorporated Hilbert’s ideas from 1900 and 1904 regarding direct consistency proofs, his conception of axiomatic systems, and also the technical developments in the axiomatization of mathematics” [91, page 415] in the work of Bertrand Russell and Alfred North Whitehead and in logical approaches presented by Paul Bernays, Moses Schönfinkel, Wilhelm Ackermann, Heinrich Behmann, John von Neumann, and Jacques Herbrand.

Hilbert and Ackermann considered the decision problem the main problem of mathematical logic. However, they did not introduce the term “Entscheidungsproblem”. That word first appeared in a talk given by Heinrich Behmann to the Mathematical Society in Göttingen on May 10, 1921, entitled “Decision Problem and Algebra of Logic”. The result Behmann reports on in this talk is that of his Habilitationsschrift “Contributions to the Algebra of Logic, in particular to the Decision Problem” [5]. In this thesis, Behmann proved, independently of Leopold Löwenheim and Albert Thoralf Skolem, that monadic second-order logic with equality is decidable.²³

²³ Behmann’s result is based on the connection between the decision problem and the elimination problem in the algebra of logic tradition. It provides a solution to the elimination problem for formulas containing monadic predicate variables and to the decision problem for the same class of formulas (c.f. [60]).

Behmann also contextualized his findings within the discussion of whether mathematics can be reduced to mechanical calculations and to deterministic, computational procedures. Behmann compared mathematicians to poets, and artists. Each needed imagination, sensitivity, and insight. The difference, according to Behmann, is that the mathematician must not only assert theorems, but also must prove them. However, Behmann remains unsure whether methods of finding proofs can be transformed to deterministic and computational procedures:

At first glance the development of a general, deterministic procedure for proving mathematical propositions may seem like a hopeless, indeed ludicrous endeavor, perhaps comparable to the medieval quest for the philosopher's stone.²⁴

He then describes the decision problem as the more specific problem of finding a deterministic, computational procedure to decide any mathematical claim:

If a logical or mathematical assertion is given, the required procedure should give complete instructions for determining whether the assertion is correct or false by a deterministic calculation after finitely many steps. The problem thus formulated can be called the general decision problem. It is essential to the character of this problem that the only allowable method of proof is entirely mechanical calculation according to given instructions, without any activity of thinking in the narrower sense. If one wanted to, one might speak of mechanical or machine-like thinking. (One day it might even be carried out by a machine.)²⁵

²⁴ “Auf den ersten Blick mag freilich die Aufstellung eines allgemeinen, zwangläufigen Verfahrens für das Beweisen mathematischer Aussagen als ein hoffnungsloses, ja wahnwitziges Unterfangen erscheinen, etwa vergleichbar dem mittelalterlichen Suchen nach dem Stein der Weisen” [5, page 179].

²⁵ “Ist irgend eine logische oder mathematische Aussage vorgelegt, so soll das verlangte Verfahren eine vollständige Anweisung geben, wie man durch eine ganz zwangläufige Rechnung nach endlich vielen Schritten ermitteln kann, ob die gegebene Aussage richtig oder falsch ist. Das eben formulierte Problem möchte ich das allgemeine Entscheidungsproblem nennen. Für das Wesen des Problems ist von grundsätzlicher Bedeutung, daß als Hilfsmittel des Beweises nur das ganz mechanische Rechnen nach einer gegebenen Vorschrift, ohne irgendwelche Denktätigkeit im engeren Sinne, zugelassen wird. Man könnte hier, wenn man will, von mechanischem oder maschinenmäßigem Denken reden. (Vielleicht kann man es später sogar durch eine Maschine ausführen lassen.)” [5, page 180].

The many attempts to pursue Hilbert's idea of proof theory systematically to its full extent led to serious difficulties due to the assumed notion of a proof as a completely formalized and purely syntactic deductive argument. Hilbert's conviction that all well-posed mathematical problems are solvable resulted in two specific hypotheses: that the axioms of mathematics, in particular, of number theory, are *complete*, in the sense that for every formula A , either A or $\neg A$ is provable, and secondly that the validities of first-order logic are decidable. Thus, reformulating the decision problem in terms of validity or satisfiability was an important step forward.

In the following decade Hilbert and his collaborators carried out intensive logico-mathematical research. Important steps were the development of the ϵ -calculus as definitive formalism for axiom systems for arithmetic and analysis, and John von Neumann's consistency proof for a system of the ϵ -formalism (which, however, did not include the induction axiom) in 1925 [64]. Wilhelm Ackermann then devised a new ϵ -substitution procedure (see [7]). The epsilon calculus is an extension of first-order predicate logic by the "epsilon operation" that picks out, for any true existential formula, a witness to the existential quantifier. The extension is conservative in the sense that it does not add any new first-order consequences. But, conversely, quantifiers can be defined in terms of the epsilons, so first-order logic can be understood in terms of quantifier-free reasoning involving the epsilon operation. It is this latter feature that makes the calculus convenient for the purpose of proving consistency. Suitable extensions of the epsilon calculus make it possible to embed stronger, quantificational theories of numbers and sets in quantifier-free calculi. Hilbert expected that it would be possible to demonstrate the consistency of such extensions; for further reading see [3].

In his address to the *International Congress of Mathematicians* in Bologna in 1928, "Problems of the Grounding of Mathematics", Hilbert optimistically claimed that the work of Ackermann and von Neumann had established the consistency of number theory and that the proof for analysis had already been carried out by Ackermann "to the extent that the only remaining task consists in the proof of an elementary finiteness theorem that is purely arithmetical" [59, page 229]. Thus, the realization of Hilbert's program was considered to be only a matter of overcoming certain technical difficulties. In 1930, both Hilbert and the young mathematician Kurt Gödel were in Königsberg at the same time. Gödel was attending the Second Conference for Epistemology of the Exact Sciences on 5–7 September.

The last day of the event was devoted to a roundtable discussion of the foundations of mathematics (only a day before Hilbert's famous speech [50] to which Gödel had been invited). Hans Hahn chaired the discussion, which included Rudolf Carnap, Arend Heyting, and John von Neumann [74, pages 147–148].

A few months later, in November 1930, the Leipzig Journal *Monatshefte für Mathematik und Physik* received Gödel's twenty-five-page article "On Formally Undecidable Propositions in Principia Mathematica and Related Systems I" [37]. Gödel proved that for the system of Russell's and Whitehead's *Principia Mathematica* (and even much more elementary ones, such as Zermelo-Fraenkel set theory), there will always be individual propositions in the language that are undecidable by the system, that is, statements that can neither be proved nor disproved from the system axioms provided the system is consistent (first incompleteness theorem). Furthermore, Gödel showed that the consistency of such a system cannot be proved within the system itself (second incompleteness theorem).²⁶

Having learned about Gödel's first incompleteness theorem, Hilbert proposed to add to the rules of inference, such as a simple form of the ϵ -rule, to shore up his program. This rule allows the derivation of all true arithmetical sentences but, in contrast to all rules of the first-order logic, it has infinitely many premises. According to his *Foundations of Mathematics*, which was written together with Paul Bernays:

We have shown to be erroneous the occasionally-held opinion that Gödel's results negate my Proof Theory. His result indeed shows only that for more advanced consistency proofs one must use the finite standpoint in a deeper way than is necessary for the consideration of elementary formalism.²⁷

²⁶ Gödel used a diagonalization technique that allowed him to formally express the concept of truth for number-theoretical sentences in the language of number theory itself. Thus, he was able to construct a self-referential formula without any infinite regress of definition. For a brief outline of the proof see [60]. There are dozens of introductions to Gödel's proof. A classical introduction to Gödel's theorems is the book *Gödel's Proof* by Ernest Nagel and James R. Newman [63]. A newer, competent and readable book is [35].

²⁷ "Im Hinblick auf dieses Ziel möchte ich hervorheben, daß die zeitweilig aufkommende Meinung, aus gewissen neueren Ergebnissen von Gödel folge die Undurchführbarkeit meiner Beweistheorie, als irrtümlich erwiesen ist" [52, page *vii*].

Whereas John von Neumann immediately realized that no finitary consistency proofs could be given for sufficiently strong systems (like Peano arithmetic), Kurt Gödel (who never had any correspondence with Hilbert) and Paul Bernays initially thought that the difficulty for the consistency proof of Peano arithmetic could be overcome by employing methods which, although not formalizable in Peano arithmetic, are nonetheless finitary.²⁸ Whether such methods would be considered finitary according to the original conception of finitism or they would necessarily constitute an extension of the original finitist viewpoint was and remains disputable.²⁹

Nonetheless, it eventually became a common assumption that the results did deliver a significant blow to the essence of Hilbert's program. The following developments and analysis led Hermann Weyl to remark:

The ultimate foundations and the ultimate meaning of mathematics remain an open problem; we do not know in what direction it will find its solution, nor even whether a final objective answer can be expected at all. "Mathematizing" may well be a creative activity of man, like music, the products of which not only in form but also in substance defy complete objective rationalization. The indecisive outcome of Hilbert's bold enterprise cannot fail to affect the philosophical interpretation [86, page 219].

The question of whether Gödel's theorems undermined Hilbert's program remained the subject of lively debate throughout the 1930s [19, 71].

In 2011 a new German-English dual text version of Hilbert's and Bernays' *Grundlagen der Mathematik* was published [53]. It was edited by Dov Gabbay, Michael Gabbay, Jörg Siekmann, and Claus-Peter Wirth, who also wrote a preface. The translation from the second German edition of 1968 was made by Claus-Peter Wirth. Also included are the translations, with annotations, of all deleted parts of the first German edition of 1934. According to the website <http://wirth.bplaced.net/p/hilbertbernays/>, last accessed on January 22, 2019, the goal of the Hilbert Bernays Project is to publish a complete and commented bilingual edition of the second German edition of Hilbert's and Bernays' *Grundlagen der Mathematik*.

²⁸ In his 1931 paper, Gödel left open the possibility that there could be finitary methods which are not formalizable in these systems and which would yield the required consistency proofs [37].

²⁹ Gerhard Gentzen, for example, argued that the consistency of elementary number theory can in fact be verified by means of techniques which, in part, no longer belong to elementary number theory, but which can nevertheless be considered to be more reliable than the doubtful components of elementary number theory itself [36, page 139].

Before the unsolvability of the decision problem could be proved and the connection between Gödel's theorems and undecidability results clarified, the informal term "effectively calculable" had to be formally defined. This was done in the mid-1930s.

6. From the Decision Problem to the Church-Turing Thesis

In the mid-1930s several different answers to the decision problem were offered. In 1936, Alonzo Church and Alan Turing published independent papers showing that a general solution to the Entscheidungsproblem is impossible. In "A Note on the Entscheidungsproblem", Church proved that there is no computable function which can decide whether two given λ -calculus expressions are equivalent or not [12, 13].

Upon hearing of Church's paper, Alan Turing quickly showed that his proposal in fact described the same set of functions [79, pages 263ff.]. In his 1936 paper "On Computable Numbers, with an Application to the Entscheidungsproblem" Turing tried to capture this notion formally with the introduction of a model, later called "Turing machine", that is, a device operating on a one-way finite (but potentially unbounded) linear tape divided into squares, meant to be deterministic and sequential, comparable to "a man in the process of computing a real number [in that it is] only capable of a finite number of conditions" [79, page 231].

Fifteen years before the publication of Church's and Turing's works, Emil Post had also found that certain decision problems closely related to the Entscheidungsproblem are unsolvable. Post inferred from these results that any finite system of symbolic logic relative to a certain class of systems must be incomplete. He thus anticipated results similar to the fundamental results by Gödel, Church, and Turing, who independently proved that there exists no finite procedure that can decide for any given formula in first-order predicate calculus whether it is deducible within this calculus [18].³⁰ In his 1937 review of Turing's paper, Church wrote:

³⁰ Lisbeth de Mol has given a detailed historical picture of how Post, Church, and Turing each arrived at their theses and how different interpretations were attached to these theses by some of the leading logicians at that time [21, 22].

As a matter of fact, there is involved here the equivalence of three different notions: computability by a Turing machine, general recursiveness in the sense of Herbrand-Gödel-Kleene, and λ -definability in the sense of Kleene and the present reviewer. Of these, the first has the advantage of making the identification with effectiveness in the ordinary (not explicitly defined) sense evident immediately [...] The second and third have the advantage of suitability for embodiment in a system of symbolic logic [14, page 42].

Given that “effective calculability” refers to an informal notion, and “computable” means “computable by a Turing machine”, then Church’s and Turing’s analyses of effective calculability ultimately led to (i) Church’s thesis: A function of positive integers is effectively calculable if and only if it is recursive, and (ii) Turing’s thesis: Any function is effectively computable if and only if it can be computed by a Turing machine. When restricted to functions of positive integers, both Church’s thesis and Turing’s thesis are equivalent. Thus it has become commonplace to speak of the Church-Turing thesis (CTT) as the thesis that the intuitive notion of “effectively calculable” is captured by the functions computable by a Turing machine, or equivalently, by those expressible in the lambda calculus:

So Turing’s and Church’s theses are equivalent. We shall usually refer to them both as *Church’s thesis*, or [...] as the *Church-Turing thesis* [56, page 232].

The CTT describes the class of λ -definable functions, recursive functions and Turing Machine-computability as extensionally equivalent to the notion of a function that can be calculated by an effective procedure or algorithm.

The CTT has generated a vast literature. The crucial point is that the CTT establishes a relation between a formal concept (Turing computable functions, general recursive functions) and an intuitive (informal) concept (effectively calculable function).³¹ To discuss this point in detail would go beyond the scope of the work; the interested reader is referred to [17, 15].

³¹ “This is a thesis rather than a theorem, in as much as it proposes to identify a somewhat vague intuitive concept with a concept phrased in exact mathematical terms” [56, page 232].

7. What Remains of Hilbert's Axiom of Solvability?

In discussing his conviction that every mathematical problem should have a solution, Hilbert allowed for the possibility that the solution could be a proof that the original problem is impossible. In this sense, Gödel's theorems and the negative results of the decision problem presented by Alonzo Church [12, 13], Alan Turing [79], and others confirmed Hilbert's *non-ignorabimus*.

But is this really the case? Gödel's incompleteness results showed that no finitary consistency proof of arithmetic can be given. Does this mean that there exist so-called "absolutely undecidable" statements, whose truth value can never be known? In retrospect, the Austrian mathematician Wilhelm Blaschke played down the relation between impossibility proofs and the *ignorabimus* when he wrote:

The so-called impossibility proof has nothing to do with the famous *ignorabimus*; it only shows that certain tools are not sufficient to achieve a given target, such as a saw is unsuitable for shaving.³²

Blaschke argued: Impossibility proofs, no matter what they look like, can neither verify nor falsify the *ignorabimus*. From his perspective, the optimistic conviction that every mathematical problem is solvable (Hilbert's *non-ignorabimus*) indeed remains entirely untouched and should not be confused with an algorithmic procedure according to which every mathematical proposition is formally decidable. Was Blaschke right?

Gödel himself shared Hilbert's "rationalistic optimism" (to use Hao Wang's term [83, page 219]). Gödel considered that in spite of the undecidability of the "Entscheidungsproblem" Hilbert's axiom of solvability "remains entirely untouched":

So the result is rather that it is not possible to formalise mathematical evidence even in the domain of number theory, but the conviction about which Hilbert speaks remains entirely untouched. Another way of putting the result is this:

³² "Ein solcher Beweis hat mit dem berühmten 'ignorabimus' nichts zu tun, er zeigt nur, daß gewisse Mittel nicht dazu ausreichen, ein vorgegebenes Ziel zu erreichen, etwa wie eine Brettersäge zum Rasieren ungeeignet ist" [9, page 15]. The English translation is my own.

It is not possible to mechanise mathematical reasoning, i. e., it will never be possible to replace the mathematician by a machine, even if you confine yourself to number-theoretic problems [38, page 164].

From his theorems, Gödel drew the conclusion that formal (syntactical) provability cannot be treated as an analysis of (semantical) truth, that the former is weaker than the latter. From his Platonist point of view, Gödel argued for the “inexhaustibility” of mathematics, in the sense of the never ending need for new axioms. However, one should distinguish between Gödel’s own interpretations of his theorems (which changed over time) and the theorems themselves. The interpretations are open to rebuttal; the theorems are here to stay.

Today, we know that an absolute proof of consistency cannot be realized in a finitistic way as Hilbert had hoped to achieve. We do know that arithmetic cannot be completely formalized. What does this mean for the *(non) ignorabimus* conviction, if at all? In order to give an answer to this question it might be helpful to remember that Gödel’s main contribution to the discussion at hand consists of two theorems. The first refers to the existence of undecidable propositions in any formal system, the second to the impossibility of proving consistency statements within such systems.

The consequences of these theorems are as follows: (i) there are undecidable propositions in any consistent, recursively axiomatized, sufficiently strong theory; (ii) the decision problem is undecidable. The results of Gödel, Church, and Turing showed that there is not one all-encompassing formal system that can solve all mathematical questions, let alone a mechanical method that allows us to compute the answer. Thus, the strong form of the *non-ignorabimus* (which Hilbert may or may not have had in mind) is certainly false. But the results have also shown that creativity is not just important in mathematical problem solving, but also in generating new systems that allow the formulation and solution of new questions.

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