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NOTES

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A Combinatorial Proof of Vandermonde's Determinant

Arthur T. Benjamin and Gregory P. Dresden

We offer a combinatorial method of evaluating Vandermonde's determinant,

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} = \prod_{0 \le i < j \le n} (x_j - x_i),$$

that is as easy as playing cards. Let V_n denote the Vandermonde matrix with (i, j)th entry $v_{ij} = x_i^j$ $(0 \le i, j \le n)$. Since the determinant of V_n is a polynomial in $x_0, x_1, \dots, x_n$, it suffices to prove the identity for positive integers $x_0, x_1, \dots, x_n$ with $x_0 \le x_1 \le \dots \le x_n$. We define a *Vandermonde card* to possess a suit and a value, where a card of Suit i has a value from the set $\{1, \dots, x_i\}$. (In our examples, we will let Suits 0, 1, 2, 3, and 4 be represented by suits $\odot$, $\clubsuit$, $\diamondsuit$, $\heartsuit$, and $\spadesuit$, respectively.) Hence there are $x_0 + x_1 + \dots + x_n$ different Vandermonde cards, but we have at our disposal an unlimited supply of each. First we do some card counting.

Card Counting Question 1. How many ways can Vandermonde cards be arranged in $n+1$ rows, where row 0 is empty, row 1 has one card of Suit 1, row 2 has two cards of Suit 2, row 3 has three cards of Suit 3, ..., and row n has n cards of Suit n? The order of the cards is important, and we are allowed to repeat values of cards within each row. We call such an arrangement a *Vandermonde table associated with the identity permutation* $\pi = 012\dots n$, an example of which is given in Figure 1.

	Col 1	Col 2	Col 3	Col 4		permutation π
Row 0						$\pi(0) = 0 = \odot$
Row 1	c_{11} $\clubsuit$				$c_{11} \in \{1......x_1\}$	$\pi(1) = 1 = \clubsuit$
Row 2	c_{21} $\diamondsuit$	c_{22} $\diamondsuit$			$c_{2j} \in \{1.........x_2\}$	$\pi(2) = 2 = \diamondsuit$
Row 3	c_{31} $\heartsuit$	c_{32} $\heartsuit$	c_{33} $\heartsuit$		$c_{3j} \in \{1............x_3\}$	$\pi(3) = 3 = \heartsuit$
Row 4	c_{41} $\spadesuit$	c_{42} $\spadesuit$	c_{43} $\spadesuit$	c_{44} $\spadesuit$	$c_{4j} \in \{1...............x_4\}$	$\pi(4) = 4 = \spadesuit$

Figure 1. A Vandermonde table associated with the identity permutation $\pi = 01234$ (or $\pi = \odot\clubsuit\diamondsuit\heartsuit\spadesuit$). Each of the i cards in row i has Suit i and a value from $\{1, \dots, x_i\}$. Such a table can be created in $x_1 x_2^2 x_3^3 x_4^4$ ways.

Answer. For $i = 0, 1, \dots, n$, the i cards in row i all have Suit i, so their values can be assigned x_i^i ways. Hence, the number of arrangements is $1x_1x_2^2x_3^3 \cdots x_n^n$, which is the product of the diagonal entries of V_n.

Card Counting Question 2. This is the same as Question 1, but now we are given a permutation π of the numbers 0 through n, say $\pi = a_0a_1 \dots a_n$. Here, row i must contain i cards from Suit $\pi(i) = a_i$. We call such an arrangement a *Vandermonde table with permutation* π. A typical table is shown in Figure 2.

Answer. Counting row by row again, there are $1x_{\pi(1)}^1 x_{\pi(2)}^2 x_{\pi(3)}^3 \cdots x_{\pi(n)}^n$ such tables, which is the product of the $n+1$ entries of the form $v_{\pi(i),i}$ from V_n.

	Col 1	Col 2	Col 3	Col 4		permutation π
Row 0						$\pi(0) = 3 = \heartsuit$
Row 1	c_{11} ♠				$c_{11} \in \{1 \dots\dots\dots\dots x_4\}$	$\pi(1) = 4 = $ ♠
Row 2	c_{21} $\odot$	c_{22} $\odot$			$c_{2j} \in \{1 \dots x_0\}$	$\pi(2) = 0 = \odot$
Row 3	c_{31} $\diamondsuit$	c_{32} $\diamondsuit$	c_{33} $\diamondsuit$		$c_{3j} \in \{1 \dots\dots\dots x_2\}$	$\pi(3) = 2 = \diamondsuit$
Row 4	c_{41} ♣	c_{42} ♣	c_{43} ♣	c_{44} ♣	$c_{4j} \in \{1 \dots\dots x_1\}$	$\pi(4) = 1 = $ ♣

Figure 2. A Vandermonde table associated with permutation $\pi = 34021$ (or $\pi = \heartsuit$♠$\odot\diamondsuit$♣). Each of the i cards in row i has Suit $\pi(i)$ and a value from $\{1, \dots, x_{\pi(i)}\}$. Such a table can be created in $x_4x_0^2x_2^3x_1^4$ ways.

Card Counting Question 3. Same as Question 2, but now π is not prescribed in advance, so π can be any permutation of $\{0, \dots, n\}$. As before, each row is assigned a different suit and row i contains i cards of the assigned suit. For this unrestricted problem, such an arrangement is simply called a *Vandermonde table*.

Answer. Sum the answer to Question 2 over all possible permutations of $0, \dots, n$. In other words, the number of ways to create a Vandermonde table is the *permanent* of V_n.

Card Counting Question 4. Question 3 again, but now we count those arrangements with even permutations positively and those arrangements with odd permutations negatively.

Answer. By definition, this is the *determinant* of V_n.

It remains to show that the answer to Question 4 also equals $\prod_{0 \le i < j \le n}(x_j - x_i)$. For a given Vandermonde table C let the cards of row i be denoted by $C_{i1}, C_{i2}, \dots, C_{ii}$, with values $c_{i1}, c_{i2}, \dots, c_{ii}$. We say that card C_{ij} is *small* if $c_{ij} \le x_{j-1}$. For example, any card in column 1 with a value less than or equal to x_0 (such as any card of Suit 0) is small.

Card Counting Question 5. How many Vandermonde tables have no small cards?

Answer. Let C be a Vandermonde table with no small cards. Since column 1 must not contain any cards of Suit 0, Suit 0 must be assigned to the empty row 0. Next, since column 2 must not contain any cards with value less than or equal to x_1 (such as any card of suit 1), we must assign suit 1 to row 1. Continuing this reasoning, row 2 must have Suit 2, ..., and row n must have Suit n. Thus C must be associated with the identity permutation. Furthermore, to avoid small cards in the first column, the values of the cards $C_{11}, \ldots, C_{n1}$ can be assigned in $(x_1 - x_0)(x_2 - x_0)(x_3 - x_0) \cdots (x_n - x_0)$ ways. Likewise, the values of the cards in the second column can be assigned in $(x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1)$ ways, and so on down to the single card of Suit n in the last column, with a value that can be assigned in $x_n - x_{n-1}$ ways. We conclude that there are $\prod_{0 \le i < j \le n}(x_j - x_i)$ Vandermonde tables with no small cards.

We say that a Vandermonde table is *good* if it has no small cards and is *bad* if it has at least one small card. Note that since the identity permutation is even, all of the good tables are counted *positively* in the determinant of V_n.

To complete the proof of Vandermonde's expansion, it suffices to show that every bad Vandermonde table can be paired up with another bad Vandermonde table with a permutation of opposite parity. Thus, when the determinant of V_n sums over all Vandermonde tables, the bad tables cancel each other out. When the dust settles, only the good tables (all counted positively) remain standing.

Now let C be a bad Vandermonde table with permutation $\pi = a_0 a_1 \ldots a_n$. We define the *first small card* of C to be the small card c_{ij} where j is as small as possible, and if column j has more than one small card, then we choose i to be as large as possible. In other words, we look for small cards from bottom to top, beginning in column 1.

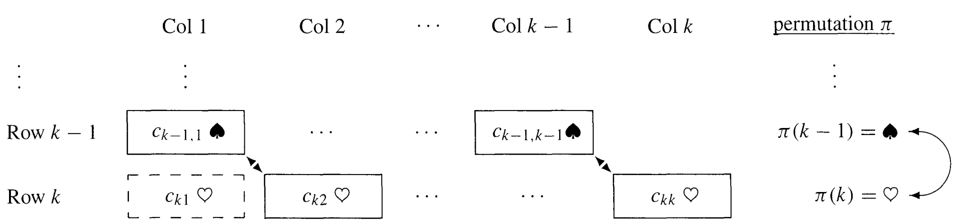

Figure 3. When the first small card occurs in the first column at card C_{k1}, simply swap the cards of row $k-1$ with the cards $C_{k2}, \ldots, C_{kk}$, and change the suit of card C_{k1}.

Suppose that the first small card of C occurs in column 1, say card C_{k1} for some $1 \le k \le n$. Then, since C_{k1} is small, $c_{k1} \le x_0$, and since it is the first small card, there are no small cards below it; that is, when $i > k$, $c_{i1} > x_0$. For definiteness, suppose that the cards in row $k-1$ have Suit $\pi(k-1) =$ ♠ and that the cards in row k have Suit $\pi(k) =$ ♡. (We make no assumptions about the suit number for hearts or spades.) Now consider the Vandermonde table C' obtained by swapping all $k-1$ spade cards with all of the heart cards except for card C_{k1}. Then change the suit of card C_{k1} from hearts to spades. The suit change from hearts to spaces is legal because $c_{k1} \le x_0$, which is a legal value for all suits. (Here we are exploiting the fact that $x_0 \le x_1 \le \cdots \le x_n$.) Notice that C_{k1} is still the first small card of C', albeit with a new suit, and thus if we apply the swapping procedure to C', we obtain C. That is, $(C')' = C$. Furthermore, C' has permutation $\pi' = a_0 a_1 \ldots a_k a_{k-1} \ldots a_n$. Permutations π and π' have opposite parity since they differ by the transposition of hearts and spades (see Figure 3).

Now suppose that the first small card of C occurs in column j with $j \ge 2$, say at card C_{kj}. Then $c_{kj} \le x_{j-1}$, and there are no small cards anywhere in columns 1 through $j-1$ nor below card C_{kj} in column j. As before, suppose that the cards of row k have the heart suit and that the cards of row $k-1$ have the spade suit. Create a new Vandermonde table C' by swapping the first $j-1$ cards of rows $k-1$ and k,

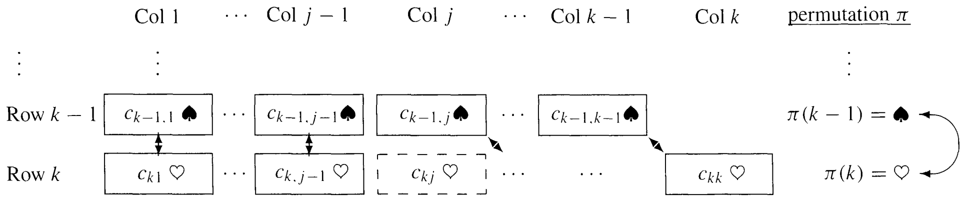

Figure 4. When C_{kj} is the first small card and $j \geq 2$, then swap the first $j-1$ cards of row $k-1$ with the first $j-1$ cards of row k, change the suit of card C_{kj}, then swap the remaining cards of rows $k-1$ and k. In the new Vandermonde table, card C_{kj} remains the first small card.

leaving card C_{kj} in its place, but changing its suit from hearts to spades, then swapping the remaining $k-j$ cards of rows $k-1$ and k, as in Figure 4.

Why is it legal to change the suit of card C_{kj} from hearts to spades? Since C_{kj} was the first small card, then the spade card $C_{k-1,j-1}$ is not small and therefore has a value strictly greater than x_{j-2}. Thus all spade cards can take on values less than or equal to x_{j-1}. Since C_{kj} is small, its value is at most x_{j-1}, so changing it from hearts to spades is allowable.

As before, C_{kj} remains the first small card of C', so $(C')' = C$ and C' has permutation π', which has opposite parity of π since they differ by a transposition. Thus there is a one-to-one correspondence between the positively counted Vandermonde tables with small cards and the negatively counted Vandermonde tables with small cards. Therefore the determinant of V_n is the number of Vandermonde tables with no small cards, namely, $\prod_{0 \leq i < j \leq n}(x_j - x_i)$, as desired.

Remark. For another combinatorial proof of Vandermonde's determinant, where the cancellation occurs in the product instead of the sums, see the short paper by Ira Gessel [**1**].

REFERENCES

1. I. Gessel, Tournaments and Vandermonde's Determinant, *J. Graph Theory* **3** (1979) 305–307.

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Evaluation of Some Improper Integrals Involving Hyperbolic Functions

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In this note I present a result that seems elementary enough to be added to the list of tricks for evaluating integrals taught in a complex variables course, but one to which I have been unable to find any reference. It gives a straightforward procedure that can be used to evaluate a class of integrals some of which do not appear in [**1**] and for which *Mathematica* 5.1 [**2**] generates expressions involving exotic special functions that it cannot simplify further.