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Modular spaces and K -widths

A. G. Aksoy and G. Lewicki

Abstract. In this paper, we show that the ball measure of noncompactness of a modular space X_ρ is equal to the limit of its K -widths when ρ is a left continuous, s -convex modular function, without any Δ_2 -condition. We also obtain a similar result for SF-spaces, when the SF-norm \mathcal{N} is uniformly continuous.

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1. Notation and definitions

Throughout the following X is a linear space over a field K ($K = \mathbb{R}$ or $K = \mathbb{C}$).

I. A function $\rho : X \rightarrow [0, \infty]$ is called *modular* if the following hold for arbitrary $x, y \in X$:

1. $\rho(x) = 0$ iff $x = 0$.
2. $\rho(\alpha x) = \rho(x)$ if $\alpha \in K, |\alpha| = 1$.
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $\alpha, \beta \geq 0, \alpha + \beta = 1$.

If in place of 3) we have

$$\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y) \quad \text{for } \alpha, \beta \geq 0, \alpha^s + \beta^s = 1$$

for an $s \in (0, 1]$, then ρ is called s -convex modular (convex if $s = 1$).

The set

$$X_\rho = \{x \in X : \lim_{\alpha \rightarrow 0} \rho(\alpha x) = 0\}$$

is called a modular space.

Each modular space X_ρ may be equipped with an F-norm given by the formula

$$|x|_\rho = \inf\{u > 0 : \rho(x/u) \leq u\} \quad \text{for } x \in X_\rho.$$

A modular ρ in X is called left continuous if $\lim_{\lambda \rightarrow 1^-} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$.

It is known (see [8]) that, if ρ is left continuous, s -convex modular in X , then the inequalities $|x|_\rho^s \leq 1$ and $\rho(x) \leq 1$ are equivalent for every $x \in X_\rho$. Here by $|x|_\rho^s$ we mean

$$|x|_\rho^s = \inf\{u > 0 : \rho(\frac{x}{u^{1/s}}) \leq 1\}.$$

A particular class of modular spaces are Orlicz–Musielak spaces. To define these spaces, let (E, Σ, μ) be a measure space and let $f : E \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfy the following conditions.

1. $f(t, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nondecreasing, continuous function such that $f(t, 0) = 0$ and $f(t, u) > 0$ for $u > 0$.
2. $f(\cdot, u) : E \rightarrow \mathbb{R}_+$ is a Σ -measurable function for all $u > 0$.
3. $\int_A f(t, u) d\mu(t) < \infty$ for every $u > 0$ and $A \in \Sigma, \mu(A) < \infty$.

Suppose that X is the space of all real (or complex) valued Σ -measurable functions defined on E . For $x \in X$, set

$$\rho_f(x) = \int_E f(t, |x(t)|) d\mu(t).$$

From the above 1) and 2), it is clear that ρ_f is modular. The modular space X_ρ is called Orlicz–Musielak space (and Orlicz space, if the function f is independent of the variable t).

We say the function f satisfies a Δ_2 -condition if

$$f(t, 2u) \leq Kf(t, u) + h(t)$$

for all $u \geq 0, t \in E$ where $h \in L^1(E, \mu), h \geq 0$ and K is a positive constant independent of the variables t, u .

For further theory of modular spaces we refer to [8].

II. Assume a function $\mathcal{N} : X \rightarrow \mathbb{R}_+$ satisfies the following conditions.

1. $\mathcal{N}(x) = 0$ iff $x = 0$.
2. $a_n \rightarrow a$ and $\mathcal{N}(x_n - x) \rightarrow 0$ then $\mathcal{N}(a_n x_n - ax) \rightarrow 0$ for all sequences $\{a_n\} \subset K$ and $\{x_n\} \subset X$.
3. If $\mathcal{N}(x_n - x) \rightarrow 0$ and $\mathcal{N}(y_n - y) \rightarrow 0$ then $\mathcal{N}(x_n + y_n - x - y) \rightarrow 0$ for all sequences $\{x_n\}, \{y_n\} \subset X$.
4. $\mathcal{N}(ax) = \mathcal{N}(x)$ for every $x \in X$ and $a \in K, |a| = 1$;
5. $\mathcal{N}(x_n - x) \rightarrow 0$ then $\mathcal{N}(x_n) \rightarrow \mathcal{N}(x)$ for every $\{x_n\} \subset X$.
6. The space X is complete with respect to the topology induced by the family $C = \{B(x; r) : x \in X, r > 0\}$ where

$$B(x; r) = \{y \in X : \mathcal{N}(x - y) < r\}.$$

The pair (X, \mathcal{N}) is said to be an SF-space and the function \mathcal{N} will be called an SF-norm.

Note that each F-space (in particular each Banach space) is an SF-space. If f satisfies a Δ_2 -condition, then each Orlicz–Musielak space (X_{ρ_f}, ρ_f) is an SF-space. However, there exist SF-spaces which are neither F-spaces nor modular spaces as can be seen by the following

Example ([7]). Assume $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies the following conditions:

1. f is continuous and $f(t) = 0$ iff $t = 0$.
2. There exists $d > 0$ such that $f|_{[0,d]}$ is strictly increasing.
3. There exists $d_1 > 0$ and $M > 1$ such that

$$f(t+s) \leq M(f(t) + f(s)) \text{ for } s, t \in [0, d_1].$$

4. There exists $\lim_{t \rightarrow \infty} f(t) \in (0, \infty)$.

Let (E, Σ, μ) be a measure space, $\mu(E) < \infty$ and let $M(E) = \{x : E \rightarrow \mathbb{R}, x \text{ is } \Sigma\text{-measurable}\}$. For $x \in M(E)$ define $\mathcal{N}(x) = \int_E f(|x(t)|) d\mu(t)$. By Fatou's lemma, the Riesz and the Lebesgue theorems, one can show that the pair $(M(E), \mathcal{N})$ is an SF-space. It is clear that for nonmonotonic f the function \mathcal{N} is not modular, and if we additionally assume

$$\lim_{t \rightarrow +\infty} f(t) < \frac{1}{2} \sup\{f(t) : t \in \mathbb{R}_+\}$$

then the pair $(M(E), \mathcal{N})$ can not be an F-space.

The SF-norm \mathcal{N} is called *nondecreasing* iff for any $t_1, t_2 \in \mathbb{R}$, $|t_1| \leq |t_2|$ implies $\mathcal{N}(t_1 x) \leq \mathcal{N}(t_2 x)$ and given D in an SF-space (X, \mathcal{N}) , we say D is bounded iff $\lambda_n \rightarrow 0$, $x_n \in D$ implies $\mathcal{N}(\lambda_n x_n) \rightarrow 0$. Analogously $D \subset X_\rho$ is ρ -bounded iff $a_n \rightarrow 0$, $x_n \in D$ implies $\rho(a_n x_n) \rightarrow 0$. The SF-norm \mathcal{N} is called *uniformly continuous* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\text{if } \mathcal{N}(f - g) < \delta \text{ then } |\mathcal{N}(f) - \mathcal{N}(g)| < \varepsilon.$$

Note that in the definition of the SF-space, \mathcal{N} is only a continuous function.

Proposition 1. Let (X, \mathcal{N}) be an SF-space with \mathcal{N} being uniformly continuous. Then for every $\varepsilon > 0$ there is V , an open neighbourhood of 0, such that

$$B^{\mathcal{N}}(0, r) + V \subset B^{\mathcal{N}}(0, r + \varepsilon) \quad (*)$$

where $B^{\mathcal{N}}(x, r) = \{y \in X : \mathcal{N}(x - y) \leq r\}$.

The proof is straightforward.

III. Let (X, \mathcal{N}) be an SF-space. For $V \subset X$, $V \setminus \{0\} \neq \emptyset$ put

$$R_{\mathcal{N}}(V) = \inf\{\sup\{\mathcal{N}(tv) : t \in \mathbb{R}_+\} : v \in V \setminus \{0\}\}.$$

This number which may be equal to $+\infty$, is called the radius of the set V ([7]). As shown in the following example, it may also occur that $R_{\mathcal{N}}(V) = 0$.

Example. Let X be the space of all complex sequences equipped with an SF-norm \mathcal{N} defined by

$$\mathcal{N}(x) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \frac{|x_i|}{1 + |x_i|} \quad \text{for } x \in X.$$

Let for $n \in \mathbb{N}$, $V_n = \{x \in X : x_j = 0 \text{ for } j > n\}$. Then clearly $R_{\mathcal{N}}(V_n) = 2^{-n}$ and consequently $R_{\mathcal{N}}(X) = 0$.

In [7] it is shown that if X_{ρ} is a modular space with ρ s -convex modular, then $R_{\rho}(X_{\rho}) = +\infty$.

K -widths are extensively studied in the context of approximation theory [10]. Our aim in this paper is to connect K -widths with measures of noncompactness. Such connections are not only useful in fixed point theory (see [3], [11]) but also in the study of the radius of the essential spectrum (see [6], [9]). Measures of noncompactness for Orlicz spaces are studied in [1], [2] and [4]. In [5] one can find fixed point theorems for Orlicz modular spaces.

2. K -widths in SF-spaces

Let (X, \mathcal{N}) be an SF-space, D be bounded set in X . Then the ball measure of noncompactness of D , $\alpha(D)$, is

$$\alpha(D) = \inf\{r > 0 : D \subset \bigcup_{k=1}^n B^{\mathcal{N}}(x_k, r)\}$$

and K -th width of D , $d^k(D)$, is defined as

$$d^k(D) = \inf\{r > 0 : D \subset B^{\mathcal{N}}(0, r) + A_k \text{ dim}(A_k) \leq k\}.$$

Theorem 1. Let D be a bounded subset of an SF-space (X, \mathcal{N}) . Suppose that \mathcal{N} is nondecreasing, $R_{\mathcal{N}}(X) = \infty$ and for every $r > 0$, $\varepsilon > 0$ and E finite dimensional subspace of X , there is an open neighborhood V of 0 in E such that

$$B^{\mathcal{N}}(0, r) + V \subset B^{\mathcal{N}}(0, r + \varepsilon). \quad (*)$$

Then $\alpha(D) = \lim_{k \rightarrow \infty} d^k(D)$.

Proof. Let D be a fixed bounded set in X . If there is $r > 0$ and $k \in N$ such that $D \subset \bigcup_{i=1}^k B^{\mathcal{N}}(x_i, r)$, then

$$D \subset B^{\mathcal{N}}(0, r) + \bigcup_{i=1}^k \{x_i\} \subset B^{\mathcal{N}}(0, r) + \text{Span}(x_1, x_2, \dots, x_k).$$

Therefore, if $\alpha(D) < \infty$ then $d = \lim_k d^k(D)$ is finite and $d \leq \alpha(D)$.

To obtain the other inequality, assume that $d < +\infty$, fix $k \in N$, $\varepsilon > 0$ and $A_k \subset X$ with $\dim A_k = k$ such that

$$D \subset B^{\mathcal{N}}(0, d^k(D) + \varepsilon) + A_k.$$

Let us define

$$D_1 = \{g \in B^{\mathcal{N}}(0, d^k(D) + \varepsilon) : \text{there is } h \in A_k, g + h \in D\}$$

$$D_2 = \{h \in A_k : \text{there is } g \in B^{\mathcal{N}}(0, d^k(D) + \varepsilon), g + h \in D\}.$$

Obviously $D \subset D_1 + D_2$. Now we claim that D_2 is a bounded set in (X, \mathcal{N}) . Assume on the contrary that there is $\{h_n\} \subset D_2$ and $\{\lambda_n\} \subset \mathbb{R}$, $\lambda_n \rightarrow 0$ such that $\mathcal{N}(\lambda_n h_n)$ does not tend to zero. Since $\dim(A_k) = k$, one has

$$\lambda_n h_n = \sum_{i=1}^k r_i^n y_i$$

where y_1, \dots, y_k is a fixed basis of A_k . Consider the following cases.

Case 1: $\sup_{n \in N} (\max_{1 \leq i \leq k} |r_i^n|) < +\infty$. Then passing to a subsequence, if necessary, by the properties of \mathcal{N} (II.2 and 3) we may assume $\mathcal{N}(\lambda_n h_n - h) \rightarrow 0$ for some $h \in X$, $h \neq 0$.

On the other hand, by the definition of D_2 , for every $n \in N$ there is $g_n \in D_1$ with $f_n = g_n + h_n \in D$. Since $R_{\mathcal{N}}(X) = +\infty$, there is a $t > 0$ such that $\mathcal{N}(th) > d^k(D) + \varepsilon$. By the boundedness of D , $\mathcal{N}(\lambda_n f_n) \rightarrow 0$. By II.1, II.2 and II.5 of the definition,

$$\mathcal{N}(t\lambda_n g_n) \rightarrow \mathcal{N}(th) > d^k(D) + \varepsilon.$$

But for $n \geq n_0$, $t\lambda_n < 1$. Since \mathcal{N} is nondecreasing

$$\mathcal{N}(t\lambda_n g_n) \leq \mathcal{N}(g_n) \leq d^k(D) + \varepsilon \quad \text{for } n \geq n_0,$$

a contradiction.

Case 2: there is $i_0 \in 1, \dots, k$ with $\lim_{n \rightarrow \infty} \inf |r_{i_0}^n| = +\infty$. Passing to a subsequence, if necessary, we may assume that for each $i \in \{1, 2, \dots, k\}$,

$$c_i = \lim_n \frac{r_i^n}{r_{i_0}^n} \quad (c_{i_0} = 1).$$

By the Definition II.1, we have

$$\mathcal{N} \left[\left(\frac{\lambda_n}{r_{i_0}^n} \right) h_n - \sum_{i=1}^k c_i y_i \right] \rightarrow 0.$$

We can set $h = \sum_{i=1}^k c_i y_i$, since y_1, \dots, y_k is a basis of A_k with $c_{i_0} = 1$, $h \neq 0$. Considering the subsequence $\lambda_n/r_{i_0}^n$ instead of λ_n and reasoning as in Case 1, we obtain a contradiction.

Therefore D_2 is a bounded set with respect to the topology defined by \mathcal{N} . Since each SF-space, as a complete, topological linear space with a countable basis of neighborhoods of 0 is metrizable, the SF-space (A_k, \mathcal{N}) is metrizable. But A_k is finite dimensional, the topology induced by \mathcal{N} in A_k is the same as any norm topology in A_k . Hence, D_2 is bounded in any norm in A_k . Consequently \bar{D}_2 (the closure of D_2 in A_k in any norm) is a compact set.

Now by the assumption (*), there is V an open neighborhood of 0 in A_k such that

$$B^{\mathcal{N}}(0, d^k(D) + \varepsilon) + V \subset B^{\mathcal{N}}(0, d^k(D) + 2\varepsilon).$$

Since \bar{D}_2 is a compact set $\bar{D}_2 \subset \bigcup_{i=1}^l x_i + V$. Note that

$$\begin{aligned} D &\subset D_1 + D_2 \\ &\subset B^{\mathcal{N}}(0, d^k(D) + \varepsilon) + \bigcup_{i=1}^l \{x_i + V\} \\ &\subset \bigcup_{i=1}^l \{x_i + B^{\mathcal{N}}(0, d^k(D) + 2\varepsilon)\} \\ &= \bigcup_{i=1}^l B^{\mathcal{N}}(x_i, d^k(D) + 2\varepsilon). \end{aligned}$$

Consequently, $\alpha(D) \leq d^k(D) + 2\varepsilon$ for every $\varepsilon > 0$ and $k \in N$, which yields

$$\alpha(D) \leq d = \lim_k d^k(D).$$

Remark. Condition (*) in Theorem 1 is satisfied if \mathcal{N} is uniformly continuous, which covers the case of F-spaces. Moreover, if one assumes $\lim_{u \rightarrow \infty} f(t, u) = +\infty$ then the corresponding ρ_f F-norm $|\cdot|_\rho$ satisfies the assumptions in Theorem 1.

3. K -widths in modular spaces

Let X_ρ be a modular space and $D \subset X_\rho$ be a ρ -bounded set. Let B_ρ be the closed ρ -ball in X_ρ i.e., $B_\rho = \{x \in X_\rho : \rho(x) \leq 1\}$. K -width of D in X_ρ , $d_\rho^k(D)$, defined

as

$$d_\rho^k(D) = \inf\{\lambda > 0 : D \subset \lambda B_\rho + H : \dim H \leq k\},$$

and ρ -ball measure of noncompactness of D , $\alpha_\rho(D)$, defined as

$$\alpha_\rho(D) = \inf\{\lambda > 0 : D \subset \bigcup_{i=1}^k \lambda(x_i + B_\rho) : x_1, \dots, x_k \in X_\rho\}.$$

Note that if ρ is a norm, then the notations d_ρ^k and α_ρ above coincide with the classical definitions of d^k and α .

The following are some basic properties of $\alpha_\rho(D)$.

Proposition 2. a) *Monotonicity:* $D_1 \subset D_2$ implies $\alpha_\rho(D_1) \leq \alpha_\rho(D_2)$.

b) *Semi-additivity:* $\alpha_\rho(D_1 \cup D_2) = \max\{\alpha_\rho(D_1), \alpha_\rho(D_2)\}$.

c) *Invariance under translation:* $\alpha_\rho(D + x_0) = \alpha_\rho(D)$ for any $x_0 \in X_\rho$.

d) *Algebraic semi-additivity:* If ρ is convex, then $\alpha(D_1 + D_2) \leq \alpha(D_1) + \alpha_\rho(D_2)$.

e) $\alpha_\rho(D) = \alpha_\rho(\overline{D})$ where the closure is taken with respect to F -norm. Moreover if ρ is convex, then $\alpha(D) = \alpha(c_0(D))$ where $c_0(D)$ denotes the convex hull of D .

f) If B_ρ is a $|\cdot|_\rho$ -bounded set, then $\alpha_\rho(D) = 0$ iff D is $|\cdot|_\rho$ -compact.

Proof. a) through e) follow from the definition. To prove f), set $B_{|\cdot|_\rho}(0, r) = \{x \in X_\rho : |x|_\rho \leq r\}$ and $\alpha_{|\cdot|_\rho}(D) = \inf\{r > 0 : D \subset \bigcup_{i=1}^k [x_i + B_{|\cdot|_\rho}(0, r)]\}$, we claim that $\alpha_\rho(D) = 0$ implies $\alpha_{|\cdot|_\rho}(D) = 0$ and hence D is $|\cdot|_\rho$ -compact. Since B_ρ is bounded set, $\lambda_0 \cdot B_\rho \subset B_{|\cdot|_\rho}(0, r)$ for some λ_0 . Then since $D \subset \bigcup_{i=1}^k \{x_i + \lambda_0 \cdot B_\rho\} \subset \bigcup_{i=1}^k [x_i + B_{|\cdot|_\rho}(0, r)]$, we have the desired result. On the other hand, if D is $|\cdot|_\rho$ -compact set, then for every $\lambda > 0$, $D \subset \bigcup_{i=1}^k [x_i + \lambda \cdot \text{int}(B_\rho)]$. Hence, for every $x \in D$ contained in the open set $x_i + \lambda \cdot \text{int}(B_\rho)$ since λ was arbitrary $\alpha_\rho(D) = 0$.

Theorem 2. Suppose ρ is a modular and $R_\rho(X_\rho) = +\infty$. Suppose that the Luxemburg norm $|\cdot|_\rho$ satisfies the following conditions:

1. For every $\varepsilon > 0$, there is $\delta > 0$ with

$$B_{|\cdot|_\rho}(0, 1 + \delta) \subset (1 + \varepsilon)B_{|\cdot|_\rho}(0, 1).$$

2. $|x_n| \rightarrow 0$ iff $\rho(x_n) \rightarrow 0$.

3. $\lim_{c \rightarrow 1} \rho(cx) = \rho(x)$ for $x \in X_\rho$.

Then $\lim_{k \rightarrow \infty} d_\rho^k(D) = \alpha(D)$.

Proof. As in the proof of Theorem 1, one can easily establish that $\lim_k d_\rho^k(D) \leq \alpha(D)$.

To prove the converse, consider $\varepsilon > 0$ and λ such that

$$\lim_k d_\rho^k(D) < \lambda < \lim_k d_\rho^k(D) + \varepsilon.$$

Define analogously as in Theorem 1,

$$D_1 = \{x \in B_{|\cdot|_\rho}(0, 1) : \text{there is } h \in H_{k_0} \text{ with } \lambda x + h \in D\};$$

$$D_2 = \{h \in H_{k_0} : \text{there is } x \in B_\rho, \lambda x + h \in D\}.$$

Note that by the Condition 3) in the Theorem 2 above, $B_\rho = B_{|\cdot|_\rho}(0, 1)$. Let H_{k_0} be a subspace of X_ρ with $\dim H_{k_0} \leq k_0$ and $D \subset \lambda B_\rho + H_{k_0}$. Reasoning as in Theorem 1, together with the Conditions 2) and 3) above, one can show that D_2 is a bounded set.

Now for $\varepsilon > 0$, choose $\delta > 0$ such that Condition 1) in the above Theorem 2 is satisfied. Since D_2 is a bounded set, $D_2 \subset \bigcup_{i=1}^k \lambda B_{|\cdot|_\rho}(x_i, \delta)$. Therefore,

$$\begin{aligned} D &\subset \lambda D_1 + D_2 \\ &\subset \lambda B_\rho + \bigcup_{i=1}^k \lambda B_{|\cdot|_\rho}(x_i, \delta) \\ &= \lambda [B_\rho + \bigcup_{i=1}^k \{B_{|\cdot|_\rho}(0, \delta) + x_i\}] \\ &\subset \lambda \left[\bigcup_{i=1}^k \{B_{|\cdot|_\rho}(0, 1 + \delta) + x_i\} \right] \\ &\subset \lambda \left[\bigcup_{i=1}^k \{(1 + \varepsilon)B_\rho + x_i\} \right] \\ &= \lambda(1 + \varepsilon) \left[\bigcup_{i=1}^k \left\{ B_\rho + \frac{x_i}{1 + \varepsilon} \right\} \right] \end{aligned}$$

Thus, $\alpha_\rho(D) \leq \lambda + \lambda\varepsilon < \lim_k d_\rho^k(D) + \varepsilon + \lambda\varepsilon$ for every $\varepsilon > 0$. Hence $\alpha_\rho(D) \leq \lim_k d_\rho^k(D)$ as desired.

Remark. Condition 2) in Theorem 2 is equivalent to Δ_2 condition in Musielak–Orlicz spaces (see [8]). The measure in the definition of Musielak–Orlicz spaces must be *sigma*-finite and atomless.

The following theorem applies to any s -convex, $0 < s \leq 1$, modular function, not just to Orlicz spaces.

Theorem 3. Let ρ be left continuous s -convex modular, $0 < s \leq 1$. Then

$$\lim_k d_\rho^k(D) = \alpha_\rho(D).$$

Proof. We need to show that the assumptions of Theorem 2 are satisfied. By the [7], we already know that $R_\rho(X_\rho) = +\infty$. In the s -convex case, the Conditions 2)

and 3) in Theorem 2 are not necessary to prove that D_2 is a bounded set. Since in this case $B_{1|\rho}(0, 1) = B_\rho$ (here the left continuity is needed) is a ρ -bounded set ($a_n \rightarrow 0, \rho(x_n) \leq 1$ then for $n \geq n_0, \rho(a_n x_n) \leq a_n^s \rho(x_n) \leq a_n^s \rightarrow 0$). Consequently, D_1 is a ρ -bounded set and this implies that D_2 is a ρ -bounded set. To see this, consider $a_n \rightarrow 0, h_n \in D_2$, then $\lambda g_n + h_n = f_n \in D, \rho(g_n) \leq 1$.

Note that, by I.3,

$$\begin{aligned} \rho(a_n h_n) &= \rho(a_n(f_n - \lambda g_n)) \\ &= \rho\left(\frac{1}{2}[2a_n f_n + (-2\lambda a_n g_n)]\right) \\ &\leq \rho(2a_n f_n) + \rho(2\lambda a_n g_n). \end{aligned}$$

Since D_2 is ρ -bounded and λB_ρ is ρ -bounded, the last two terms tend to 0 as $n \rightarrow \infty$.

To complete the proof, we need to establish Condition 1) in Theorem 2, for any s -convex modular ρ .

First observe that $B_{1|\rho}(0, 1 + \varepsilon) = (1 + \varepsilon)^{1/s} B_\rho$. Let $\varepsilon > 0$ be fixed, choose $\delta > 0$ such that $(1 + \delta)^{1/s} \leq 1 + \varepsilon$. Then

$$B_{1|\rho}(0, 1 + \delta) = (1 + \delta)^{1/s} B_\rho \subset (1 + \varepsilon) B_\rho.$$

Remark. The above theorem is an improvement of Theorem 2 in [1], which clarifies the solution in the s -convex case without any Δ_2 -condition.

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