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Students Studying Students and Reasoning about Reasoning: A Qualitative Analysis

Salvatore J. Petrilli
Adelphi University

Grant Clark
Adelphi University

Nicholas DeMarco
Adelphi University

Jack Esposito
Adelphi University

Brianne Giuliano
Adelphi University

See next page for additional authors

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Cover Page Footnote

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Authors

Salvatore J. Petrilli, Grant Clark, Nicholas DeMarco, Jack Esposito, Brianne Giuliano, Sara Greiss, Emily Harris, Alessia Merritts, Kyle Murray, Mateusz Piekut, Brian Seidl, Scott Shannon, Nicole Silva, Christina Sullivan, Brittany Willoughby, and Yile Zhou

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Adelphi University, New York, USA
petrilli@adelphi.edu

Abstract

In this work, a faculty member takes a journey along with students as they enhance their understanding of how people solve mathematical problems through a mainly qualitative statistical project. Student authors of this paper registered for a problem solving seminar led by the faculty author, and then created and analyzed self-built assessment tools to explore problem solving techniques. Here we share our findings and recommendations, which we hope will inspire others to explore novel pedagogical techniques in the teaching of mathematical problem solving. We incorporate into our presentation our voices, reflecting on how we and others solve problems.

Key words: mathematics education; problem solving; qualitative statistics.

1. Introduction and the Mathematics Honors Seminar

Problem solving has always been a central part of the mathematics curriculum and that emphasis has only increased, which is clearly stated by the Committee on Undergraduate Programs in Mathematics [5]. In that same document, the CUPM has also stated the importance of data analysis having a large role in mathematics curricula.

Many studies exist, such as [9, 11, 13], which explore different pedagogical ideas for the teaching and learning of problem solving. However, here is a wild idea: create a project that explores problem solving techniques from a data analysis point-of-view. As the students perform the statistical analysis, they will enhance their understanding of problem solving! This paper describes a one-credit seminar that intertwines problem solving and data analysis.

Our institution Adelphi University is a predominantly undergraduate institution located in Garden City, New York, on Long Island.[1] The mathematics program is offered by a joint Department of Mathematics and Computer Science. In addition to two undergraduate degrees in mathematics (B.A. and B.S.), we also offer a B.S. in Statistics, a B.S. in Computer Science, and, jointly with the School of Business, a B.S. in Information Systems. At the graduate level, the Department offers an M.S. in Computer Science and an M.S. in Applied Mathematics and Statistics.

The Mathematics Honors Seminar (MTH 290) was initially created as a one-credit seminar that was mandatory for an undergraduate student to graduate with Departmental Honors.[2] The content of the seminar depended on the faculty member who taught the course. Topics in the past included: Complex Analysis, The Mathematics Behind Origami, Great Theorems in Mathematics, and Combinatorics.

In Fall 2010, SJP, the first author of this paper, taught this seminar, where the topic was Problem Solving.[3] In this course, we read the great work of George Polya, *How to Solve It* [8]. Additionally, we explored the well-known work of Brown and Walter, *The Art of Problem Posing* [4]. The most important part of this course was that we solved lots of problems as a group!

The course covered the following problem solving techniques:

- Related Problems, Heuristics, and Analysis and Synthesis
- Working Backwards

[1]You can learn more about Adelphi University at: `https://about.adelphi.edu/overview/quick-facts/fact-sheet/`.

[2]The other requirement was a major GPA of at least 3.50. The MTH 290 requirement was dropped in Fall 2009, due to insufficient staff to run the course every semester. The course now runs as an optional elective and gives no credit towards the mathematics major.

[3]This seminar was inspired by a three-credit lecture course that SJP had taken at Teachers College, Columbia University, under the instruction of Dr. J. Philip Smith.

- Induction and Mathematical Induction
- Generalization and Specialization
- Parity
- Symmetry
- Can you Derive the Result Differently?

Before each class, students would read the associated problem solving technique from Polya [8] and supporting readings from Brown and Walter [4]. Then, they would attempt to solve problems that were related to the technique we were covering that week. Students would come in and make presentations on the problems that had been assigned. As a class, we would discuss the validity of the solutions, as well as possible alternate solutions. Also, on many occasions, we would apply the "What-If-Not" strategy of Brown and Walter [4] to add new perspectives to the problems.

At the end of the semester, three students wrote the following about the Honors Seminar:

> *I gained a lot of knowledge about myself, how I approach problems, and new useful techniques I can apply to other math classes.*
>
> *Great insight to the mathematical world.*
>
> *I've grown a lot as a critical thinker in class.*

Given the enthusiasm of the students, in Spring 2018, SJP decided to run the problem solving seminar again. However, he felt the need to spice up the course a bit. He was inspired by the CUPM's strong recommendation:

> Mathematical sciences major programs should include concepts and methods from data analysis. [5]

From a statistical-point-of-view, many of the recommendations made by the CUPM are clearly geared towards quantitative data analysis. Qualitative data analysis always appears to get the short end of the stick; however, qualitative data adds a lot of insight that quantitative data cannot provide. In particular, analyzing students' solutions to problems from a qualitative point-of-view adds much insight into the learning of problem solving. Therefore, SJP decided to incorporate a qualitative data analysis project into this new version of the course.

2. The Project

The Spring 2018 course followed the same content and structure as the Fall 2010 course described above; however, a week was devoted to qualitative trend analysis. This naturally carried the students into the qualitative analysis project that is the topic of this paper.

At the start of the semester, each student was assigned to a group. The class had fifteen students enrolled, and four groups (three groups of four and one group of three) were formed. Each group was assigned a specific problem solving technique and was asked to select a grouping variable so a comparative analysis could be conducted (see Table 1).

Group Members	Problem Solving Technique	Grouping Variable
Group 1: Grant, Jack, Alessia, Christina	Related Problems and Heuristics	Class (Freshman, Sophomore, Junior, Senior)
Group 2: Brittany, Sara, Brianne, Nicholas	Induction and Mathematical Induction	Major (Math, Math Education, Other)
Group 3: Jireh, Nicole, Scott, Mateusz	Working Backwards	Major (Mathematics, Computer Science)
Group 4: Brian, Kyle, Emily	Symmetry	Play a Musical Instrument (Yes, No)

Table 1: Student groups with their problem solving assignments.

Each group created an assessment tool to evaluate their specific problem solving technique (see Appendices A-D). We claim no originality for the problems included in these surveys. We listed in our References the sources of at least some of the problems, and others were adaptations of classic problems done in class.

Participants were self-selected Adelphi mathematics students. The surveys were shared with the student body through an e-mail request by the Chair of the Department. Students enrolled in MTH 290 were not permitted to participate. So none of the participants in this experiment had been formally exposed to the problem solving content covered in this seminar.

The groups utilized the problem solving rubric [3] developed by the Association of American Colleges & Universities (AAC&U) to assess the survey responses. To ensure a valid assessment, each group did a norming session with the rubric.

In the next four sections of this paper, you will take a journey with each group in the class as they explore how people solve problems, reflect on their experiences, and formally present their results.

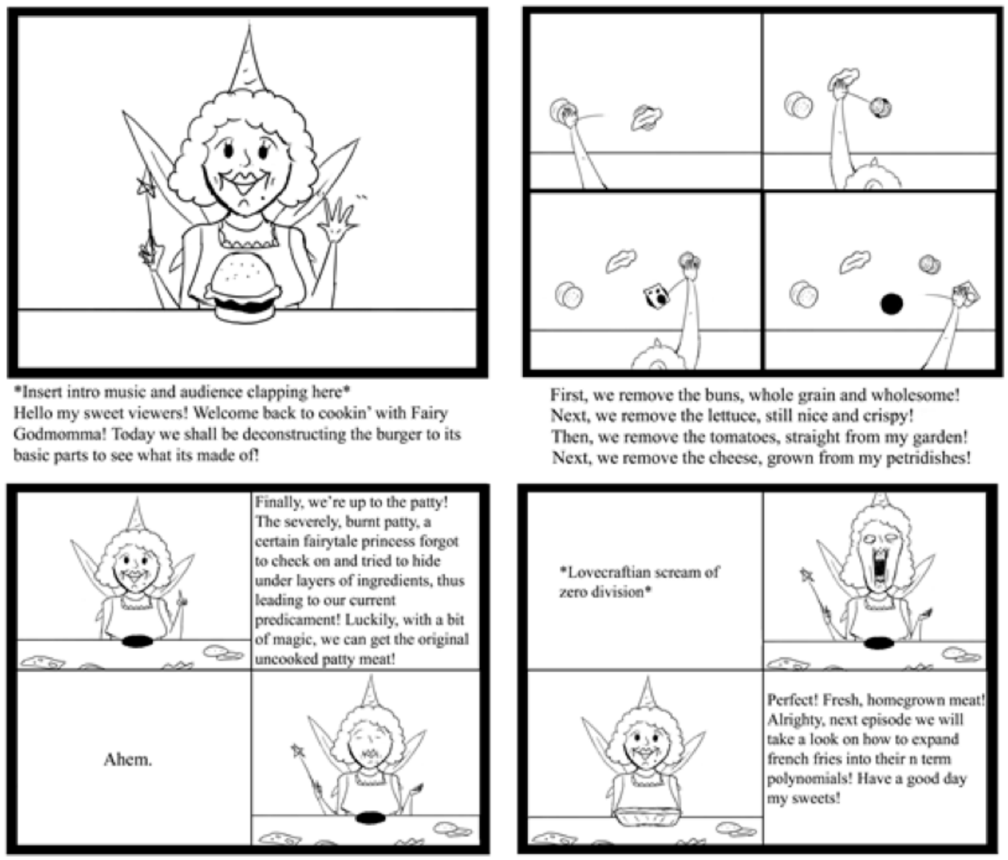


Figure 1: Decomposing a Problem. Cartoon image created by Mateusz Piekut.

3. Heuristic Reasoning and Related Problems (Group 1)

3.1. What is the Heuristic/Related Problems Problem Solving Technique?

Heuristics can be described as a practical means of finding a solution. It does not rely on perfection or optimization. Polya [8] describes heuristics as a problem solving technique based on intuitive thinking and rational shortcuts.

The process of solving a problem is therefore sped up, and a satisfactory answer is achieved. It is important to not associate rigor to heuristic thinking because it is an imperfect process that is often dependent on educated guesses.

Heuristics connects well with the "related problems" problem solving technique. Polya [8] encourages us to use related problems to draw back on previous solutions, methods, or overall experiences. To demonstrate this, we will provide a student-teacher dialogue.

> **Student:** *So I drew a parallel line to one side of a triangle through the opposite vertex. I do not see how this is helpful in proving the angle sum of a triangle.*
>
> **Teacher:** *Write down anything you know about all angles created. If you're stuck, tell me your thought processes.*
>
> **Student:** *Well I know we created three supplementary angles, and I know those sum to* $180°$. *Interesting,* $180°$ *comes up again.*
>
> **Teacher:** *So can you use a previous theorem to help you complete this problem?*
>
> **Student:** *It just came to me! We create two pairs of congruent alternate interior angles and can conclude the angle sum of a triangle is* $180°$.

3.2. Methodology

Participants: Our sample size was n = 10 students, subsequently broken down into two groups of 5 students: Upper-classmen (Juniors and Seniors) and Under-classmen (Freshmen and Sophomores).

Materials: Our first question was a geometry question testing participants' ability to use a 30-60-90 special right triangle. They were given the standard proportions of the triangle and were instructed to use this to solve for another right triangle. Our second question was simple at the surface, yet demanding to solve. It was to find the area of a triangle given two side lengths of 8 and 6 and an angle measuring 60 degrees between. See Appendix A for our survey.

Procedure: We conducted a norming session using the AAC&U Problem Solving VALUE Rubric [3] to grade the pilot exams. After conducting the norming session we plotted the averages of the first question vs. the averages of the second in R. From this we were able to get a reliability index of 0.7. After this, we conducted our survey by having the participants complete the questions with a twenty-minute time limit. This was done over the span of one week in the library. Next, we graded each person's survey using the same AAC&U Rubric. Lastly, we analyzed qualitative trends for similarities and differences between the two groups, which is discussed further in the following two sections.

3.3. Results

Using the AAC&U Problem Solving VALUE Rubric, we obtained the following scores for each group: Our Upper-classmen averaged 3.75 on Question One and 2.62 on Question Two and our Under-classmen averaged 3.35 on Question One and 2.31 on Question Two.

The immediate analysis was that across both groups, question one was answered correctly and was solved efficiently. For the most part, all participants succeeded in visualizing and applying what was needed to solve the problem. Everybody utilized equations, diagrams, and justifications to their advantage in order to adequately solve Question One. Our ultimate goal was to qualitatively look at how participants would use what they learned in Question One to aid in the solving of Question Two. That is, we were looking for instances of participants using Heuristic Reasoning and Related Problems.

Firstly, we noticed right away that the Upper-classmen were more willing to apply Heuristic Thinking (see Figure 2 below), which would seem to come from their mathematical maturity. This is because they all seemed to have some sort of thought process stemming from Problem One whereas there were some Under-classmen that did not reference Problem One whatsoever. This was particularly highlighted by the fact that two of our underclassmen used the formula

$$\text{Area} = \frac{1}{2}ab\sin C$$

to find the triangle's area. No Upper-classmen solely relied on that formula to answer Question Two. We found this very interesting because this formula was not given to our participants. Also, this formula is not related to 30-60-90 special right triangles, which was what Question One was testing.

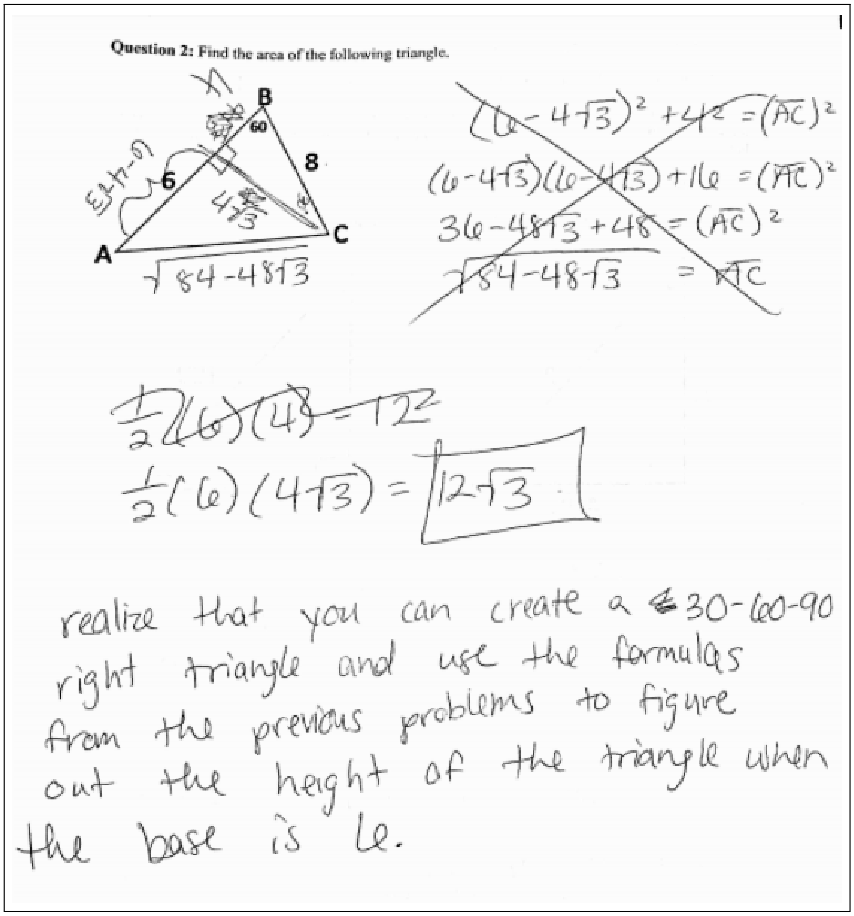


Figure 2: Example Upper-classman work.

Under-classmen seemed to rely on valuable geometric formulas learned in high school and upperclassmen seemed to rely on their ability to solve math problems applying thinking required to grow as a math major. This includes, but is not limited to: drawing diagrams, providing justifications, and being concise. One last observations would be how upperclassmen all drew some sort of altitude to create a 30-60-90 right triangle involving the given 60 degree angle. This shows a direct relation to using related problems because they were trying to make the difficult Problem Two into a question like Problem One. Some of the underclassmen did this, but it was not unanimous like the upperclassmen.

In terms of providing explanations, the Upper-classmen were more thorough with their justifications. They were also more willing to provide explanations. The Under-classmen were more direct and to the point in just giving a formula and the answer (See Figure 3). This is particularly interesting because actually more under-classmen got the correct answer of 123 than Upper-classmen did. However, the Upper-classmen still got higher scores. This is because qualitatively the Upper-classmen provided more justification to their work. Their equations and diagrams were all justified with clear thoughts. The Under-classmen, the ones that provided no explanations at all, were more jumbled with their thoughts. They seemed more willing to rely on formulas and instincts rather than well-thought out explanations.

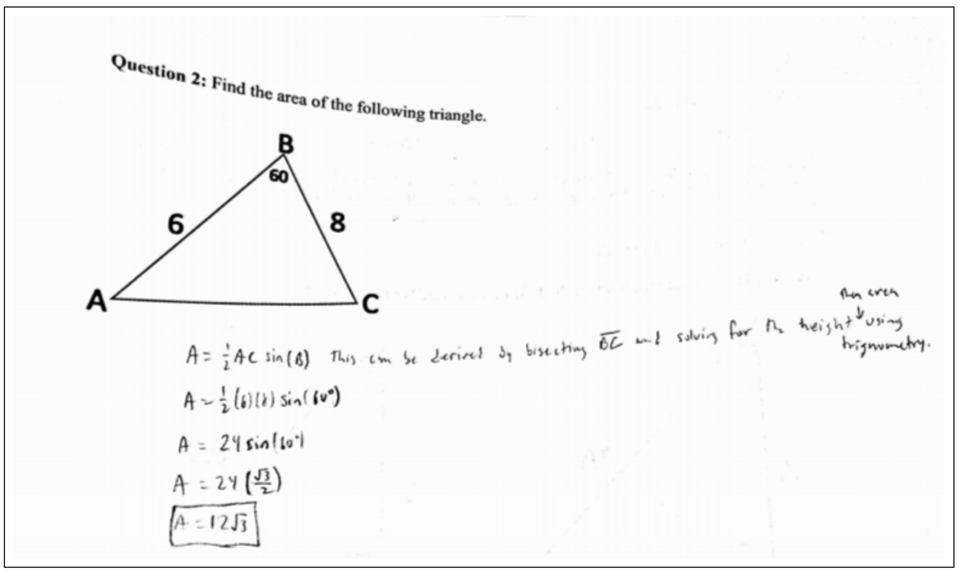


Figure 3: Example Under-classman work.

3.4. What Did We Learn About Problem Solving?

In terms of solving problems, we learned a few things. Firstly, we discovered that when people are approached with a math problem, they often stick to what they know. This was evident in the fact that so many of our participants drew an altitude to the triangle from the top of the triangle. The problem called for drawing the altitude from one of the base corners, however, in high school students are primarily taught to draw altitudes from the top corner.

This skewed the expected method of the solution for this problem. We have now learned that solving problems requires "outside of the box" thinking. We must challenge our traditional modes of thinking and analyze all angles and different perspectives a problem gives us.

Some techniques to analyze problems include drawing diagrams, labeling everything, writing down key information, justifying all formulas, and checking your solutions. Completing all of these actions demonstrates that we are dedicated to the problem. It also will increase our ability to solve problems in the future. We must take every problem with gratitude as it can be useful in solving future problems.

Let us take a look at a dialogue between two of own group members: Jack and Christina. We have different futures in math: Jack a future high school math teacher and Christina a future applied mathematician.

> **Christina:** *Jack, I know that you are studying to be a high school math teacher. How do you think our results could be useful in the classroom?*
>
> **Jack:** *Our results for problem solving can be strategically used to education my future students. For one, I learned that I must repeatedly remind my students to not think one-dimensionally. Looking through and appreciating different perspectives is a necessity.*
>
> **Christina:** *I completely agree! Not only are those techniques useful for math students, but for any professional in any field, particularly in math (like myself). If one approach does not work, changing the way you look at the problem opens up new opportunities to reach a solution.*
>
> **Jack:** *Couldn't have said it any better. That's my future goal as an educator; to inspire creativity in my students to help them think critically. I also have to make sure my students always have a drive to learn new things.*
>
> **Christina:** *Absolutely! Students must give it their all when trying to solve problems, as it will make them career and college ready. I wish you the best of success in your future classroom! You're going to mold some great future problem solvers.*

4. Induction (Group 2)

4.1. What is the Induction Problem Solving Technique?

Induction is the process of solving easier problems to develop the skills needed to solve a more difficult problem. Induction utilizes general observations and is then combined with the information given in a particular problem [8]. The Principle of Mathematical Induction (PMI) is a formalized method of induction commonly used in mathematics. Before a problem can be solved with PMI, it first must be proved that the base case works and then from there a student can solve for what the question is asking using the same method.

> **Brittany:** *Bella texted me saying that she was having trouble understanding induction. Before I respond back to her, I wanted to run my thoughts by you guys. I think of induction as a process that helps students understand the basis of what the question is asking and then from there helps them build upon their knowledge to solve more difficult problems.*
>
> **Sara:** *Exactly Brittany! Induction simplifies things; you use what you already know to try to prove things about all natural numbers- it's like building blocks you have to use what is already there to get to the next level.*
>
> **Nick:** *Remember the example that we talked about in our proofs class where we found the sum of this series:*
>
> $$1 + 2 + 3 + ... + 200?$$
>
> *First we would start with looking at partial sums to find a pattern eventually working our way to determining that we can apply the formula,*
>
> $$S_n = \frac{n(a_1 + a_n)}{2};$$
>
> *which then we proved using PMI to show that it always worked for every arithmetic series. It's a classic example that always helps me when doing induction!*

> **Brianne:** *Mhmm! So Bella should start off by proving the "base case." This is the scenario of the first number in the set of numbers you are trying to prove this for. For example, in the set of natural numbers, you would start with $n = 1$. This is the most "basic" case of the proof. Then you would assume the $n = k$ case and prove the $n = k + 1$ case to show that the theorem always holds for the next level of blocks in the building!*

4.2. Methodology

Participants: Our sample consisted of n = 11 people, 6 of them were strictly math majors, and 5 of them were math majors who were also a part of the education program at Adelphi University.

Materials: For our survey we selected two questions, one more straightforward and one that required more advanced knowledge on PMI. The first question was created by us and the second was inspired by a similar question that was given for an assignment for a class at Carnegie Mellon University; see Appendix B for a copy of the survey.

Procedure: We first gathered the two questions that we wished to use within our survey. To make sure that the questions we selected were deemed to be fair, we conducted a pilot using the same questions. We then selected five students outside of Adelphi to be apart of our pilot. We gave all students in the pilot and our sample twenty minutes to complete the survey and told them to write as much as possible.

After we administered our pilot, we did a norming session using the AAC&U grading rubric [3]. In order to ensure reliability, we needed to verify that the correlation coefficient of the scores of question two vs question was above the minimum of 0.5 which was established by Dr. Petrilli; the reliability index of our survey was 0.63.

Our sample was given the same instructions as the pilot: twenty minutes, no calculator, and write as much as possible. After they completed the survey, we had them answer basic questions regarding their experience in an introduction to proofs course and their academic year and programs. We then graded the surveys using the AAC&U grading rubric. After all of the surveys were graded, we examined their quantitative score and then conducted a qualitative analysis of their actual solutions.

4.3. Results

The mean total score for the math majors in the education program was 3.4 and for the math majors not in the education program the mean was 3.6.

> **Brianne:** *So we're able to see that there is a subtle difference in the way mathematics majors and mathematics majors in the education program answered this question. Which way of thinking is better?*
>
> **Nick:** *Both are the best! Non-education math majors lean more towards using straightforward calculations and specific formulas. Instead of explaining their work by writing paragraph responses, they use just a few lines of calculation to justify their answer. It's like the student in Figure 4.*

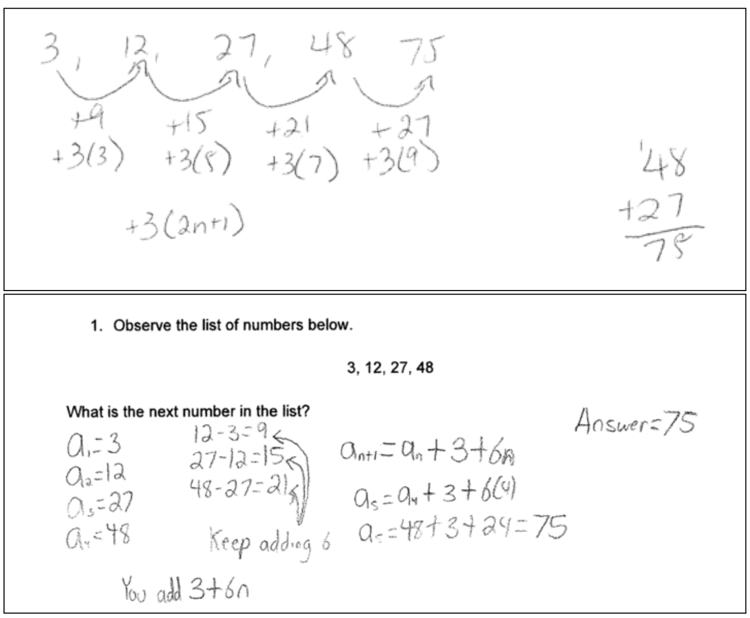


Figure 4: Two examples of work done by non-education math majors.

> **Brittany:** *And future teachers are trained to be more descriptive. That's why in their responses they use writing and pictures to explain their answer. It's how we're taught to teach our students so it's how we're taught to solve problems ourselves!*

This can be seen in how Student 3 answered and explained, using words, problem 1 in Figure 5.

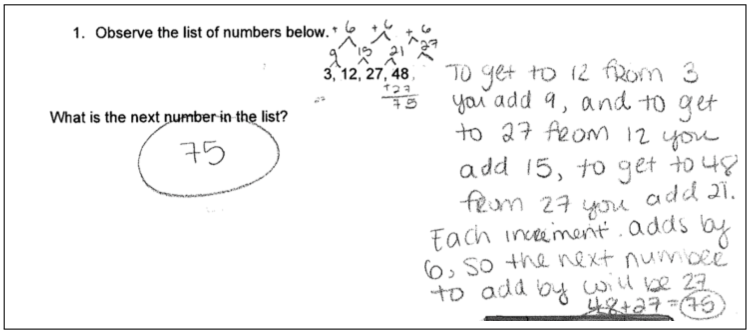


Figure 5: Example of a math education major.

Sara: *Precisely! Both mathematics majors and mathematics majors in the education program are able to come to the same conclusion or answer, where the mathematics education students just use more words.*

4.4. What Did We Learn About Problem Solving?

We discovered that math majors that are in a teaching program are more inclined to justify their reasoning in solving a problem by either writing clear explanations or drawing a picture. Math majors not enrolled in a teaching program focus more on just simply doing the problem and relying on known formulas.

Brittany: *As a future math teacher, the process of induction has taught me to solve difficult problems using the most basic form of that example first. Then, using the same thought process, use this method to solve more complex problems. I believe that induction should be used more in high school math, especially since the process of induction is in line with the goals of the New York State Common Core, which is to think critically and explain why things happen and their thought process. The students that we surveyed did just that, they used their words to justify their reasoning in solving a problem by either writing clear explanations or drawing a picture.*

Nick: *For an aspiring college professor, it was interesting to me to notice the difference in the ways each student communicated their answer to each problem. It was interesting to see the math education majors actually write paragraphs similar to a written response question on an english language arts exam to answer a problem that can be done in a few lines of calculations. Even though they may have used fewer calculations and formulas, and more writing to justify their answer, they were able to communicate the same methodology as using a simple formula and substituting values.*

Brianne: *As a future math teacher, my goal is to be as explanatory, yet direct as possible. Students need to understand the steps used in order to solve a problem correctly. Especially since New York State rolled out the Common Core standards, students are now required to explain their answers using words. This means that math STEP*[4] *majors are used to being very detailed, and this showed in our research.*

Sara: *As a math major going into financial mathematics, this process has taught me, personally, that math majors not in the STEP program approach problem solving differently. They try to solve most problems they face using less words and more concise math. I will most likely use this method when I go into Analytics Consulting because it is all about getting the results efficiently, so it was very interesting to see the comparison between math STEP and math non- STEP.*

5. Working Backwards (Group 3)

5.1. What is the Working Backwards Problem Solving Technique?

Problems are encountered day in and day out and it is in our human nature and sense of curiosity to find different approaches and ways to solve them.

[4]The Scholar Teacher Education Program (STEP) is a combined bachelors/masters program for undergraduate students preparing to teach at the childhood and adolescent grade levels.

Generally speaking, when a problem arises, we begin by searching for a solution that is unknown based upon initial given information. It seems atypical for one to begin with a solution and somehow manipulate their way back to the givens because of the way our brains are programmed to problem-solve, however this technique is known as Working Backwards.

Polya discusses in *How to Solve it* [8] the Working Backwards technique as a method where we start from our end goal and assume what we are looking for is already solved. Utilizing this strategy requires one to reverse each operation already performed in a step-by-step procedural fashion in order to get back to the beginning, whatever it may be. Another case of working backwards, which is commonly used in word problems, is when you use opposite operations to solve a problem [12]. In order to convey this idea, we will examine a scenario where a teacher prompts his or her students to think in this abstract manner.

> **Do Now: Write out some basic walking directions from where you live to school.**
>
> **Student:** *Beginning Destination- Home, Turn left on Mineola Blvd, Continue straight for two miles, Turn right on Byrd Street, Continue on Byrd Street for a mile, Turn right onto Jefferson Avenue, Final Destination- School.*
>
> **Teacher:** *Now that you have your directions from home to school, I want you to give directions from school back home.*
>
> **Student:** *Why? Won't it be the same thing backwards?*
>
> **Teacher:** *Why is it that we are working backwards?*
>
> **Student:** *Because in order to get home from school we need to go the opposite way that we went in order to get to school.*
>
> **Teacher:** *I like the way you're thinking, write out the directions and see if your method is correct.*
>
> **Student:** *Final Destination- School, Turn left onto Byrd Street from Jefferson Avenue, Continue on Byrd Street for a mile, Turn left onto Mineola Blvd, Continue straight for Two miles, Turn right onto Cleveland Avenue, Beginning Destination- Home.*

This "do now" activity allowed the student to use the method of working backwards to get to their destination of interest. With the foundation of this idea in place, the student can now be able to understand and utilize this technique in mathematical contextual problems.

5.2. Methodology

Participants: We acquired a random sample n = 10 students, four of which were Mathematics majors and the other six being Computer Science majors.

Materials: Our survey was composed of three different parts. Prior to beginning the working backwards aspect of the survey, participants were required to identify whether they were Mathematics or Computer Sciences majors. Once establishing a major, two questions of interest were proposed, one which builds off of the other from simple to complex pertaining to real life situations (See Appendix C for our survey).

Procedure: Before we were able to distribute our survey to participants, it was crucial that we initially executed our survey as a pilot to random individuals. This gave us the correlation needed to then conduct our survey with participants of interest. As a group, we went to a location on campus where students of both majors gathered. Once at this location, we sampled those who were willing to participate and allocated fifteen minutes for completing the given problems. The directions were read aloud, and participants were asked to show all work and thought processes to the best of their abilities. After the fifteen minutes elapsed we collected the surveys and began grading using the AAC&U rubric [3]. We then tallied the scores of each student's survey and analyzed the numbers to look for particular trends or patterns in data.

5.3. Results

Overall, quantitatively we saw that Mathematics majors seemed to score higher based upon the AAC&U rubric and our collaborative grading technique. The scores between Mathematics and Computer Science majors varied substantially on a scale from 0-4 with the average MA major scoring roughly 3.354 and the average CS major scoring about 2.083. Although quantitative data are crucial in most studies, for this particular survey they weren't as significant as the qualitative differences found between our two groups of

Mathematics majors and Computer Science majors. The methods by which each individual solved the given problems were of interest, which is why numerically we weren't as concerned.

Upon initial inspection, it seemed that both groups approached each question similarly, until we went deeper and discovered differences between strategies. Many of the Mathematics majors approached both questions one and two by developing a formula, which helped them ultimately obtain their solutions, whereas Computer Science majors used methods similar to coding logic, mainly recursion and arrays. These two approaches were interesting to examine because in many mathematics courses students are trained to find or derive a formula to come to a solution, while in computer science courses the process of recursion is used to break a computation to its smallest parts, then work back up using the results. The math students used variables naturally as part of their formulas (see Figure 6), which they developed, whereas the computer science students structured their data into an array-like format, storing the individual values into cells to be referenced.

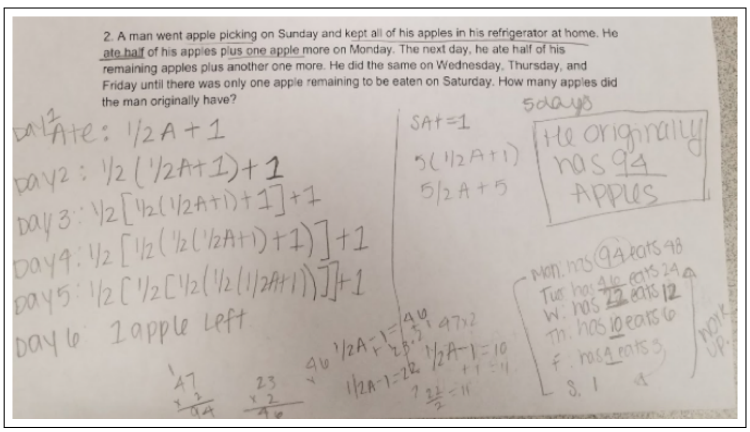


Figure 6: Visual of formula use for math majors.

Another qualitative observation made revealing the difference between the two groups took into account the steps undergone to reach a solution. Each Mathematics major labeled their steps to show a procedural strategy, which is often taught and seen in mathematics courses. The Computer Science majors did not label their steps procedurally, but instead wrote down solutions without showing preceding scratch work and relied on drawing arrows from each block of work to show their train of thought.

This is a key difference between the two groups because mathematics courses train students to show all work, while computer science courses rely on programs to do all of the work, without focusing on the procedural aspect.

We observed that some of the math majors used simple visuals for the first question however; the visual appeared as a single mass to represent the parking lot. For computer science majors we noticed that the visuals drawn were similar in appearance to what an array in computer science normally would look like, this leading to a specific kind of visual that they may be used to dealing in with other problems presented dealing with arrays (see Figure 7). These kinds of diagrams are actually very useful for visualizing for this type of question involving the removal or switching of individual items. These differences in problem solving ultimately led us to our conclusion that Computer Science majors rationalize and approach problems in a different manner than Mathematics majors.

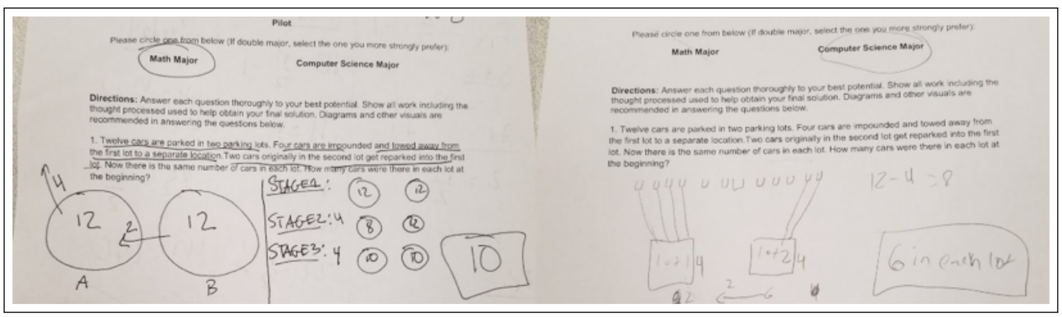


Figure 7: Comparison between math and computer science visuals.

5.4. What Did We Learn About Problem Solving?

In order to truly reflect on the results obtained, it was necessary to examine the work of each participant with both a mathematical and a computational lens. The purpose of this survey was to investigate the different approaches a Mathematics major would take in comparison to a Computer Science major when proposed a problem. With this being the topic of interest, it only made sense that the members of our group reflected on the results based upon our individual majors, along with past and possible future experiences.

> **Mathematics Perspective - Jireh:** *Being both a mathematics and finance student, I feel that working backwards is not only used in reference to math problems, but is also seen as part of daily life.*

Although the survey questions were designed in a mathematical way, the context and components of each problem pertain to real life scenarios. In the financial world, when analyzing prospects of money and funds, the basic foundation used to model regression, being linear and logistic, stems from the technique of working backwards. We observed our participants working from the known to the unknown in a variety of different ways for mathematical purposes, and yet we were still able to recognize individualistic strategies used to solve real world situations. Overall, it is apparent that we utilize mathematical methods across various disciplines of our everyday lives.

Mathematics Perspective- Nicole: *The discipline of mathematics is generally extremely abstract and unclear for many individuals. As a current student and future educator, it is crucial to find various methods to transmit or process a concept to oneself or an audience. Generally, these methods consist of differentiation paired with distinct techniques at arriving to the same solution. The system of working backwards seemed far out of this world to me when first introduced because traditional schooling had trained my brain to work from left to right, whereas working backwards required that I go from right to left. Although this concept was sprinkled in here and there in high school, the primary exposure to this notion was introduced in college whereas I was already accustomed to solving problems procedurally from start to finish. Because of my lack of exposure earlier on, I encountered obstacles and much frustration with this idea; it just didn't register with me. Time and patience were pivotal in reaching that "Eureka" moment, where it all seemed to finally make sense. I needed that extra push and extra time. Establishing the idea of working backwards into the high school curriculum is crucial for students because it will broaden their perspective and allow them to utilize their toolbox, not only in the classroom, but also the outside world. The teaching profession is impactful; an educator will encounter students at all different levels with varying degrees of prior knowledge at their belt and it is their responsibility to ensure that each individual is as successful as possible in their learning acquisition through life.*

Computer Science Perspective- Scott: *My experience with working backwards as a computer science major has led to me usually rely on recursion, hence I would have to dive into the problem further and further and backtrack in the computational sense to then actually piece everything together. Normally working backwards can be very confusing for me since recursion can be a tricky method for working backwards. However, in practical use simply looking at the problem from the perspective of the end goal and breaking it down in reverse is a different way of working backwards then what I am used to. As a computer science major I have the tendency that when I am given a problem I dive straight in and "spitball" ideas until something sticks rather than starting from our goal and working in reverse.*

Computer Science Perspective- Mateusz: *As a computer science major, my main task when creating a program is figuring out the logic behind how the program would actually work before starting to write the code, such as what steps will be carried out when, how they would be done, and why I would want to build the program that way. Usually when programming, we know what our final intended goal is supposed to be, and we have to code the program from scratch. Figuring out the logic has roots in working backwards due to the fact our goal is already known, all we have to do is visualize the steps that would have to come before hand, all the way down to the smallest part, that would lead to our goal. While it's not guaranteed I will find all the steps in the logic sequentially every time, such as possibly determining some of the base blocks first before some later step, by the end I have a framework that I can use to start coding. Once I have the logic figured out, I start writing the code at the base blocks of the program and build off them until I reach my goal.*

As we came together and merged our observations, it was apparent that we have all seen the technique of working backwards across varying aspects of our lives.

6. Symmetry (Group 4)

6.1. How is Symmetry Applied in Problem Solving?

Polya [8] states that the general concept of symmetry relies on the presence of interchangeable parts within a problem. When using symmetry to solve a problem, one should look for parts of the problem that mirror each other. Finding these relationships in problems help lead to generalization of an approach to problems with the same qualities. Through this technique, the problem is solved using prior knowledge, related problems and other problem solving skills.

When examining symmetry through our research, we wanted to see a common theme where students were using parts of the problem or prior knowledge in order to find a solution. An important part of being a mathematics student is having the ability to build upon prior knowledge. We can use problems we have done in the past to solve new problems that are presented to us.

6.2. Methodology

Participants: In our study, the participants included five non-instrument playing mathematics majors and five instrument-playing mathematics majors chosen at random.

Materials: In our study, the participants were given a two-question survey. Our survey can be found in Appendix D.

Procedure: First, we did a sample survey that was administered to five people outside of the parameters of our study. Once we completed the sample survey, we performed a norming session in order to make sure that our grading styles were aligned with the AAC&U Problem Solving Value Rubric [3]. Our reliability statistic is 0.69. Afterwards, we chose students at random to complete the two question survey in a twenty minute time span. The students that completed the survey then handed them to us anonymously for grading. We graded the surveys together in order to make sure that our grading was fair and reasonable. Once the ten surveys were graded, we examined the surveys qualitatively rather than quantitatively. First, we separated the instrumental and non-instrumental surveys and examined similarities and differences. Then, we looked at the surveys as a whole to find different measurable trends within the symmetry problem solving technique.

6.3. Results

The results of our survey did not necessarily match up with our original hypothesis that students who played an instrument would use the symmetry problem solving technique more effectively than students who did not play an instrument. When first creating the survey, we were hoping to see students using symmetry in order to make the problem easier to solve. However, even when the problem was solved and explained correctly, there was no evidence of symmetry used in the solution given. On the other hand, many students that completed the survey used different techniques such as contradiction in order to find the answer. This led us to believe that symmetry was not something that students think to use from the start. Many students do not use symmetry outside of the geometry curriculum. There was no relationship found when examining the answers between the answers and the symmetry problem solving technique. Here is an example of a solution to the first question:

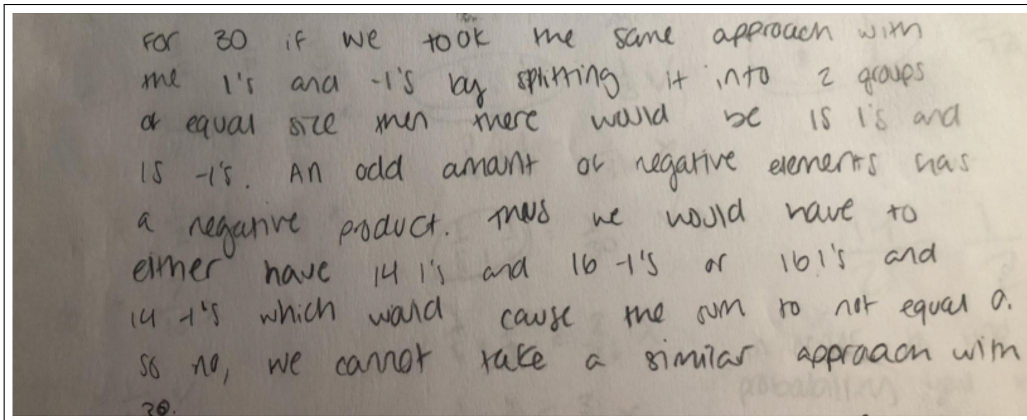
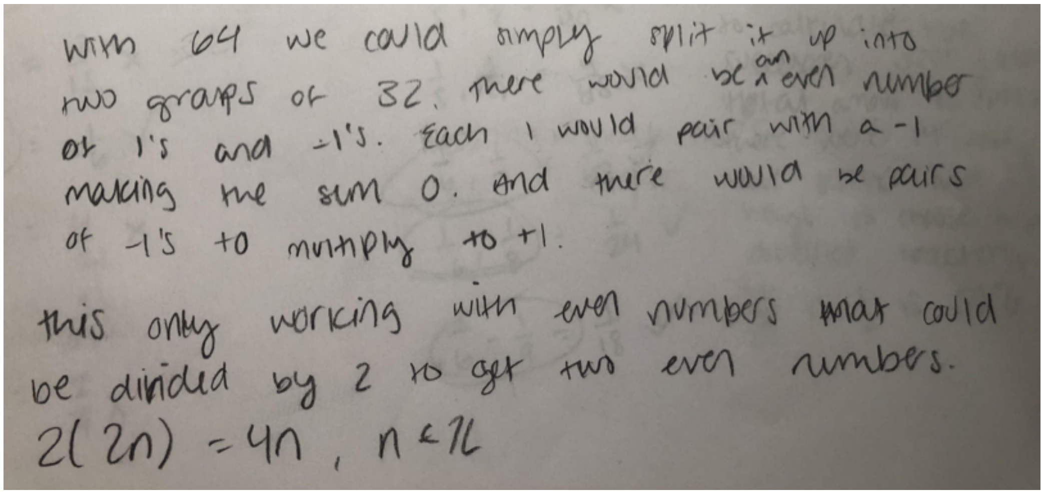

For 30 if we took the same approach with the 1's and -1's by splitting it into 2 groups of equal size then there would be 15 1's and 15 -1's. An odd amount of negative elements has a negative product. Thus we would have to either have 14 1's and 16 -1's or 16 1's and 14 -1's which would cause the sum to not equal 0. So no, we cannot take a similar approach with 30.

Figure 8: Work of a student that plays an instrument.

In Figure 8, the person completing the survey plays an instrument. The person that completed the survey used contradiction in order to prove that 30 would not work. In doing this, they were able to find a generalization. They used a lot of description in their approach in order to show their thought processes. Notice the symmetric thought process used when examining 64 integers Figure 9). The symmetric step we expected was for the student to notice 64 is two sets of 32, which they can deduce since 32 worked two sets of 32 would also work. This student is one of the very few who tried to use a symmetry style solution to solve this problem. Many people examined 64 as a completely separate question.


with 64 we could simply split it up into two groups of 32. there would be an even number of 1's and -1's. Each 1 would pair with a -1 making the sum 0. And there would be pairs of -1's to multiply to +1.

this only working with even numbers that could be divided by 2 to get two even numbers.

$2(2n) = 4n,\ n \in \mathbb{Z}$

Figure 9: Work of a student that plays an instrument using the Symmetric Step.

The most common solution was noticing that the number of integers is divisible by four (as in Figure 8). Now, let us examine the second question that was answered by a participant that does not play an instrument: see the next page.

In Figure 10, the student did the expected symmetric step which was not present in most of the other surveys. The student examined each of the differences and noticed that using absolute value they did not need to examine the other half. For example,

$$\left|\frac{1}{2}-\frac{1}{3}\right| = \left|\frac{1}{3}-\frac{1}{2}\right|.$$

Many students did not follow this thought process. The other students who got it correct had simply computed every other possible combination rather than using the symmetric step.

6.4. What Did We Learn About Problem Solving?

After implementing the survey and analyzing the results, we had a much different view of the symmetry problem solving technique. We discussed our results and opinions on the technique as a whole:

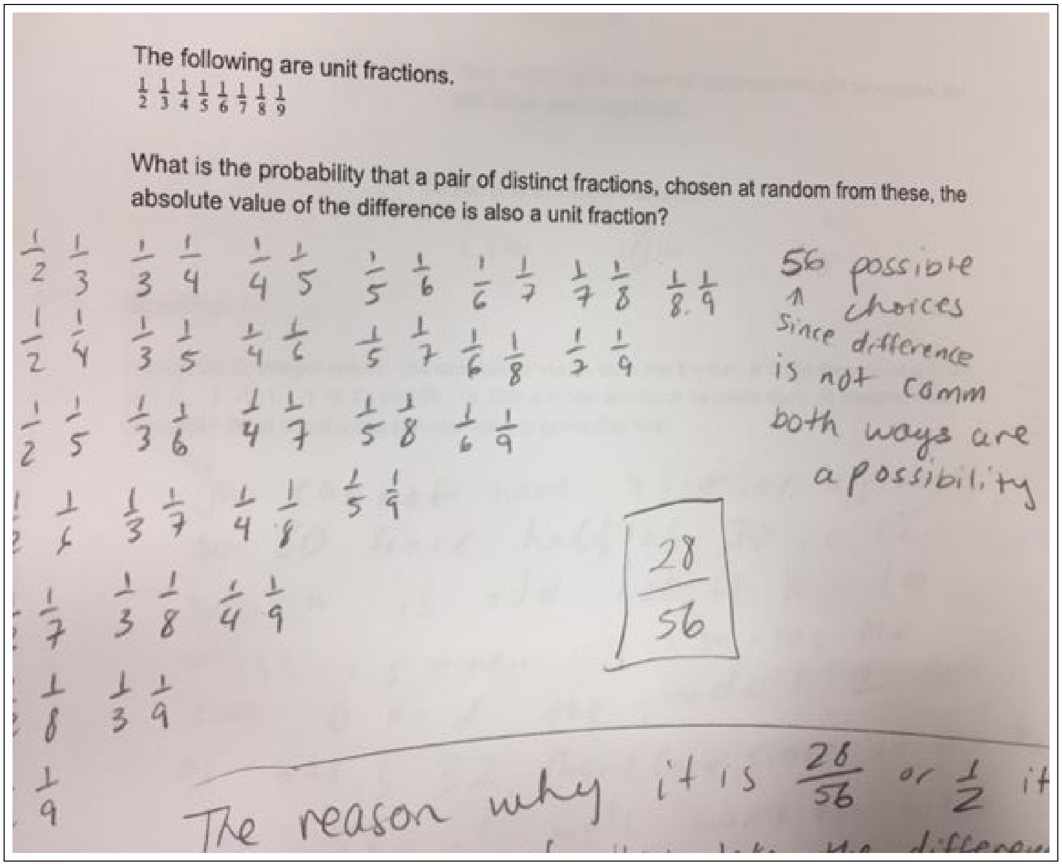


Figure 10: Work of a student that does not play an instrument using the Symmetric Step.

Brian: *Why do you think that symmetry is so underutilized in the survey results?*

Emily: *Maybe it is because we did not use a question that involved geometry? Is that how most people tend to use symmetry?*

Kyle: *It may be because the symmetry problem solving technique is much more difficult when it is applied to other types of problems outside of geometry.*

Emily: *As a future mathematics teacher, I know that symmetry is presented and applied in K-12 geometry. Students may have a hard time applying it in a way that they have not seen before.*

Brian: *Maybe these techniques should be presented earlier on in high school education in a more effective way?*

Emily: *I agree that there should be more of a presence in high school mathematics. However, do you find yourself using symmetry often? Is it worth being presented earlier?*

Kyle: *In both my mathematics and computer science classes, I do not use symmetry unless I am seeing an image or something more visual.*

Brian: *I agree with Kyle. In my thought process, I tend to use symmetry if there is something that I can visualize. However, most cases in mathematics and computer science that are not geometry based do not call for this thought process.*

Emily: *I think that if it was presented more often in contexts other than geometry, students would be more willing to work out the questions that were given to them using symmetry. However, the difficulty level of symmetry may still deter students from using symmetry.*

Brian: *Do we think that it is actually really difficult or just that people do not really know how to use it?*

Kyle: *I feel like most people do not really know how to use it. It is not that much harder, I think it makes some problems simple.*

Emily: *What makes symmetry so tedious?*

Brian: *Also, if students find a way to solve a problem without using symmetry, they will often use the alternate technique.*

Emily: *I agree. We did find in our results that people mostly found ways around using symmetry to solve problems.*

Overall, we found that symmetry is not as prevalent as other problem solving techniques. It is difficult for students to use symmetry in problem solving. However, this may just be because students do not know how to use symmetry properly due to a lack of it being taught in grades K-12. Although it is not as prevalent, some students do still use symmetry to solve problems. Studying symmetry has made us more aware of how and when to use it and we all believe that we will be using it more in our problem solving thought process.

7. Conclusions and Recommendations

There are many pedagogical techniques for the teaching of mathematical problem solving, such as student presentations of problems and instructors providing written feedback to solutions of problems. All the students in this course experienced the problem solving techniques of Heuristics, Induction, Working Backwards, and Symmetry in their Introduction to Proofs course, where these techniques were examined from a proofs based perspective. These students received the two methods of feedback previously mentioned. These students explored the same four techniques to solve problems in this course. From the instructor's point of view, the two assessment methods have the students reflect on the solutions to problems; however, it was not the level of reflection that was encountered in this project.

This data-driven method really enabled students to explore the psychology behind problem solving and more importantly reflect on their own problem solving abilities. Examining this project from that perspective, it appears that this is a successful activity for enhancing the understanding of mathematical problem solving. However, this project needs to be examined from the practical point of view as well. It took an entire semester for the groups to get to the point of reflection that they reached, but that was only with one problem solving technique. It was a very time-consuming project; however, the benefits were worth it. It would be interesting for instructors to explore smaller "bite-size" type projects that could do similar reflection on a much smaller scale. Much research has been written about writing during the problem solving process, such as [14]; however, there needs to be more than just writing. Such writing must include a substantial component devoted to reflection. A data analysis point-of-view could be that missing component.

On the last day of class, we had two faculty members who joined us for our final meeting, where the students reflected on their experiences. We end with sharing these faculty perspectives.

> **Professor LJS:** *The students enrolled in the seminar course gained new perspectives on how other students solve problems, which in turn improved their own ability to reflect on how they approach problems, which will undoubtedly make them stronger students, better equipped to tackle a variety of challenges. I was particularly impressed by the students' ability to speak extempora-*

neously and in depth about problem solving strategies; their conversations showed that they had attained a very high level of mathematical maturity. The seminar students observed that students in different tracks (such as education or finance) would explain solutions differently — they recognized that someone's career trajectory influence their learning style and communication ability — an astute observation that will serve them well, as they will need to adapt to interact with people from different walks of life in the future. In previous classes, these students had taken "proofs" or "bridges" courses where they solved problems in a variety of ways and reflected on their techniques; in this course, they reflected on their own and other students' style of reflection, a type of meta-analysis that will begin a feedback loop of continual improvement throughout their professional careers.

Professor JPH: *I think that this experience was invaluable for the MTH 290 students. The level of pride that these students exhibited in their work is what I strive for in every class I teach. They expressed an enthusiasm for their research problem that I have never before witnessed. What was incredible was the depth of thought that each student had on her particular technique. I have no doubt that every single student truly understood the context that made her "tricks" relevant. I think this experience led the pupils to engage in a self-reflection that will aid them throughout their mathematical career at Adelphi and beyond. Particularly, the math-education students seemed to have internalized the importance of communicating problem solving techniques effectively. I hope this is the first of many such classes at Adelphi.*

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A. Appendix A

Below is the assessment tool created by Group 1. See Table 1.

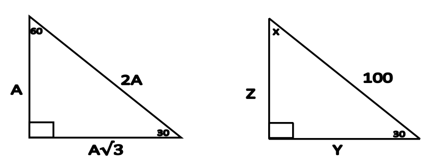
Select the following that applies to you:

[ ] Freshman [ ] Sophomore [ ] Junior [ ] Senior

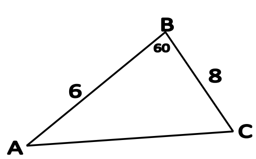
Directions: Show **ALL** work and write out in words your reasoning and logical steps. Complete all of the following problems. If unsure, write any and all thoughts/thought process you might have.

Figures are not drawn to scale

Question 1: Using the triangle on the left, find the values of *x*, *y*, and *z*.



Question 2:[7] Find the area of the following triangle.



[7] Question taken from Jacobsen (2008).

B. Appendix B

Below is the assessment tool created by Group 2. See Table 1.

Check the box that applies to you.

[ ] Math Major [ ] Math in STEP Major [ ] Other

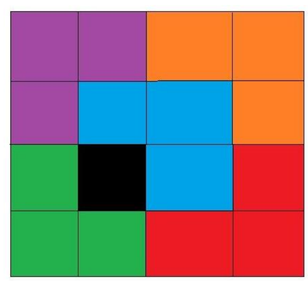
Read the following questions and answer them to the best of your abilities. Show ALL work and explain your thought process whenever possible.

1. Observe the list of numbers below.

3, 12, 27, 48

What is the next number in the list?

2.[8] Prove that any $2n\times 2n$ grid with one square blacked out can be tiled with L shaped pieces of 3 units area. Here is an example of the 4×4 case to the right.



[8] This question was taken from www.cs.cmu.edu/afs/cs.cmu.edu/academic/class/15251-s05/Site/Materials/Review/s04-q1-sol.pdf.

C. Appendix C

Below is the assessment tool created by Group 3. See Table 1.

Please circle one from below (If double major, select the one you more strongly prefer):

Math Major **Computer Science Major**

Directions: Answer each question thoroughly to your best potential. Show all work including the thought processed used to help obtain your final solution. Diagrams and other visuals are recommended in answering the questions below.

1. Twelve cars are parked in two parking lots. Four cars are impounded and towed away from the first lot to a separate location.Two cars originally in the second lot get reparked into the first lot. Now there is the same number of cars in each lot. How many cars were there in each lot at the beginning?

2. A man went apple picking on Sunday and kept all of his apples in his refrigerator at home. He ate half of his apples plus one apple more on Monday. The next day, he ate half of his remaining apples plus another one more. He did the same on Wednesday, Thursday, and Friday until there was only one apple remaining to be eaten on Saturday. How many apples did the man originally have?

[9] These questions were inspired by problems done in class.

D. Appendix D

Below is the assessment tool created by Group 4. See Table 1.

Please answer all questions to the best of your ability. Show all work and thought processes. **Do not leave anything blank.**

Do you play a musical instrument?

[ ] Yes [ ] No

Question 1:

If you take 32 integer values, you can choose values such that the sum is 0 and the product is 1, like so 1,1,...1 -1,-1...-1 16 1's and 16 -1's. Can a similar approach be made about 30 integers? How about 64? What conclusions can you make to generalize this?

Question 2:[10]

The following are unit fractions.

$\frac{1}{2}\frac{1}{3}\ \frac{1}{4}\ \frac{1}{5}\ \frac{1}{6}\ \frac{1}{7}\ \frac{1}{8}\ \frac{1}{9}$

What is the probability that a pair of distinct fractions, chosen at random from these, the absolute value of the difference is also a unit fraction?

[10] Question taken from *A Difference of Two Fractions*.