Differential Equations of Love and Love of Differential Equations

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Synopsis

In this paper, simple ordinary differential equations are discussed against the background of William Shakespeare's *Romeo and Juliet*. In addition, a version of this relationship in a somewhat opposite setting is considered. It is proposed that engineering mathematics courses include this topic in order to promote additional interest in differential equations. In the final section it is shown that vibration of a single-degree-of-freedom mechanical system can be cast as a love-hate relationship between its displacement and velocity, and dynamic instability identified as a transition from trigonometric love to hyperbolic.

1. Introduction

According to Galileo Galilei "Mathematics is the language in which God has written the universe". Hence, whether we talk about attraction and repulsion of physical bodies or love as an interrelation between two people we use the same mathematical language. Indeed, Richard Feynman teaches us: "The physicist needs a facility in looking at problems from several points of view. The exact analysis of real physical problems is usually quite complicated, and any particular physical situation may be too complicated to analyze directly by solving the differential equation ... There is only one precise way of presenting the laws, and that is by means of differential equations. They have the advantage of being fundamental and, so far as we know, precise. If you have learned the differential equations, you can always go back to them. There is nothing to unlearn." In this paper we offer the differential equations of love, as first principles of human interrelationship of special kind.

Indeed, the summary of love according to George Gershwin: "When you want'em,/ You can't get 'em,/ When you've got 'em/ You don't want 'em" sounds and looks like a physics law, with the attendant need in constructing appropriate differential equations.

Can love be described in terms of physics or rather mathematics? The answer is a resounding Yes. Indeed, renowned mathematical physicist Vladimir Arnold teaches us in [1]: " ... it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic." Indeed, methods of mathematical physics are widely used in economics. Why not use them in psychology? Look at the titles of papers published in physics journals: (a) " Econophysics: Can Physicists Contribute to the Science of Economics?" [34], (b) "Challenges in Network Science: Applications to Infrastructures, Climate, Social Systems and Economics" [14], (c) "Dynamics of Two-Actor Cooperation-Competition Conflict Models" [21], and numerous others.

Derek Bok, a former president of Harvard University, expressed the following idea (see [4]):

> "The college that takes students with modest entering abilities and improves their abilities, substantially contributes more than the school that takes very bright students and helps them develop only modestly. We really need to take the focus off entering scores and put it more on how much value is added."

Indeed, in non-elite colleges, faculty members ought to spend more time than at say, Harvard — motivating students whose attention span has been stunted under the impact of the man-made "aliens" — TV and computer games. (This does not imply that students at elite institutions have been spared the cerebral damage inflicted by "His Excellency the Computer", originally intended, if you recall, to serve us.)

The situation actually has a positive corollary: not taking students for granted, not assuming that they would understand everything on their own necessitates constant thinking about how to make the presentation of the material more exciting and inspiring.

I recall that in Russia they used to say: "A university professor who is a bad teacher is a great asset, because students are then forced to study from books on their own, and thus master the subject better." This of course is a joke, but on a more serious note, non-elite universities cannot possibly afford an indifferent faculty, since then students are likely to goof off completely.

Only after consistent "F" or near "F" grades would the system learn that the particular student took his or her freedom too literally.

This must have been what Providence had in mind specifically when in 1988 Professor Steven Strogatz (then at MIT, now at Cornell) published a single page item about the differential equations of love [36]. It was intended "...to suggest an unusual approach to teaching of some standard material about systems of coupled ordinary differential equations. The approach relates the mathematics to a topic that is already on the minds of many college students: the time-evolution of a love affair between two people. Students seem to enjoy the material, taking an active role in the construction, solution, and interpretation of the equations."

2. Hyperbolic Love According to Shakespeare

When posing to students the idea of modeling a love relationship between a man (let us call him Romeo) and a woman (let us call her Juliet), I refer to the rate of change of whichever functions would be involved. The students suggest these to be the measure of Romeo's love for Juliet, denoted $R(t)$, and the that Juliet's for Romeo, denoted $J(t)$. When the question is then raised — how to measure love — I remind my audience of the more immediate problem — how to define love. Students volunteer their own definitions, and there is wide scatter in their choices, (which range from "Love is everything that allows breathing" to "Love is suffering"), and none of them helps solve the measurement problem. At this juncture I write on the blackboard Piet Hein's (1905-1996) verse

> "Love is like a pineapple,
> Sweet and undefinable."

If it is indefinable, what do the Beatles mean when they sing "All you need is love"? Still everyone agrees that love exists. Otherwise how can anyone fall in love or need an assurance that they are loved?

Faced with this impasse, I suggest that we seek the solution to the measurement problem elsewhere.

Giralamo Cardano (1501-1576) in his *Ars Magna* maintained that complex numbers (not yet termed so at that time) are "senseless" [5]. Two centuries later, Leonhard Euler assumed that they at least "exist in our imagination" and went ahead with derivation of the formula for e^{ix}, in other words, the number e

to a "senseless" power. His genius enabled him to find that $e^{ix} = \cos x + i \sin x$ and its corollary, obtained by substituting $x = \pi$, $e^{i\pi} + 1 = 0$, considered by some to be the most beautiful formula in mathematics.

I suggest that we follow Euler in our quest for a "yardstick" for love. Pushing our own poetic license to the extreme, we assume that Romeo and Juliet have cell phones. We can now ask Romeo what is the value of the function $R(t)$; likewise, in order to know $J(t)$, we just have to place a phone call to Juliet. We can then record values $R(t)$ and $J(t)$ at discrete time intervals t_k and apply interpolation or extrapolation. Alternatively, we can write differential equations that will predict, if a solution exists and if it is unique, the whole dynamics of the love relationship between Romeo and Juliet.

The pace of Romeo's love to Juliet $\frac{dR}{dt}$ is proportional to her love for him; analogously, Juliet's love derivative $\frac{dJ}{dt}$ towards Romeo is proportional to his love for her. Thus, we obtain the following set of differential equations:

$$\frac{dR}{dt} = aJ; \qquad \frac{dJ}{dt} = bR, \tag{1}$$

We also have to stipulate the initial conditions. Romeo and Juliet meet each other at the party given by the Capulet family. Since Romeo's family, the Montagues, are enemies of the Capulets, Romeo cannot risk being recognized. Still, he is able to see Juliet. Were we to call him at that time, he would probably respond:

$$R(0) = 1. \tag{2}$$

Romeo is transfixed. He put himself in the hands of "He who hath the steerage of my course," and exclaims:

> "Beauty too rich for use, for earth too dear ... ...
> Did my heart love till now? Forswear it, sight!
> For I ne'er saw true beauty till this night."

Let us listen to what modern (that is, late twentieth century) Romeo has to say to modern Juliet:

> "The first time I ever saw your face,
> I thought the sun rose in your eyes,
> And the moon and the stars were
> The gifts you gave ... ."
> (*First Time I Ever Saw Your Face*, Roberta Flack)

or yet another:

> "I took one look at you,
> That's all I meant to do
> And then my heart stood still
> My feet could step and walk,
> My lips could move and talk,
> And yet my heart stood still."
> (*My Heart Stood Still*, Frank Sinatra)

Juliet had not seen Romeo prior to the party. Had we placed a phone call to her, she would state that her love for a person she doesn't know is nonexistent, zero:

$$J(0) = 0. \tag{3}$$

At this juncture, I ask students to specify the coefficients a and b. In the most recent class, the following values were suggested:

$$a = 5, \qquad b = 2. \tag{4}$$

We now substitute:

$$R(t) = Ce^{mt}, \qquad J(t) = De^{mt} \tag{5}$$

into Equation (1) with our chosen a and b values, leading to the auxiliary equation:

$$\begin{vmatrix} m & -5 \\ 2 & -m \end{vmatrix} = 0, \tag{6}$$

or $-m^2 + 10 = 0$, whence

$$m = \pm\sqrt{10}. \tag{7}$$

We can now solve the linear system (1) of differential equations:

$$R(t) = \cosh\sqrt{10}t; \tag{8}$$

$$J(t) = \frac{\sqrt{10}}{5}\sinh\sqrt{10}t. \tag{9}$$

I ask for conclusions. Some students say that we could verify the model by using our cell phone connection and checking whether the results of the model agree with the true feelings of Romeo and Juliet.

So, our mathematically minded Romeo could say to Juliet: "My love is hyperbolic. It increases as the hyperbolic cosine!" Juliet, who is not mathematically challenged either, would respond: "My love for you is hyperbolic too. It intensifies as the hyperbolic sine!" Indeed, Juliet maintains that her "bounty is as boundless" and her "love as deep" as the sea. Hence,

"The more I give to thee
The more I have, for both are infinite"
    (*Act 2 Scene 2 Lines 134-135*).

How do the functions $R(t)$ and $J(t)$ vary as time passes? In the hyperbolic functions involved:

$$\cosh \sqrt{10}t \;=\; \frac{1}{2}\left(e^{\sqrt{10}t} + e^{-\sqrt{10}t}\right) \text{ and} \tag{10}$$

$$\sinh \sqrt{10}t \;=\; \frac{1}{2}\left(e^{\sqrt{10}t} - e^{-\sqrt{10}t}\right), \tag{11}$$

the second term $e^{-\sqrt{10}t}$ tends to zero, and so:

$$R(t) \sim \tfrac{1}{2}e^{\sqrt{10}t} \qquad \text{and} \qquad J(t) \sim \tfrac{\sqrt{10}}{10}e^{\sqrt{10}t}. \tag{12}$$

Thus each of them can tell the other: "I love you exponentially!"

To support the possibility of measuring love, we quote from Shakespeare's (1564-1616) Sonnet 102:

"My love is strengthened though more weak in seeing,
I love not less though less the show appears."

The concepts of time and love are likewise intertwined. In his Sonnet 116, the Bard claims:

"Love alters not with brief hours and weeks
But hears it out to the edge of doom
If this be error and upon me proved,
I never writ, nor no man ever loved."

Infinite love is indicated in the verses of 17th century English poet John Donne (1572–1631). In his poem quite mathematically titled "Love's Growth", he says:

"And since my love doth every day submit
New growth, thou shouldst have new rewards in store."

He soon gets even more specific: "My love was infinite, if spring make it more."

3. Trigonometric Love According to Steven Strogatz

Not all love stories proceed hyperbolically in time. Most unfortunately, in some cases, as both literature and experience suggest, love doesn't always grow exponentially either; indeed it doesn't necessarily improve, like old wine. Strogatz [36, 37] has his own version of Romeo and Juliet. He does borrow Juliet from Shakespeare so that the second equation of the set is preserved, but his Romeo behaves differently from the original one, and is characterized by Strogatz as a "fickle lover": the more Juliet loves him, the more he runs away and hides. But when Juliet gets discouraged and backs off, Romeo begins to find her strangely attractive. Juliet, on the other hand, tends to echo him: she warms up when he loves her and grows cold when he does not. The differential equations become:

$$\frac{dR}{dt} = -mJ; \qquad \frac{dJ}{dt} = nR, \tag{13}$$

The only thing that has changed is the sign in the second differential equation in the set of equations (1). The initial conditions in Equations (2) and (3) remain the same. This situation is reminiscent of the one described in the "anti-love poem" by Paley [26]:

> "Sometimes you don't want to love
> The person you love
> You turn your face
> Away from that face."

When we explored this case in class, my students specified $m = 3$ and $n = 5$. In these new circumstances, the roots of the auxiliary equation:

$$m^2 + 15 = 0 \tag{14}$$

are purely imaginary:

$$m = \pm i\sqrt{15}. \tag{15}$$

Note indicentally that we get the square root of minus fifteen in the expression for the roots. This value also appears in the subtitle of the 2003 book [23] by Harvard's Barry Mazur. At this juncture therefore, I encourage my students to obtain a copy of this book on their way to growing their own library of professional books.

The solution thus derived is:

$$R(t) = \cos\sqrt{15}t; \tag{16}$$

$$J(t) = \frac{\sqrt{15}}{3}\sin\sqrt{15}t. \tag{17}$$

These are no longer hyperbolic functions, but trigonometric ones. In the new circumstances, the love functions $R(t)$ and $J(t)$ can take both positive and negative values. John Donne actually titled one of his poems "Negative Love":

> "If that be simply pefectest,
> Which can by no way be express'd.
> But negatives, my love is so.
> To all, which all is love, I say no."

The negative-valued love can be described as hate. Strogatz's Romeo, the fickle lover, causes the love relationship to be transformed into a love-hate relationship. The 'time to love' is unfortunately followed by a 'time to hate', and so on ad infinitum, were the lovers to live forever.

For the functions describing the variation in time of $R(t)$ and $J(t)$, we see that the amplitude of love equals that of hate. It is apropos to recall the English proverb: "The greatest hate springs from the greatest love." When Juliet's love of Romeo is at a maximum, Romeo's love to her vanishes! Juliet's love rate becomes negative as a result, and her love turns to hate. This makes Romeo's love become positive, and when Juliet's hate crosses the horizontal axis of indifference, with zero values of $R(t)$ and $J(t)$, Romeo's love is at its apogee.

Here is an interesting question: What is the percentage of time when both Romeo and Juliet love each other, i.e. when $R(t) > 0$ and $J(t) > 0$?

To answer this question, it is instructive to derive the geometric representation of Equations ((16) and (17). This way we get

$$\cos \sqrt{15}t \;=\; R(t); \tag{18}$$

$$\sin \sqrt{15}t \;=\; 3J(t)/\sqrt{15}. \tag{19}$$

We add the squares of the left-hand sides of these equations and get:

$$R^2(t) + \frac{3}{5}J^2(t) = 1, \tag{20}$$

which represents an ellipse. The relationship starts at $t = 0$, when $R(0) = 1$ and $J(0) = 0$, i.e. at the top point of the ellipse. With time elapsing, the relationship follows the path indicated by an arrow. It is seen that one-quarter of the time Romeo and Juliet are in love; one-quarter of the time they hate each other; and the rest of the time one loves and the other hates.

Thus Romeo's fickleness results in the situation that three-quarters of the time the couple is frustrated, the product of their love being negative:

$$R(t)J(t) < 0.$$

As one student remarked, three quarters of the time their relationship "stinks". Such a relationship cannot be characterized as love in the first place. As the Spanish dramatist Pedro Calderon de la Barca (1600-1681) claimed, "Cuando amor no es locura, no es amor" (when love is not madness, it is not love). He expresses himself metaphorically, but at least a hint of the truth is there.

Note that if $m = 1$ and $n = 1$, the solution reads

$$R(t) = \cos t; \qquad J(t) = \sin t.$$

The phase portrait is then represented by a circle of radius unity:

$$R^2 + J^2 = 1.$$

The magnitude of their love equals unity again and they are in the state of synchronized love $(R > 0, J > 0)$ only a quarter of the time.

At this time in class I assign my students the homework of checking the case of equal excitement at the time of meeting $(R(0) = J(0) = a$ with $|a| < 1)$ and comparing the magnitudes of the developing love. Likewise, they are asked to draw the phase portrait in such a case. I also ask them to comment on the statement by Mignon McLaughlin: "In the arithmetic of love, one plus one equals everything, and two minus one equals nothing." Typically an interesting discussion ensues.

4. Comparison of Shakespeare's and Strogatz's Models

As *Romeo and Juliet* is a play written in verse, I thought it would be appropriate to ask my students to compare Shakespeare's and Strogatz's models in verses, not just in prose. Below I share two samples from my students' work for this assignment.

The first is a poem by Mr. Charles Bingham (now an engineer) entitled "Hundred vs. Twenty-Five":

> If you are looking for an equation
> Of the differential persuasion
> Look no further than Strogatz's model
> For a love and hate occasion.

Shakespeare only presented true love
Romeo and Juliet were like a dove,
Love, not hate, was all he meant,
Unlike Strogatz's twenty-five percent.

The second poem, "First Order Love" was written by Chip Greene (presently an engineer):

"Mathematicians, / As a rule you can say,
Like describing things, / Their own special way.

They do it with symbols / And numbers that make
Equations depicting / The subject at stake.

If one's so inclined / Human relations
Can be modeled with / Differential equations.

Take two young lovers / And their feelings for
Each other over / A time span of yore.

The most famous story / Of romance I know
Involves Juliet / and her love Romeo.

Their tale was first told / By William Shakespeare
A bard very famous / Also for "King Lear."

In the bard's story / Their love is sublime
It always keeps growing / To the end of time.

A model of this $\dot{R} = aJ$
When coupled with $\dot{J} = Rk$

When solving this system / Of coupled equations
We get the result: / The situation

Where R is his love / And J stands for hers,
When graphed they both are / Exponential curves.

Now fiction it's said / Is the stuff of dreams
But life isn't like/ Fairy tales it seems.

No, in real life / Love's not so kind
It fluctuates, / It's not sublime

Strogatz made a model / With R and with J
Just like the first one / But for real life let's say.

J stays the same / Strogatz said, although
R equals minus / aJ and so

When his system's solved / The curves are much different
They don't keep increasing / Without any restraint.

As it turns out / Three fourths of the time
They don't love each other / They're sine and cosine.

Love ebbs and love flows / With each pair it seems
Tomorrow one yawns / wile the other one beams.

As time goes on / The love vacillates
From love to like / To just plain hate.

It's sad but it's true, / That positive feelings
Aren't always there / For each other's beings.

Mathematical models / Of all types abound
They can show romance / And it's ups and downs.

Now fiction it's said / Is the stuff of dreams.
But life isn't like / Fairy tales, it seems.

During these conversations, I bring my students an example that inspired Milton Friedman (the Chicago economist who wrote the book with the provocative title *There is No Such Thing as a Free Lunch*) to pursue science. In their joint book with his wife Rose, he reminisces:

> "From 1924 to 1928, I went to Rahway High School. One teacher who had a lasting influence on me was Mr. Cohan, who taught political science. He also taught Euclidean geometry simply because he loved his subject. And he succeeded in passing that love onto his students, including me. Nearly seven years later, I still recall him putting the classic proof of the Pythagorean theorem (the square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides) on the blackboard, and stressing what a beautiful proof it was by quoting from Keats's 'Ode to a Grecian Urn': "Beauty is truth, truth is beauty — that is all ye know on earth, and all ye need to know" - thereby instilling love simultaneously for mathematics and poetry." [9, page 24]

As the readers of this journal will agree, mathematics and poetry can go hand in hand. I do hope at this point that my students also see the possibility of such cooperative coexistence. Most importantly, my students do the poetry assignments without the protest anticipated by one of my good colleagues ("This is not Harvard. Stop teaching love!").

Note that Sebnem Arsu [2] reported in the New York Times that the oldest poem dates apparently over 4,000 years! A small Sumerian clay tablet exhibited at the Istanbul Museum of the Ancient Orient records a love poem. Here is an excerpt:

> "Bridegroom, dear to my heart,
> Greedy is your beauty, honeysweet
> You have captivated me, let me stand trembling before you..."

Arsu notes that some phrases in the tablet are similar to the text appearing in the Song of Songs by the King Solomon in the Bible.

Love and poetry are certainly interconnected. Mathematics and poetry are compatible as well. It is my modest proposal here to use poetry as a means towards inspiration and love of mathematics. The article [15] makes a similar argument.

5. Refining the Shakespeare Model

Next I ask my students to compare Shakespeare's model with a totally different problem, namely that of population dynamics, whose simplest possible model is that of Thomas Robert Malthus (1766-1834):

$$\frac{dP}{dt} = kP(t). \tag{21}$$

In the process, the students come up with the following differences:

1. In Shakespeare's model we have a set of ordinary differential equations whereas Malthus model represents a single equation.

2. The population $P(t)$ cannot take on negative values, whereas the measures of love $R(t)$ and $J(t)$ can take on both positive and negative values, as in Strogatz model.

3. The coefficient k in the Malthus model can be determined by census data, whereas the coefficients a and b in our Shakespeare model, and the coefficients m and n in the Strogatz model have to be determined by asking Romeo and Juliet themselves, and hoping they would respond.

4. In the Malthus model the rate of change of population is proportional to the function itself, whereas in our Shakespeare model, the derivative of one function is proportional to the value of the other function.

I ask if any of these differences constitutes a drawback in the Shakespeare model. The students respond that realistically the rate of Romeo's love should also depend upon his own love for Juliet at the moment, not solely on Juliet's love for him. Thus when asked to modify Shakespeare's model with some suggested coefficients, the students come up with the following:

$$\frac{dR}{dt} = aR + bJ; \qquad \frac{dJ}{dt} = cR + dJ. \tag{22}$$

If $a = d = 0$, $b > 0$, $c > 0$, we get our original Shakespeare model back. If $a = d = 0$, $b < 0$, $c > 0$ we get the Strogatz model. The set of equations (22) was written by Strogatz [37]. Here the coefficients a and b represent Romeo's romantic mode whereas c and d represent Juliet's counterpart coefficients. The coefficient a is associated with the extent to which Romeo is encouraged by his own feelings to Juliet, and b is the extent to which he is encouraged by Juliet's feelings.

Strogatz [37] and his students coined four romantic styles depending on the signs of the coefficients:

Eager beaver: $a > 0, b > 0$, indicating that Romeo is encouraged both by his own feelings and by Juliet's.

Narcissistic nerd: $a > 0, b < 0$, signifying that Romeo is encouraged by his own feelings, but responds negatively to Juliet's.

Cautious lover: $a < 0, b > 0$, implying that Romeo is discouraged by his own feelings but responds positively to Juliet's.

Hermit: $a < 0, b < 0$, indicating that Romeo is discouraged both by his own feelings as well as Juliet's.

Juliet can also belong to one of the four styles, creating altogether sixteen possible combinations, each with its own "solution", or development of love over time.

Various possible scenarios may occur, and these can be assigned as homework. Sprott notes:

> "Two romantic clones ($c = b$ and $a = d$) have eigenvalues $\lambda = a \pm b$ and dynamics that depend on a and b. Cautious lovers with $|a| < |b|$ and eager beavers end up in a love fest or war depending on initial conditions. Hermits with $|a| < |b|$ and narcissistic nerds end up with one loving and one hating. Cautious lovers and hermits with $|a| > |b|$ end up in a state of mutual apathy. Oscillations are not possible." [33]

In the semi-autobiographical novel *The Sorrows of Young Werther*, Johann Wolfgang Goethe (1749–1832) puts the following words in Werther's mouth: "All learned professors and doctors are agreed that children do not comprehend the causes of their desires". But, as we see above, we can comprehend the results, at least.

6. What is Love Then?

Erich Fromm (1900–1980) in his famous book notes:

> "There is hardly any activity, any enterprise, which starts with such tremendous hopes and expectations, and yet, which fails so regularly, as love." [10]

This was written half a century ago! Wouldn't he be much more pessimistic today?

As Gila Manolson writes in her essay "What is Love?" [22], many define love as follows: "Love is that feeling you get when you meet the right person. Consciously or unconsciously, they believe love is a sensation (based on a physical and emotional attraction) that magically, spontaneously, generates when Mr. and Mrs. Right appears. And just as easily, it can spontaneously degenerate when the magic 'just isn't there' anymore. You can fall in love, you can fall out of it."

Philosophers could not abandon the issue of defining love. Baruch Spinoza (1632–1677) defines it as follows: "Love is nothing but Joy with the accompanying idea of external cause" [32, proposition 13]. Are you excited by this definition? Probably not. Harry Frankfurt (b.1929), modern-day moral philosopher at Princeton, gives his own definition:

> "Love is, most centrally, disinterested concern for the existence of what is loved, and for what is good first. The lover desires that his beloved flourish and not harmed; and he does not desire this just for the sake of promoting some other goal ... It is important to avoid confusing love — as circumscribed by the concept that I am defining — with infatuation, lust, obsession, possessiveness, and dependency in their various forms." [8, page 42]

`Wikipedia` (who can put it aside?) provides its own definition: "love is an intense feeling of affection related to a sense of strong loyalty or profound oneness".

We have seen in Section 5 that if the (linear) mathematical treatment is applied, there are sixteen possible pairings of Romeo and Juliet under consideration, and only in special circumstances can they end up with permanent love ($R > 0$, $J > 0$). Mere chemistry, unsupported by character, is not a guarantor of everlasting love. It is possible that what we have discussed is not love, but infatuation. This immediately implies that in order to achieve the goal of loving and being loved, some additional elements are needed in life, hence in the differential equations that describe it.

Most relevantly, we find in literature that the character of the partner is an important factor: "The value these couples placed on the partners' moral qualities was an unexpected finding" [38]. Likewise, Manolson argues: "Love is a choice" [22].

According to Erich Fromm [10] giving leads to love, and true giving consists of four elements:

1. care, demonstrating active concern for the recipient's life and growth;

2. responsiveness to others' expressed needs;

3. respect, "the ability to see a person as he [or she] is, to be sure of his [or her] unique individuality, and wanting the other to 'grow and unfold' as he [or she] is, to be aware of his [or her] unique individualities and wanting the other to 'grow and unfold' as he or she is";

4. knowledge of the other: one can care for, be responsive to, and respect only to the extent of knowing the other.

Likewise, Murray claims: "Love is a behavior" [25].

Thus it appears that for a successful love relationship, one needs additional components that can be collectively described as commitment. Ursula Le Guin explains [20]: "Love doesn't lay there. It has to be made like bread; re-made all the time, made new." The first such reference appears to be in the Bible: "God said: It is not good for man to be alone. I will make help-opposite for him" (Genesis 2:18). Help-opposite is a translation of a Hebrew term "Ezer k'negdo", a phrase used to describe the creature to be provided for Adam in order to conquer his loneliness. Rabbi Solomon ben Isaac, better known as Rashi (1040–1105), the supercommentator of the Bible, writes, "If a man is worthy, then his wife will be an "ezer" (a helper), and if he is unworthy, she will be "k'negdo", (against him, an opposite force)". Control, we learn, creates war, not the ideal relationship of complement.

7. Love Triangles

Another possible topic that might be of interest to students is love triangles. I suggest choosing from the classical cases (not from those "discussed" on the TV screen!). Such triangles have existed from antiquity to modern times; there is even an astronaut love triangle.

In one of my classes a student brought up the relationship between Jacob, Rachel, and Leah from the Book of Genesis in the Bible. Here it is important to note that in the Biblical context, love and hate are not perceived by women in absolute terms. The Bible notes "he [Jacob] loved Rachel more than Leah ... and the Lord saw that Leah was hated ..." This "hate" can be expressed as $J_R > J_L$, i.e. Jacob's love for Rachel exceeds his love for Leah, even though both quantities J_R and J_L may be positive. This is where the concept of relativity comes into play!

The Bible probably does not imply that Jacob should have maintained the principle of equality, namely

$$J_R(t) = J_L(t) \tag{23}$$

continuously, since as Kass writes:" ... preference is probably unavoidable in any bigamous marriage" [17]. Perhaps the requirement could be relaxed to imply:

$$\int_0^\tau \left(J_R(t) - J_L(t) \right) dt = 0 \tag{24}$$

over the time span τ. In other words, it is the time-average of the love emotions, of Jacob for Rachel, and Jacob for Leah, that ought to be equal.

Perhaps it was the inherent difficulty of maintaining Equation (23) or Equation (24) that led Rabbi Gershom *Meor-Hagolah* to prohibit polygamy some one thousand years ago because in the terminology of Kass: "preference . . . promises trouble in the house" [17]. The same moral comes out of the traditional tale of a man who had two wives: an old one and a young one. The old wife would pluck the black hair from her husband's head, to keep him more like her; the young wife would pluck the white hairs so her husband would look younger. You can guess the result: the husband became bald!

Let us get back to the Bible and the story of Jacob and his affairs of love. Jacob had been cheated by Laban, the father of Rachel and Leah. He had wanted only the hand of Rachel. Leah could well have recited the verse from Donne's poem "Lovers' Infiniteness":

> "If then thy the gift of love were partial,
> That some to me, some should to others fall,
> Dear, I shall never have the all."

But Leah is not frustrated! "Even if God does not simply favor Leah over Rachel, even if He is only compensating Leah out of pity for her being unloved, this intervention highlights for us the tension between love and generation . . ." [17]. Leah believes, and repeatedly says, that her husband should dwell with her because she is the fertile source of his children.

Here are some other love triangles my students came up with:

(a) Mario, Peach, and Bowser,

(b) Popeye, Brutus, and Olive,

(c) Rose, Cal, and Jack from the James Cameron's motion picture epic *Titanic*,

(d) The DreamWork's loving "trio", Shrek, Lord Farquaad, and Princess Fiona,

(e) Theseus, Phaedra, and Hippolytus from Greek mythology,

(f) Jose Arcadio, Susan, and Melquiades from *One Hundred Years of Solitude* by Colombian writer Gabriel Garcia Marquez,

(g) Miss Piggy, Kermit the Frog, and Gonzo from The Muppet Show,

(h) Anna, Karenin, and Vronsky from Leo Tolstoy's *Anna Karenina*,

(i) Arthur, Guinevere, and Lancelot from *King Arthur and the Knights of the Round Table*.

8. Concluding Remarks

In his book *Mathematics and Romantics* [19] Kovantsov does not discuss love relationships but rather love for science. He notes: "One can love science for the strong agreement of its truths, for its strength and its multifacetedness; and one can, contrarily, express his hate for its dryness and apparent unnecessary complexity, for unmotivated artificiality of its constructions and so on." The students respond with enthusiasm to topics so close to their own existence. They begin to understand, if they did not earlier, that mathematics describes life, that it is a human endeavor that aims to understand our human existence. One student (Mr. Mas) says this in rhyme:

> Professor Strogatz's model he one day would create,
> Gave all mathematics a more interesting fate.
> For all of us who know Shakespeare —
> It helps our math, of which we fear.

Examples for mathematical content like the ones explored in this paper neutralize the fear mentioned in the verses above, and if we are lucky, some of our students might even drop the prefix "dis" from the word "dislike". Some of them might even conclude that mathematics is a lovable subject!

Instructors who might wish to incorporate a discussion of love into their mathematics classes might find other references useful and inspiring in a range of ways. I personally found news items such as [13, 18, 27, 28] and research works such as [12, 16, 24, 29, 30, 31, 35] helpful as I worked to develop material to explore themes of love in my differential equations classes.

It must be stressed that in this paper, we used so-called "hard" models in the terminology of Vladimir Arnold [1], i.e. we utilized differential equations with constant coefficients. "Soft" models would treat the coefficients such as a and b in the system (1)) as functions of dependent variables $R(t)$ and $J(t)$, which could lead to even more realistic descriptions of the subject in question. Some examples of such "soft" models are those of Pierre Francois Verhulst (1804–1849) and Benjamin Gompertz (1779–1865) in population dynamics.

As a curious fact, let us note that the recent novel [7] by a non-mathematician, the great-great-granddaughter Emma Darwin of Charles Darwin, is entitled *The Mathematics of Love*. Clio Cresswell [6] was bolder in her choice of a book title: *Mathematics and Sex*. She argues (page *viii*) that "mathematics is the study of patterns; their discovery, their interactions, and their implications." John Gottman *et al.* [11] provide deep insights on the mathematics of marriage.

As Galileo Galilei (1564–1642) maintained, "Nature's great book is written in mathematical symbols." It turns out that human nature — not just the physical nature around us — is also describable in mathematical terms.

References

[1] Arnold V.I., "Hard' and 'Soft' Mathematical Models", *Priroda* (Nature), Issue 4 (1998), pages 3-14 (in Russian).

[2] Arsu, S, "After 4,000 years, love keeps poem alive", *New York Times*, 14 February 2006, available online at https://www.nytimes.com/2006/02/14/world/africa/14iht-POETRY.html, last accessed on July 19, 2019.

[3] Binder P.M., "Chaos and Love Affairs", *Mathematics Magazine*, Volume **73** Number 3 (2000), pages 235-236.

[4] Brush S., "Q&A: Derek Bok: Fixing Undergrad education", *U.S. News & World Report*, March 6, 2006, page 28.

[5] Cardano G., *The Rules of Algebra; Ars Magna*, 1545.

[6] Cresswell C., *Mathematics and Sex*, Allen & Unwin, Crows Nest, 2003.

[7] Darwin E., *Mathematics of Love*, William Morrow, New York, 2007.

[8] Frankfurt H.G., *The Reasons of Love*, Princeton University Press, 2004.

[9] Friedman M. & R.D., *Two Lucky People: Memoirs*, The University of Chicago Press, Chicago, 1998.

[10] Fromm E., *The Art of Loving*, Harper Perennial, New York, 1956.

[11] Gottman J.M., Murray J.D., Swanson C.C., Tyson R., and Swanson K.R., *The Mathematics of Marriage*, MIT Press, Cambridge, MA, 2002.

[12] Gragnini, A., Rinaldi S., and Feichtlinger G., "Cyclic Dynamics in Romantic Relationships", *International Journal of Bifurcation and Chaos*, Volume **7** (1997), pages 2611-2619.

[13] Harakas, M., "Words to Love by", *Fort Lauderdale Sun-Sentinel*, February 14, 2006 pages E1, E4.

[14] Havlin S., Kenett D.Y., Ben-Jacob E., Bunde A., Cohen R., Hermann H., Kantelhardt J.W., Kertész J., Kirkpatrick S., Kurths J. and Portugali J., "Challenges in Network Science: Applications to Infrastructures, Climate, Social Systems and Economics", *The European Physical Journal* Special Topics, Volume **214** Number 1 (2012), pages 273-293.

[15] Karaali, G., "Can Zombies Write Mathematical Poetry? Mathematical poetry as a model for humanistic mathematics", *Journal of Mathematics and the Arts*, Volume **8** Issue 1-2 (2014), pages 38–45.

[16] Kaminisky, M.P. and Ayyub B.M., "Terrorist Population Dynamics Model", *Risk Analysis*, Volume **26** Number 3 (2006), pages 747-752.

[17] Kass L.R., "Love of Woman and Love of God: The Case of Jacob", *Commentary*, March 1999, pages 46-54.

[18] Kennedy, L., "Where is Love?", Associated Press, *South Florida Sun-Sentinel*, Dec. 23, 2003.

[19] Kovantsov N.I., *Mathematics and Romantics*, Vishcha Shkola Publishers, Kiev, 1976 (in Russian).

[20] LeGuin, U. K, *The Planet of Exile*, Hachette, UK, 2015.

[21] Liebovitch, L. S., Naudot, V., Vallacher, R., Nowak, A., Bui-Wrzosinska, L. and Coleman, P., "Dynamics of Two-Actor Cooperation-Competition Conflict Models", *Physica A: Statistical Mechanics and its Applications*, Volume **387** Number 25 (2008), pages 6360-6378.

[22] Manolson, G. *Head to Heart: What You Need to Know Before Dating and Marriage*, Targum Press, 2004.

[23] Mazur, B., *Imaginary Numbers (Particularly the Square Root of Minus Fifteen)*, Picador, Farrar, Straus, and Giroux, New York, 2003.

[24] McDill, J.M. and Felsager B., "The Lighter Side of Differential Equations", *The College Mathematics Journal*, Volume **25** Number 5 (1994), pages 448-451.

[25] Murray, J., *But I Love Him: Protecting Your Teen Daughter from Controlling, Abusive Dating Relationships*, Regan Books, New York, 2006.

[26] Paley, G., "Anti Love Poem", *New Yorker Magazine*, Dec.12, 2005, page 64.

[27] Peterson, Ch., "As Usual, Boy + Girl = Confusion", Chicago Tribune, March 30, 1998, Sect 2, page 1.

[28] Peterson, I., "Love's Dynamics", *Math Trek* blog, posted on September 7, 1998. Last accessed from http://www.maa.org/mathlanf/mathtrek9_7_98.html on 2/14/2006.

[29] Radzicki, M.J., "Dyadic Processes, Tempestuous Relationships, and System Dynamics", *System Dynamics Review*, Volume **9** (1993), pages 79-94.

[30] Rinaldi, S., "Laura and Petrarch: An Intriguing Case of Cyclical Love Dynamics", *SIAM Journal of Applied Mathematics*, Volume **58** Number 9 (1998), pages 1205-1221.

[31] Rinaldi, S., Della Rossa, F., Dercole, F., Gragnini, A. and Landi, P., *Modeling Love Dynamics*, World Scientific, Singapore, 2016.

[32] Spinoza, B., *Ethics*, part III, Scholium, 1677.

[33] Sprott, J.C., "Dynamical Models of Love", *Nonlinear Dynamics, Psychology, and Life Science*, Volume **8** Issue 3 (2004), pages 303-313.

[34] Stanley, H.E., Amaral, L.A.N., Canning, D., Gopikrishnan, P., Lee, Y. and Liu ,Y., "Econophysics: Can Physicists Contribute to the Science of Economics?", *Physica A: Statistical Mechanics and its Applications*, Volume **269** Number 1 (1999), pages 156-169.

[35] Sternberg, R.J., "The Triangular Theory of Love", *Psychological Review*, Volume **93** (1986), pages 119-135.

[36] Strogatz, S., "Love Affairs and Differential Equations", *Mathematics Magazine*, Volume **61** Number 1 (February 1988), page 35.

[37] Strogatz, S., *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering*, Addison-Wesley, Reading MA, 1994.

[38] Wallerstein, J. and Blakeslee, S., *The Good Marriage: How and Why Love Lasts*, Grand Central Publishing, New York, 1996.