1-1-1993

A Bifurcation Theorem and Applications

Alfonso Castro
Harvey Mudd College

Jorge Cossio
Universidad Nacional de Colombia

Recommended Citation
Castro, Alfonso and Jorge Cossio. "A bifurcation theorem and applications", Dynamic Systems and Applications, Vol. 2 (1993), pp. 221-226.

Dynamic Systems and Applications 2 (1993) 221-226

A BIFURCATION THEOREM AND APPLICATIONS

Alfonso Castro[1] and Jorge Cossio[2]

[1]Department of Mathematics, University of North Texas,
Denton, Texas 76203-5116, USA.

[2]Departamento de Matemáticas, Universidad Nacional de Colombia,
Apartado Aéreo 3840, Medellín, Colombia.

ABSTRACT: In this paper we give a sufficient condition on the nonlinear operator N for a point (λ, u) to be a local bifurcation point of equations of the form $u + \lambda L^{-1}(N(u)) = 0$, where L is a linear operator in a real Hilbert space, L has compact inverse, and $\lambda \in \mathbb{R}$ is a parameter. Our result does not depend on the variational structure of the equation or the multiplicity of the eigenvalue of the linear operator L. Applications are made to systems of differential equations and to the existence of periodic solutions of nonlinear second order elliptic equations.

AMS (MOS) subject classification. 35B32, 34A34, 47H12, 47H15.

1. INTRODUCTION

The purpose of this paper is to provide a sufficient condition for bifurcation of equations of the form

$$u + \lambda L^{-1}(N(u)) = 0, \tag{1.1}$$

where L is a linear operator in a real Hilbert space H, L has a compact inverse, $\lambda \in \mathbb{R}$ is a parameter, and $N\colon H \to H$ is of class C^2 with $(L^{-1}N)'(0) = L^{-1}$, $N(0) = 0$. Let $-\Lambda \neq 0$ be an eigenvalue of L such that

$$H = Ker(L + \Lambda I) \oplus Range(L + \Lambda I) := X \oplus Y.$$

Let $K \subset X$ be a closed cone such that $K \neq X$, and P the projection onto X across Y. Our main result is :

Theorem A. *If for some $r_1 > 0$*

$$P(N - I)(\{u \in H : |u| \leq r_1\}) \subset K \tag{1.2}$$

then $(\Lambda, 0)$ is a bifurcation point of (1.2). Moreover, for $r > 0$ sufficiently small and $K \cap (-K) = \{0\}$ the equation (1.1) has two different solutions (α, u), (β, v) with $\|Pu\| = \|Pv\| = r$, $|\alpha - \Lambda| < r$ and $|\beta - \Lambda| < r$.

Unlike the classical results on bifurcation theory (see [2, p.204], [4, p.381], [9, p.70], and references therein) our result does not depend on the multiplicity of the eigenvalue Λ or the variational structure of equations where Theorem A applies.

It is well known that the map $F: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(\lambda,(x,y)) = (x+\lambda(x-y^3), y+\lambda(y+x^3))$$

has a singularity at $(-1,(0,0))$, however $(-1,(0,0))$ is not a bifurcation point of

$$\begin{bmatrix} x \\ y \end{bmatrix} + \lambda \begin{bmatrix} x-y^3 \\ y+x^3 \end{bmatrix} = 0. \tag{1.3}$$

Indeed the set of solutions of (1.3) is $\{(\lambda,(0,0)) \colon \lambda \in \mathbb{R}\}$. Here we observe that if in (1.3) we replace y^3 by y^2 and x^3 by x^2 then by Theorem A $(-1,(0,0))$ is point of bifurcation. In this case $K = \{(x,y) \colon x \le 0, y \ge 0\}$. This example shows that our theorem is sharp in the sense that for r small there are exactly two solutions with norm r.

2. PROOF OF THE MAIN RESULT

Let Q denote the projection onto Y across X. By standard Lyapunov-Schmidt arguments we know that there exist $\epsilon > 0$, $\delta > 0$, and a function

$$\psi \colon [\Lambda-\epsilon,\Lambda+\epsilon] \times \{x \in X \colon \|x\| \le \delta\} \longrightarrow \{y \in Y \colon \|y\| \le \delta\}$$

of class C^2 such that

$$\psi(\lambda,x) + \lambda L^{-1}(Q(N(x+\psi(\lambda,x)))) = 0. \tag{2.1}$$

Also (λ,u) is a solution to (1.1) with $\|Pu\| \le \delta$ iff $u = x+\psi(\lambda,x)$ for some $x \in X$ and

$$x + \lambda L^{-1}(P(N(x+\psi(\lambda,x))) = 0. \tag{2.2}$$

Moreover

$$\lim_{\|x\|\to 0} \frac{\|\psi(\lambda,x)\|}{\|x\|} = 0. \tag{2.3}$$

In order to prove Theorem A, first we show that for r small enough the equation (2.2) has a solution (α,u) with $\|Pu\| = r$, and $\|\alpha-\Lambda\| < r$. Since $P(\psi(\lambda,x)) = 0$, the equation (2.2) is equivalent to

$$(\lambda-\Lambda)x + \lambda P(N-I)(x+\psi(\lambda,x)) = 0. \tag{2.4}$$

Let h be defined by $h(\lambda,x) := (\lambda-\Lambda)x + \lambda P(N-I)(x+\psi(\lambda,x))$. Because N is of class C^2 then h is of class C^2. By the implicit function theorem we can assume that for all $s \in ([-\epsilon,-\frac{\epsilon}{2}] \cup [\frac{\epsilon}{2},\epsilon])$ h restricted to $\{\Lambda+s\} \times \{x \in X \colon \|x\| \le \delta\}$ is a diffeomorphism. More precisely, if $q \in X$ is small enough there exist differentiable functions f,g (depending of q) such that

$$\left\{(\lambda, x)\colon \lambda \in \left[\Lambda - \epsilon, \Lambda - \frac{\epsilon}{2}\right], h(\lambda, x) = q\right\} = \left\{(\lambda, f(\lambda))\colon \lambda \in \left[\Lambda - \epsilon, \Lambda - \frac{\epsilon}{2}\right]\right\} \tag{2.5}$$

and

$$\left\{(\lambda, x)\colon \lambda \in \left[\Lambda + \frac{\epsilon}{2}, \Lambda + \epsilon\right], h(\lambda, x) = q\right\} = \left\{(\lambda, g(\lambda))\colon \lambda \in \left[\Lambda + \frac{\epsilon}{2}, \Lambda + \epsilon\right]\right\}. \tag{2.6}$$

Let $0 < r < \min\{\epsilon, r_1\}$. Since $X - K$ is a non-empty open set, by Sard's theorem there exists a sequence of positive real numbers $\{q_n\}$ converging to 0, and $\xi \in X - K$ such that $\{s\xi\colon s \in \{q_n\}\}$ is a set of regular values of h. Since $\{q_n\}$ converges to 0, without loss of generality we can assume that for each $n = 1, 2, 3, \ldots$ there exists a unique z_n with $\|z_n\| < r$ such that $h(\Lambda - \epsilon, z_n) = q_n\xi$; similarly there exists w_n with $\|w_n\| < r$ such that $h(\Lambda + \epsilon, w_n) = q_n\xi$. Let Γ_n and Σ_n be the connected component of $h^{-1}(q_n\xi)$ containing $(\Lambda - \epsilon, z_n)$ and $(\Lambda + \epsilon, w_n)$ respectively. Because the $q_n\xi's$ are regular values then the $\Gamma_n's$ and $\Sigma_n's$ are diffeomorphic to either $[0, 1]$ or S^1 (see [8, p.55]). Since both the $\Gamma_n's$, and $\Sigma_n's$ contain points in $\partial([\Lambda - \epsilon, \Lambda + \epsilon] \times \{x \in X\colon \|x\| < \delta\})$ namely $(\Lambda - \epsilon, z_n)$ and $(\Lambda + \epsilon, w_n)$ (see also (2.5) and (2.6)), we see that the sets Γ_n and Σ_n are diffeomorphic to $[0, 1]$.

Let $\Phi_n\colon [0, 1] \to \Gamma_n$ be a diffeomorphism with $\Phi_n(0) = (\Lambda - \epsilon, z_n)$. Let $\Phi_n(1) = (\alpha_n, x_n)$. Since $q_n\xi \notin K$ and Γ_n is connected (see (1.2), (2.4)) we see that Γ_n is contained in $[\Lambda - \epsilon, \Lambda] \times \overline{B_X(r)}$ (similarly Σ_n is contained in $[\Lambda, \Lambda + \epsilon] \times \overline{B_X(r)}$). Thus $\Phi_n(1) \in \partial((\Lambda - \epsilon, \Lambda) \times B_X(r))$. Since (see (2.4)) $\alpha_n < \Lambda$ and h restricted to $\{\Lambda\} \times \overline{B_X(r)}$ is a diffeomorphism we see that

$$\alpha_n \in (\Lambda - \epsilon, \Lambda) \quad \text{and} \quad \|x_n\| = r, \tag{2.7}$$

thus

$$(\alpha_n - \Lambda)x_n = q_n\xi - \sigma_n \quad \text{with} \quad \sigma_n \in K. \tag{2.8}$$

Similarly, for n sufficiently large we see that there exist $\beta_n \in (\Lambda, \Lambda + \epsilon)$ and $\overline{x_n}$ such that

$$(\beta_n - \Lambda)\overline{x_n} = q_n\xi - \rho_n \quad \text{with} \quad \|\overline{x_n}\| = r \quad \text{and} \quad \rho_n \in K. \tag{2.9}$$

Let (α, x) and $(\beta, \overline{x})$ be accumulation points of $\{(\alpha_n, x_n)\}$ and $\{\beta_n, \overline{x_n})\}$ respectively. By the continuity of N and ψ we see that x and $\overline{x}$ are solutions to (2.4), which proves that $(\Lambda, 0)$ is a point of bifurcation of (1.1).

If, in addition, $K \cap (-K) = \{0\}$ then $A = \{x \in X\colon \|x\| = r, x = \alpha\xi + \sigma, \alpha \leq 0, \sigma \in K\}$ and $B = \{x \in X\colon \|x\| = r, x = \alpha\xi - \sigma, \alpha \geq 0, \sigma \in K\}$ are compact and disjoint thus we see that $x \neq \overline{x}$, which proves theorem A ■

Remark 1. Doublechecking the proof of Theorem A we see it holds even if hypothesis (1.2) is relaxed to

$$P(N-I)(\{x+\psi(\lambda,x) \colon \|x\|\leq r_1, |\lambda-\Lambda|<\epsilon\})\subseteq K. \tag{2.10}$$

3. APPLICATIONS

Example 1. let us consider the system

$$\begin{bmatrix} u'' \\ v'' \end{bmatrix} + \lambda \begin{bmatrix} u+h(u,v,u',v') \\ v+\gamma h(u,v,u',v') \end{bmatrix} = 0, \tag{3.1}$$

$$\begin{bmatrix} u \\ v \end{bmatrix}(0) = \begin{bmatrix} u \\ v \end{bmatrix}(\pi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.2}$$

where $\gamma \in \mathbb{R}$ and h is a C^2 function with bounded derivative such that $h(0)=0$ and $\nabla h(0)=0$. First we observe that points of the form $(n^2,0)$ $n=1,2,3,\ldots$ are possible points of bifurcation. It is easily verified that the eigenspace corresponding to n^2 is generated by $\left[\begin{smallmatrix} \sin(nt) \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ \sin(nt) \end{smallmatrix}\right]$. Because the inverse of the operator second derivative with Dirichlet boundary condition defines a continuous operator from $L^2(0,\pi)$ into $H^2(0,\pi)\cap H_0^1(0,\pi)$ we see that (3.1)-(3.2) is in the form (1.1) with $H=H_0^1(0,\pi)\times H_0^1(0,\pi)$. Without loss of generality we can assume $|\gamma|>1$ (the case $|\gamma|<1$ is similar).

In order to verify (1.2) we observe that

$$\left\langle (N-I)\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \sin(nt) \\ 0 \end{bmatrix} \right\rangle_H = \langle h, \sin(nt)\rangle_1 \tag{3.3}$$

and

$$\left\langle (N-I)\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} 0 \\ \sin(nt) \end{bmatrix} \right\rangle_H = \gamma\,\langle h, \sin(nt)\rangle_1 . \tag{3.4}$$

Thus taking $K=\left\{a\left[\begin{smallmatrix} \sin(nt) \\ 0 \end{smallmatrix}\right]+b\left[\begin{smallmatrix} 0 \\ \sin(nt) \end{smallmatrix}\right]; a\in\mathbb{R}, b\in\mathbb{R}, |a|\leq|b|\right\}$, we see that (1.2) holds, and hence $(n^2,0)$ is point of bifurcation. Unfortunately $K=-K$, and thus we can not conclude the existence of two solutions on the sphere of radius r, for r small.

Example 2. We consider the existence of 2π–periodic solutions to the equation

$$u''-u+\lambda(u+(\sin(nt)+\cos(nt))(u')^2)=0. \tag{3.5}$$

We now apply Theorem A (see also Remark 1) to show that $(n^2+1,0)$ is a point of bifurcation. In this case we take H to be the Sobolev space $H^1_{2\pi}(\mathbb{R})$ of 2π–periodic functions in $L^2([0,2\pi])$ having derivative in $L^2([0,2\pi])$. We take L and N to be defined by

$$Lu = u'' - u$$
$$N(u) = u + (\sin(nt) + \cos(nt))(u')^2,$$

and $K = \{\alpha(\sin(nt) + \cos(nt)) + \beta(\sin(nt) - \cos(nt)); \alpha \geq 0, |\beta| \leq M\alpha\}$, where M is a positive real number to be determined.

Let $x = a(\sin(nt) + \cos(nt)) + b(\sin(nt) - \cos(nt))$ and

$$\psi(\lambda, x) = \sum_{\substack{j=1 \\ j\neq n}}^{\infty} c_j(\sin(jt) + \cos(jt)) + d_j(\sin(jt) - \cos(jt)).$$

By the definition of the inner product in H, we have

$$\begin{aligned}\langle (N-I)(x+\psi(\lambda,x)), (\sin(nt)+\cos(nt))\rangle_1 \\ \geq c\|x\|^2 - 4(n^2+1)\|x\|\,\|\psi(\lambda,x)\| - 2(n^2+1)\|\psi(\lambda,x)\|^2, \quad (3.6)\end{aligned}$$

where $c = n^2\pi$. Now by (2.3) we see that if λ is close to Λ and x is small then

$$4(n^2+1)\|x\|\|\psi(\lambda,x)\| + 2(n^2+1)\|\psi(\lambda,x)\|^2 \leq (c/2)\|x\|^2.$$

Thus we have

$$\langle (N-I)(x+\psi(\lambda,x)), (\sin(nt)+\cos(nt))\rangle_1 \geq (c/2)\|x\|^2 > 0. \quad (3.7)$$

On the other hand

$$|\langle (N-I)(x+\psi(\lambda,x)), (\sin(nt)-\cos(nt))\rangle_1| \leq (n^2+1)\|x+\psi(\lambda,x)\|^2. \quad (3.8)$$

Now by (2.3) we see that if λ is close to Λ and x is small then

$$\begin{aligned}|\langle (N-I)(x+\psi(\lambda,x)), (\sin(nt)-\cos(nt))\rangle_1| \leq \\ 4\langle (N-I)(x+\psi(\lambda,x)), (\sin(nt)+\cos(nt))\rangle_1. \quad (3.9)\end{aligned}$$

From (3.7)–(3.9) we see that (2.8) holds, which proves that $(n^2+1, 0)$ is a point of bifurcation.

Moreover, since $K \cap (-K) = \{0\}$ for every r sufficiently small the equation (3.5) has two different solutions (α, u), (β, v) with $\|Pu\| = \|Pv\| = r$.

Remark 2. Examples of second order elliptic and hyperbolic boundary value problems can be constructed mimicking example 2.

ACKNOWLEDGMENT

This reasearch was partially supported by the Texas Advanced Research Program.

REFERENCES

1. H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review **18** (1976), 620-709.
2. S. Chow and J. Hale, *Methods of bifurcation theory*, A series of comprehensive studies in Mathematics 251, Springer-Verlag, New York, 1982.
3. E. N. Dancer, *On the indices of fixed points of mappings in cones and applications*, J. Math. Anal. and Appl. **91** (1983), 131-151.
4. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
5. J. Ize, *Bifurcation theory for Fredholm operators*, Memoirs of the American Mathematical Society number 174, 1976.
6. M. A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations*, Pergamon, New York, 1964.
7. J. Keller and S. Antman, *Bifurcation theory and nonlinear eigenvalue problems*, Benjamin, New York, 1969.
8. J. Milnor, *Topology from the differentiable viewpoint*, the University press of Virginia, Charlottesville, 1965.
9. P. Rabinowitz, *Minimax Methods in critical point theory with applications to differential equations*, Regional conference series in mathematics number 65, American Mathematical Society, 1986.
10. P. Rabinowitz, *Topological methods in bifurcation theory*, Seminaire scientifique OTAN, Les presses de l'Université de Montreal, 1985.