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Abstract

In this note we present a local surjectivity result which is applicable to differential equations for which full boundary conditions may not be known. Our method uses continuous steepest descent and Sobolev gradients.

The main purpose of this note is the proof of the following.

Main Theorem. *Suppose that each of H and K is a Hilbert space, $r > 0$, $Q > 0$, F is a C^1 function from H to K which has a locally Lipschitzian derivative and $F(0) = 0$. Finally suppose that there is $c > 0$ such that if $u \in H$, $\|u\| \le r$, and $g \in K, \|g\|_K = 1$, then*

$$\langle F'(u)v, g\rangle_K \ge c, \text{ for some } v \in H \text{ with } \|v\|_H \le Q. \tag{1}$$

If $y \in K$ and $\|y\|_K < \frac{rc}{Q})$, then

$$u = \lim_{t\to\infty} z(t) \text{ satisfies } F(u) = y \text{ and } \|u\|_H \le r$$

where z is the unique function from $[0, \infty)$ to H so that

$$z(0) = 0,\ z'(t) = -F'(z(t))^*(F(z(t)) - y),\ t \ge 0. \tag{2}$$

From condition (1) we have that if $u \in H, \|u\| \le r,\ g \in K$, then

$$\|F'(u)^*g\|_H \ge \frac{c}{Q}\|g\|_K \tag{3}$$

where $F'(u)^*$ denotes the member of $L(K, H)$ so that

$$\langle F'(u)v, g\rangle = \langle v, F'(u)^*g\rangle, v \in H,\ g \in K \tag{4}$$

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Although some of the ingredients needed to prove the Theorem can be found in [3] and [4] we have elected to give a more nearly self-contained argument. References [3], [4], [5] mainly concern global results and details on numerical tracking of solutions z to equations such as (2).

Proof. Let $y \in K$. If $\phi : H \to R$ is defined by

$$\phi(x) = \|F(x) - y\|_K^2/2, \quad x \in H,$$

then

$$\phi'(x) = \langle F'(x)h, F(x) - y\rangle_K = \langle h, F'(x)^*(F(x) - y)\rangle_H, \ x, h \in H.$$

We denote $F'(x)^*(F(x) - y)$ by $(\nabla\phi)(x)$ and call this element the gradient of ϕ at the point x, $x \in H$.

Assertion 1. For each $y \in K$ the equation (2) has a unique solution defined on $[0, \infty)$. In fact, since F' is locally Lipschitzian so is $\nabla\phi$. From basic theory of ordinary differential equations there is $d_0 > 0$ so that the equation in (2) has a solution on $[0, d_0)$. Suppose now that the set of all such numbers d_0 is bounded and denote by d its least upper bound. Denote by z the unique solution to the equation to (2) on $[0, d)$. We will show that $\lim_{t\to d-} z(t)$ exists. Note that if $0 \le a < b < d$, then

$$\|z(b) - z(a)\|_H^2 = \|\int_a^b z'\|_H^2 \le (\int_a^b \|z'\|_H)^2 \le (b - a)\int_a^b \|z'\|_H^2 \tag{5}$$

Note also that if $0 \le t < d$, then

$$\phi(z)'(t) = \phi'(z(t))z'(t) = < z'(t), (\nabla\phi)(z(t)) > = -\|(\nabla\phi)(z(t))\|_H^2 \tag{6}$$

and so

$$\phi(z(b)) - \phi(z(a)) = \int_a^b \phi(z)' = -\int_a^b \|z'\|_H^2.$$

Hence

$$\int_a^b \|z'\|_H^2 \le \phi(z(a)), a \le b < d. \tag{7}$$

By the Cauchy-Schwartz inequality and (7) we see that

$$(\int_a^b \|z'\|_H)^2 \le (b - a)\int_a^b \|z'\|_H^2.$$

Therefore

$$\int_a^b \|z'\|_H \leq (d\phi(z(a)))^{1/2}, a \leq b < d.$$

But this implies that $\int_a^{d-} \|z'\|_H$ exists and so $q \equiv \lim_{t \to d-} z(t)$ exists. But again from basic theory, there is $d_1 > d$ for which there is a function f defined on $[c, d_1)$ such that

$$f(d) = q, f'(t) = -(\nabla\phi)(f(t)),\ t \in [d, d_1).$$

But the function w on $[0, d_1)$ so that $w(t) = z(t),\ t \in [0, d),\ w(d) = q$, $w(t) = f(t),\ t \in (d, d_1)$, satisfies

$$w(0) = 0, w'(t) = -(\nabla\phi)(w(t)),\ t \in [0, d_1),$$

contradicting the nature of d since $d < d_1$. Hence there is a solution to (2) defined on $[0, \infty)$. Uniqueness follows from the basic theory. This completes an argument for Assertion 1, a known result ([1] ,[5]).

Assertion 2. Let $y \in K$ and z satisfy (2). If for some $C > 0$

$$\|(\nabla\phi)(z(t))\|_H \geq C\|F(z(t)) - y\|_K,\ t \geq 0, \tag{8}$$

then

$$u = \lim_{t\to\infty} z(t) \text{ exists and } \quad F(u) = y.$$

To prove this assertion, note first that if $(\nabla\phi)(0) = 0$, then the conclusion holds (with $u = 0$). Suppose then that $(\nabla\phi)(0) \neq 0$. This last supposition has the consequence that $(\nabla\phi)(z(t)) \neq 0, t \geq 0$. [If $(\nabla\phi)(z(t_0)) = 0$ for some $t_0 \geq 0$, then the function w on $[0, \infty)$ defined by $w(t) = z(t_0),\ t \geq 0$, would satisfy

$$w' = (\nabla\phi)(w) \text{ and } w(t_0) = z(t_0) \tag{9}$$

and hence $(\nabla\phi)(w(t)) = 0, t \geq 0$. But if w is replaced in (9) by z, then (9) still holds. This violates the fact that (9) does not have two solutions.

As in (6), $\phi(z)' = -\|(\nabla\phi)(z)\|_H^2$ and so using (8)

$$\phi(z)'(t) \leq -C^2\|F(z(t)) - y\|_H^2 = -2C^2\phi(z(t)).$$

Therefore

$$\phi(z)'(t)/\phi(z(t)) \leq -2C^2, t \geq 0.$$

Accordingly,

$$\ln(\phi(z(t))/\phi(0)) \leq -2C^2 t$$

and

$$\phi(z(t)) \leq \phi(0)\exp(-2C^2 t),\ t \geq 0.$$

Therefore

$$\lim_{t\to\infty} \phi(z(t)) = 0.$$

Moreover if n is a positive integer,

$$\left(\int_n^{n+1} \|z'\|_H\right)^2 \leq \int_n^{n+1} \|z'\|_H^2 = \phi(z(n)) - \phi(z(n+1)) \leq \phi(0)\exp(-2C^2 n)$$

and so

$$\int_0^\infty \|z'\|_H \leq \phi(0)^{1/2} \sum_{n=0}^{\infty} \exp(-C^2 n) = \phi(0)^{1/2}/(1-\exp(-C^2)).$$

Consequently, $\|z'\|_H \in L_1([0,\infty)]$ and so $u = \lim_{t\to\infty} z(t)$ exists. Moreover $\phi(u) = 0$ since $0 = \lim_{t\to\infty}\phi(z(t))$. This completes an argument Assertion 2. This is essentially Theorem 1 of [3] (see also Theorem 2 of [4]).

Assertion 3. Suppose $y \in K$ and $\|y\|_K < \frac{rc}{Q}$ then $\|z(t)\|_H < r$ for all $t \in [0,\infty)$. By the Cauchy-Schwartz inequality and (1) we have

$$c\|g\|_K \leq \langle F'(u)v, g\rangle_K = \langle v, F'(u)^* g\rangle_H \leq Q\|F'(u)^* g\|_H$$

if $\|u\|_H \leq r$ and $g \in K$. Consequently

$$\|(\nabla\phi)(u)\|_H = \|F'(u)^*(F(u)-y)\|_H \geq C\|F(u)-y\|_K, u \in H, \|u\|_H \leq r,$$

where $C = c/Q$. Hence

$$\|(\nabla\phi)(z(t))\|_H = \|F'(z(t))^*(F(z(t))-y)\|_H \geq C\|F(z(t))-y\|_K \tag{10}$$

provided that

$$t \geq 0 \text{ and } \|z(t)\|_H \leq r.$$

Finally we show that if $\|y\| < Cr$ then $\|z(t)\| \leq r$ for all $t \in [0,\infty)$. To prove this assertion first note that if $(\nabla\phi)(0) = 0$, then $z(t) = 0$, $t \geq 0$, and so the conclusion holds with $u = 0$.

Suppose now that $(\nabla\phi)(0) \neq 0$. Thus $(\nabla\phi)(z(t)) \neq 0$ for all $t \geq 0$ as noted in the argument for the previous assertion.

Define α as the function which satisfies

$$\alpha(0) = 0, \quad \alpha'(t) = 1/\|(\nabla\phi)(z(\alpha(t)))\|_H, \quad t \in [0, d), \tag{11}$$

where d is as large as possible, possibly $d = \infty$.

Define v by $v(t) = z(\alpha(t))$, $t \in D(\alpha)$. Thus

$$\begin{aligned} v'(t) &= \alpha'(t)z'(\alpha(t)) = -\alpha'(t)(\nabla\phi)(z(\alpha(t))) \\ &= -(1/\|(\nabla\phi)(z(\alpha(t)))\|_H)(\nabla\phi)(z(\alpha(t))) \end{aligned} \tag{12}$$

and so

$$v(0) = 0,\ v'(t) = -(1/\|(\nabla\phi)(v(t))\|_H)(\nabla\phi)(v(t)),\ t \in D(v) = D(\alpha). \tag{13}$$

Note also that

$$\|v(t)\|_H = \|v(t) - v(0)\|_H = \|\int_0^t v'\|_H \leq \int_o^t \|v'\|_H = t, t \in D(v).$$

Hence

$$\phi(v)' = \phi'(v)v' = -\langle(\nabla\phi)(v), (\nabla\phi)(v)\rangle_H/\|(\nabla\phi)(v)\|_H = -\|(\nabla\phi)(v)\|_H,$$

and so if $t \in D(v)$ and $t \leq r$, then

$$\phi(v)'(t) = -\|(\nabla\phi)(v(t))\|_H \leq -C\|F(v(t)) - y\|_H = -C(2\phi(v(t)))^{1/2},$$

and thus

$$\phi(v)'(t)/\phi(v(t))^{1/2} \leq -C2^{1/2}.$$

This differential inequality is solved to yield

$$2\phi(v(t))^{1/2} - 2\phi(v(0))^{1/2} \leq -2^{1/2}Ct,\ t \in D(v),\ t \leq r.$$

But this is equivalent to

$$\|F(v(t)) - y\|_K \leq \|F(0) - y\|_K - Ct \leq C(r - t) \tag{14}$$

since $F(0) = 0$ and $\|y\|_K < \frac{rc}{Q}$. It follows from (14) that $d \leq r$ since if not, there would be $t \in [0, r]$ such that $F(v(t_0)) = y$ and hence $(\nabla\phi)(v(t_0)) = 0$

and consequently, $(\nabla\phi)(z(\alpha^{-1}(t_0))) = 0$, a contradiction. From this it follows that $\|v(t)\|_H \leq r, t \in D(v)$. Since $D(\alpha)$ is a bounded set, and α is increasing, it must be that $\lim_{t\to d-} \alpha(t) = \infty$ (if $\lim_{t\to d-} \alpha(t) = q < \infty$, it would follow that $(\nabla\phi)(z(q)) = 0$, a contradiction). Therefore $\|z(t)\|_H \leq r$, $t \geq 0$. This and Assertion 2 prove the Theorem.

That the pairing between

$$g \in K, \|g\| < \frac{rc}{Q}$$

and the corresponding

$$u = \lim_{t\to\infty} z(t)$$

yields a function is intended to justify the use of the term 'inverse function theorem' in the title. We hope to eventually make an extension of the present process in the direction of [2].

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