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Semilinear equations with discrete spectrum

Alfonso Castro

ABSTRACT. This is an overview of the solvability of semilinear equations where the linear part has discrete spectrum. Semilinear elliptic and hyperbolic equations, as well as Hammerstein integral equations, are used as motivating examples. The presentation is intended to be accessible to non experts.

1. INTRODUCTION

These lectures deal with the study of equations of the type

$$(1) \quad L(u) = N(u),$$

where L is linear operator with discrete spectrum $\sigma(L) = \{\lambda_i; i = 1, 2, \dots\}$. We recall that if L is a linear operator defined on a dense subset of a Hilbert space then $\sigma(L)$ is the complement in the complex plane of the resolvent $\rho(L) = \{\lambda; L - \lambda I \text{ has a bounded inverse}\}$.

A main model for equation (1) is the elliptic problem

$$(2) \quad \begin{cases} -\Delta u = f(x, u) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded region in $\mathbf{R}^n$ with smooth boundary, Δ is the Laplace operator, and $f : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ is a function of class C^1 .

We will see how the interaction of the *growth* of N , generally determined by the *numerical range* of N' , with the spectrum of L provides information on the existence, multiplicity, stability, and other properties of the solutions to (1). Such interaction introduces terms such as resonance, jumping nonlinearities, superlinearities, asymptotic nonlinearities, critical growth, and others.

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While these lectures emphasize the *variational* case, hence $\sigma(L) \subset \mathbf{R}$, digressions will be made to consider the case where L is a nonselfadjoint perturbation of a selfadjoint operator. This area is yet to be adequately explored.

2. THE SPECTRUM OF SOME LINEAR OPERATORS

Let us begin by considering the spectrum of the second order derivative operator, that is,

$$(3) \quad L(u) \equiv -u'' = \lambda u, \quad Bu = 0,$$

where $B(u)$ is a boundary condition such as $u(0) = u(\pi) = 0$ (Dirichlet boundary condition) or $u(t) = u(t + 2\pi)$ (periodic boundary condition). In the case of the Dirichlet boundary condition $\sigma(L) = \{j^2; j = 1, 2, \dots\}$, where as in the case of periodic boundary condition $\sigma(L) = \{j^2; j = 0, 1, \dots\}$, a minor difference. However when we consider a first order perturbation of the second derivative, for example

$$(4) \quad L(u) \equiv -u'' + au' = \lambda u, \quad Bu = 0,$$

the spectrum changes radically from one boundary condition to the other. For the Dirichlet boundary condition, σ is, as before, an increasing sequence of positive real numbers. Namely, $\sigma(L) = \{j^2 - (a^2/4); j = 1, 2, \dots\}$. The corresponding eigenfunctions are multiples of $u(t) = e^{at/2} \sin(jt)$. On the other hand for the periodic boundary condition, $\sigma(L)$ contains only one real number $\lambda = 0$. This has a semilinear analog. Namely, while the periodic problem

$$(5) \quad -u'' + au' = u^3, \quad u(t) = u(t + 2\pi)$$

is satisfied only by $u \equiv 0$, the equation

$$(6) \quad -u'' + au' = u^3, \quad u(0) = u(\pi) = 0$$

has infinitely many solutions. More precisely, for each nonnegative integer j the equation (6) has a solution with exactly j interior zeroes.

If $L = -\Delta$ is applied to functions defined on the smooth bounded region satisfying one of the following boundary conditions

$$(7) \quad \begin{aligned} u &= 0 \quad \text{on } \partial\Omega \quad (\text{Dirichlet}), \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{on } \partial\Omega \quad (\text{Neumann}), \text{ or} \\ \frac{\partial u}{\partial \eta} + b(x)u &= 0 \quad \text{on } \partial\Omega \quad (\text{mixed}), \end{aligned}$$

then

$$(8) \quad \sigma(L) = \{\lambda_1, \lambda_2, \dots, \lambda_k, \dots\}, \quad \lambda_k \rightarrow \infty.$$

Except for the case when Ω is an interval in $\mathbf{R}$, the eigenvalues need not be simple. However, all the eigenvalues have finite multiplicity. This implies that if $c \notin \sigma(L)$ then the inverse of $L - cI$ is compact. The reader is referred to [29] for a detailed analysis of this topic. In particular, the reader should

consult [29] for the property of eigenfunctions.

λ_1

$$(9) \quad \lambda_{k+1}$$

where H is the space of functions having corresponding eigenfunctions.

LEMMA 2. If λ_1 is the first eigenvalue then ϕ_1 is the corresponding eigenfunction.

The proof is given in [39].

When L is applied to functions on $\partial\Omega$ the condition gives

$$(10) \quad u(0, t) =$$

the spectrum

$$(11)$$

Unlike the elliptic case both to $+\infty$ and $-\infty$ with n a positive integer, verify, for example, the multiplicity of the eigenvalues of the form $\sin(kx)$ corresponding to the infinite multiplicity of the eigenvalue 0. Invertible but not invertible modulo 2π this set of eigenvalues is a number theory

consult [29] for the proof that the eigenvalues $\{\lambda_k\}$ and their corresponding eigenfunctions $\{\phi_k\}$ may be characterized by the following variational property

$$\begin{aligned} \lambda_1 &= \min \left\{ \frac{\int_{\Omega} \|\nabla u\|^2 dx}{\int_{\Omega} \|u\|^2 dx}; u \in H - \{0\} \right\}, \text{ and for } k > 1 \\ (9) \quad \lambda_{k+1} &= \min \left\{ \frac{\int_{\Omega} \|\nabla u\|^2 dx}{\int_{\Omega} \|u\|^2 dx}; u \in H - \{0\}, \right. \\ &\quad \left. \int_{\Omega} u \phi_i dx = 0 \text{ for } i = 1, \dots, k \right\}, \end{aligned}$$

where H is the subspace of the Sobolev space (see [1]) of square integrable functions having first order partial derivatives in $L^2(\Omega)$ and that satisfy the corresponding boundary condition in (7). From (9) the following properties of eigenfunctions follow.

LEMMA 2.1. *If ϕ_1 is an eigenfunction corresponding to the eigenvalue λ_1 then ϕ_1 does not change sign in Ω . If ϕ_2 is an eigenfunction corresponding to the eigenvalue λ_2 then ϕ_2 changes sign exactly once, that is, $\{x \in \Omega; \phi_2(x) > 0\}$ is connected and $\{x \in \Omega; \phi_2(x) < 0\}$ is connected.*

The proof of Lemma 2.1 uses strongly that if $u \in H$ then $|u| \in H$ (see [39]).

When L is given by the wave operator $L(u) \equiv \square(u) = u_{tt} - u_{xx}$ acting on functions defined on $[0, \pi] \times \mathbf{R}$ satisfying the Dirichlet-periodic boundary condition given by

$$(10) \quad u(0, t) = u(\pi, t) = 0, \quad u(x, t) = u(x, t + T), \quad \text{for all } x \in [0, \pi], \quad t \in \mathbf{R},$$

the spectrum of L is the closure of

$$(11) \quad \left\{ k^2 - \frac{4\pi^2}{T^2}; k = 1, 2, \dots, j = 0, 1, \dots \right\}.$$

Unlike the elliptic case ($L = -\Delta$), now the spectrum of L accumulates both to $+\infty$ and $-\infty$. If T is a rational multiple of π or of the form $\sqrt{n}\pi$, with n a positive integer, then $\sigma(L)$ is a discrete subset of $\mathbf{R}$. It is easy to verify, for example, that when $T = 2\pi$ that all the eigenvalues are of finite multiplicity, except for 0 which has infinite multiplicity. In fact functions of the form $\sin(kx) \sin(kt)$, $k = 1, 2, \dots$ are mutually orthogonal eigenfunctions corresponding to the eigenvalue 0. When $T = \sqrt{n}\pi$ all eigenvalues have infinite multiplicity (see [10]). This implies that if $c \notin \sigma(L)$ then $L - cI$ is invertible but $(L - cI)^{-1}$ is not compact. When T is not a rational multiple of 2π this set can be very complicated. Its study requires delicate tools from number theory (see [46]).

3. Critical point theory

Let $H = H_0^{1,2}(\Omega)$ be the Sobolev space of square integrable functions with first order partial derivatives in $L^2(\Omega)$ and vanishing on $\partial\Omega$ (see [1]). Finding solutions to (2) is equivalent to finding critical points of the functional of the functional $J: H \rightarrow \mathbf{R}$ defined by

$$(12) \quad J(u) = \int_{\Omega} \left\{ \frac{1}{2} |\nabla u|^2 - F(x, u) \right\} dx,$$

where $F(x, u) = \int_0^u f(x, s) ds$. The gradient and Hessian of J are given by

$$(13) \quad \langle \nabla J(u), v \rangle = \int_{\Omega} \{ \nabla u \cdot \nabla v - f(x, u)v \} dx$$

and

$$(14) \quad \langle D^2 J(u)v, w \rangle = \int_{\Omega} \left\{ \nabla v \cdot \nabla w - \frac{\partial f}{\partial u}(x, u)vw \right\} dx.$$

If u is a critical point of J , we define its **the Morse index** to be the dimension of the largest subspace in which $D^2 J(u)$ is negative definite. Also for future reference we note that if $\{\phi_1, \phi_2, \dots\}$ is a set of eigenfunctions corresponding to the eigenvalues given in (8), $X = \text{span} \{\phi_1, \dots, \phi_k\}$, and $Y = \text{span} \{\phi_{k+1}, \dots\}$ then from (9)

$$(15) \quad \begin{aligned} \lambda_k \int_{\Omega} u^2(x) dx &\geq \int_{\Omega} \|\nabla u(x)\|^2 dx \text{ for } u \in X \\ \lambda_{k+1} \int_{\Omega} u^2(x) dx &\leq \int_{\Omega} \|\nabla u(x)\|^2 dx \text{ for } u \in Y, \end{aligned}$$

the latter are known as *Poincaré's inequalities*.

The solutions to the equation

$$(16) \quad \begin{cases} \square(u) + f(x, t, u) = 0 & (x, t) \in (0, \pi) \times \mathbf{R} \\ u(0, t) = u(\pi, t) = 0, \quad u(x, t) = u(x, t+T), & \text{for all } x \in [0, \pi], t \in \mathbf{R} \end{cases}$$

may also be studied as critical points of a functional. In this case H is given by the space of functions $u: [0, \pi] \times \mathbf{R} \rightarrow \mathbf{R}$ such that $u(0, t) = u(\pi, t) = 0$ for all $t \in \mathbf{R}$, $u(x, t) = u(x, t+T)$ for all $(x, t) \in (0, \pi) \times \mathbf{R}$, $u, u_x, u_t \in L^2((0, \pi) \times [0, T])$. The functional whose critical points are the solutions of (16) is given by

$$(17) \quad W(u) = \int_{\Omega} \left\{ \frac{1}{2} (u_t^2 - u_x^2) - F(x, t, u) \right\} dx,$$

where $\Omega = (0, \pi) \times [0, T]$, and $F(x, t, u) = \int_0^u f(x, t, s) ds$. Results for J do not immediately transfer to W . A key difference between J and W is that while the integrand in J depending on the partial derivatives of u is positive definite, the one in W is positive definite in an infinite dimensional subspace, negative definite in an infinite dimensional subspace, and may be identically equal to zero in an infinite dimensional subspace. This creates many difficulties when applying arguments based on *compactness*.

Another point theory the form

(18)

where $\Omega \subset \mathbf{R}^n$ with $K(x, y)$ $\int_{\Omega} K(x, y)u(x)v(y)dx dy$ exist closed s $\mathcal{K}(Y) \subset Y$, $\langle \cdot, \cdot \rangle$ further that find self-adjoint such that $K_1 v \in Y$. Let J defined by

$$(19) \quad I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \int_{\Omega} K(x, y)u(x)v(y)dx dy$$

is such that $\mathcal{K}(Y) \subset Y$ to (18) (see [1])

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Another class of semilinear equations that may be studied via critical point theory are *Hammerstein Integral Equations*. These are equations of the form

$$(18) \quad u = \psi(x) + \int_{\Omega} K(x, y) f(x, u(x)) dx,$$

where $\Omega \subset \mathbf{R}^N$ is a bounded region, f is as above, and $K \in L^2(\Omega \times \Omega)$ with $K(x, y) = K(y, x)$ for all $x, y \in \Omega$. Since the operator $\mathcal{K}(u)(y) \equiv \int_{\Omega} K(x, y) u(x) dx$ defines a self-adjoint compact operator in $L^2(\Omega)$, there exist closed subspaces $X, Y \subset L^2(\Omega)$ such that $X \oplus Y = L^2(\Omega)$, $\mathcal{K}(X) \subset X$, $\mathcal{K}(Y) \subset Y$, $\langle \mathcal{K}(u), u \rangle \geq 0$ for $u \in X$, and $\langle \mathcal{K}(v), v \rangle \leq 0$ for $v \in Y$. Assuming further that $\mathcal{K}$ is positive definite in X and negative definite on Y one can find self-adjoint positive definite operators $K_1 : X \rightarrow X$, and $K_2 : Y \rightarrow Y$ such that $K_1(K_1(u)) = \mathcal{K}(u)$ for all $u \in X$, and $K_2(K_2(v)) = -\mathcal{K}(v)$ for all $v \in Y$. Let F be as in (12). Now the functional $I : X \oplus Y = L^2(\Omega) \rightarrow \mathbf{R}$ defined by

$$(19) \quad I(u + v) = \int_{\Omega} \left\{ \frac{1}{2}(u^2 + v^2) - F(x, K_1 u - K_2 v) - \psi(x)(u + v) \right\} dx,$$

is such that $u + v$ is a critical point of I if and only if $K_1 u - K_2 v$ is a solution to (18) (see [11].)

Hamiltonian systems provide also an important class of semilinear problems with discrete spectrum whose solutions are critical points (see [53].)

4. Equations without resonance

Assuming that $\rho(L)$ contains an interval $[a, b]$ with $a < b$, we can give a condition on N (from (1)) that guarantees the existence and uniqueness of solution to (1). In fact we have the following result.

THEOREM 4.1. *Let H be real Hilbert space, and $N : H \rightarrow H$. Suppose that $a < b$ are real numbers and that $[a, b] \subset \mathbf{R} - \sigma(L)$. For each $c \in (a, b)$, there exist $d \in (0, \infty)$ such that if $\|N(u) - cu - (N(v) - cv)\| \leq d\|u - v\|$ for all $u, v \in H$, then the equation (1) has a unique solution. Moreover if L is selfadjoint then d may be any number in $(0, \min\{c - a, b - c\})$.*

The proof of this theorem follows from the contraction mapping principle. This is because the equation (1) is equivalent to the fixed point equation

$$(20) \quad u = (L - cI)^{-1}(N(u) - cu).$$

When $(L - cI)^{-1}$ is compact, by appealing to the Schauder fixed point theorem (see [26]), we can further relax the hypotheses on N in the previous theorem. For example, the following holds.

THEOREM 4.2. *Let H be real Hilbert space. Suppose that $a < b$ are real numbers and that $[a, b] \subset \mathbf{R} - \sigma(L)$. If there exist $c \in (a, b)$ and $M \in \mathbf{R}$ such that $\|N(u) - cu\| \leq M$ for all $u \in H$, and $(L - cI)^{-1}$ is compact then (1) has a solution. This solution may not be unique (see Theorem 4.5 below).*

From the properties of the properties of the spectrum of $-\Delta$ or the wave operator $\square$ stated in the previous section, it follows that Theorem 4.1 applies to (2) and to (16). In fact the reader is invited to verify the following.

THEOREM 4.3. *Let L be the linear operator defined in (2) or (16). If f is continuous function, and $(\partial f / \partial u)(y, u) \in (a, b)$ for all (y, u) , and $[a, b] \cap \sigma(L) = \emptyset$, then the corresponding problem has a unique solution.*

While Theorem 4.2 applies to (2), it does not apply to (16). In fact, as noted in previous section the inverse of the linear operator defined by $-\Delta + cI$ subject to the boundary conditions in (2) is compact while that corresponding to $\square - cI$, with the boundary condition in (16), is not so. Applying Theorem 4.2 to (2) the following may be established.

THEOREM 4.4. *Let $f(x, u) = cu + h(x, u)$ be continuous with $\lim_{|u| \rightarrow \infty} h(x, u) = 0$, uniformly for $x \in \Omega$. If $c \in (a, b)$, and $[a, b] \cap \sigma(L) = \emptyset$, then the problem (2) has a solution.*

In [22] and [10] sufficient conditions for the existence of solutions to (16) under assumptions related to those Theorem 4.4. See also [58].

Unlike Theorem 4.3, the solution provided in Theorem 4.4 need not be unique. In fact the issue of how many solutions (2) has under the hypotheses even stronger than those in Theorem 4.4 is a matter of many open questions. In fact, let us now assume that $f(x, u) = g(u)$ with

$$(21) \quad g(0) = 0, \quad g'(0) \in (\lambda_k, \lambda_{k+1}), \quad \text{and} \quad \lim_{u \rightarrow +\infty} g'(u), \quad \lim_{u \rightarrow -\infty} g'(u) \in (\lambda_j, \lambda_{j+1}),$$

where the λ_i 's are as in (8) and $\lambda_0 = -\infty$. When Ω is an interval in $\mathbf{R}$, it can be shown that (21) implies that (2) has at least $2|k - j| + 1$ solutions. This, however, does not extend to higher dimensions. In [31], N. Dancer provided an example of where Ω is the unit ball in $\mathbf{R}^N$, $|k - j| = N - 1$ and yet (2) has only three solutions. In [44] it is shown that if $k \neq j$ then (2) has at least two solutions. Also, building on the results of [21], [16], and [13] the following theorem was established in [15].

THEOREM 4.5. *Suppose (21) holds with $k = 1$ and $j \geq 2$, and that $rg'(r) > 0$ for $r \in \mathbf{R} - \{0\}$. If there exists $a < \lambda_{j+1}$ such that $g'(r) \leq a$ for all $r \in \mathbf{R}$, then (2) has at least five solutions. Two of these solutions change sign.*

As stated in section 2, the solutions to

$$(22) \quad \begin{cases} -\Delta u = g(u) & \text{en } \Omega \\ u = 0 & \text{en } \partial\Omega, \end{cases}$$

are given by the critical points of

$$(23) \quad J(u) = \int_{\Omega} \left\{ \frac{1}{2} |\nabla u|^2 - G(u) \right\} dx,$$

where $G(t) = \int_0^t g(s) ds$ and that $u \equiv 0$ is a

(24)

where $\lambda_1 > 0$ is (24) we conclude is 0, and its local

Let $g_+(x) =$ are the solution points of ∇J_+ open set containing Schauder degree solutions is -1 .

∇J restricted to Following the changes sign exactly then the local I the set of critical of ∇J is large by solution that changes

(25)

where 1 accounts Schauder degree of negative solutions Leray-Schauder must have a second of the proof of

If $j = k + 1$ [21]). Also if j exactly two solutions

Thus it remains in terms of $|k - j|$ then number of for the hyperbolic is a rational number are needed due

5. Re

In the hypothesis

$$(26) \quad \frac{\partial f}{\partial u}(x, -c$$

where $G(t) = \int_0^t g(s) ds$. Since, trivially, $u \equiv 0$ is a solution to (22) we see that $u \equiv 0$ is a critical point of J . From (14) we see that

$$(24) \quad \begin{aligned} \langle D^2 J(0)v, v \rangle &= \int_{\Omega} [\|\nabla v(x)\|^2 - g'(0)v^2] dx \\ &\geq (1 - \frac{g'(0)}{\lambda_1}) \int_{\Omega} \|\nabla v(x)\|^2 dx, \end{aligned}$$

where $\lambda_1 > 0$ is as in (8) for the boundary condition $u = 0$ on $\partial\Omega$. From (24) we conclude that $u \equiv 0$ is a local minimum. Moreover, its Morse index is 0, and its *local degree* is +1.

Let $g_+(x) = \max\{g(x), 0\}$. Let J_+ be the functional whose critical points are the solutions to (22) when g is replaced by g_+ . Now the set of critical points of ∇J_+ is bounded and the Leray-Schauder degree of ∇J_+ on any open set containing all its critical points is equal to zero. Hence the Leray-Schauder degree of ∇J restricted to any open set containing all positive solutions is -1. Similar argument shows that the Leray-Schauder degree of ∇J restricted to any open set containing all negative solutions to (22) is -1.

Following the arguments of [16] one sees that (22) has a solution w that changes sign exactly once. Moreover such a solution is an isolated solution, then the local Leray-Schauder degree of ∇J is +1. From (21) one sees that the set of critical points of J is bounded and that the Leray-Schauder degree of ∇J is large balls centered at 0 is $(-1)^j$. Assuming that (22) has one only solution that changes sign we have then that

$$(25) \quad (-1)^j = 1 + (-1) + (-1) + 1,$$

where 1 accounts for the local Leray-Schauder of 0, (-1) for the Leray-Schauder degree of the set of positive solutions to (22), similarly the set of negative solutions to (22), and the last term in (25) accounts for the local Leray-Schauder degree of w . Since (25) is contradictory we see that (22) must have a second solution that changes sign. This concludes a sketch of the proof of Theorem 4.5.

If $j = k + 1$ and $rg''(r) \geq 0$, then (22) has exactly three solutions (see [21]). Also if $j = 2$ then, under the assumptions of Theorem 4.5, (22) has exactly two solutions that change sign (see [15]).

Thus it remains open determining the number of solutions (2) must have in terms of $|k - j|$. It is tempting to conjecture that as $|k - j|$ tends to infinity, then number of solutions must tend to infinity. For a version Theorem 4.5 for the hyperbolic problem (16) the reader is referred to [8]. Even when T is a rational multiple of π additional hypotheses on the monotonicity of f are needed due to the infinite multiplicity of the eigenvalue 0.

5. Resonance and the Landesman-Lazer condition

In the hypotheses of the previous section the terms

$$(26) \quad \frac{\partial f}{\partial u}(x, -\infty) \equiv \lim_{u \rightarrow -\infty} \frac{\partial f}{\partial u}(x, u), \quad \text{and} \quad \frac{\partial f}{\partial u}(x, +\infty) \equiv \lim_{u \rightarrow +\infty} \frac{\partial f}{\partial u}(x, u)$$

stay in an interval contained in $\rho(L)$. Thus we can take advantage of the invertibility of $L - cI$. In contrast, in [38] the authors considered the case

$$(27) \quad f(x, u) = \lambda_k u + g(u) + p(x),$$

with g bounded and monotone. For the sake of simplicity let us assume that g is monotonically increasing. Let $g(-\infty) \equiv \lim_{u \rightarrow -\infty} g(u) < \lim_{u \rightarrow +\infty} g(u) \equiv g(+\infty)$. When (27) holds we say that the nonlinearity is **in resonance** with the eigenvalue λ_k . Surprisingly the following was established in [38].

THEOREM 5.1. *Suppose λ_k is a simple eigenvalue of $-\Delta$ subject to the Dirichlet boundary condition (see (7)). Let ϕ_k be an eigenfunction associated with the eigenvalue λ_k . Let ϕ_+, ϕ_- be the positive and negative parts of ϕ_k , respectively. If*

$$(28) \quad \begin{aligned} g(-\infty) \int_{\Omega} \phi_+(x) dx + g(+\infty) \int_{\Omega} \phi_-(x) dx &< \int_{\Omega} p(x) \phi_k(x) dx \\ &< g(-\infty) \int_{\Omega} \phi_-(x) dx + g(+\infty) \int_{\Omega} \phi_+(x) dx, \end{aligned}$$

then (2) has a solution. Moreover, if $g(-\infty) < g(u) < g(+\infty)$ for all $u \in \mathbf{R}$, then (28) is a necessary condition for the solvability of (2).

Theorem 5.1 not only pioneers the solvability of nonlinear problems with non invertible linear part ($L = -\Delta + \lambda_k I$), but also motivates important developments in variational methods. In fact, the **saddle point principle** proved by P. Rabinowitz in [52] owes its origin to a variational interpretation of the hypothesis (28), also known as Landesman-Lazer condition.

Let us state a simplified version of the saddle point principle. Let X, Y of H such that $H = X \oplus Y$, and $\dim X < \infty$. Suppose that $\lim_{\|x\| \rightarrow \infty, x \in X} I(x) = -\infty$, $\lim_{\|y\| \rightarrow \infty, y \in Y} I(y) = \infty$, and I is bounded below on Y . Let $r > 0$ be so large that

$$(29) \quad \begin{aligned} \max\{I(x); x \in X, \|x\| = r\} &< \inf\{I(y); y \in Y\}, \text{ and} \\ \inf\{I(y); y \in Y, \|y\| = r\} &> \max\{I(x); \|x\| = r, x \in X\}. \end{aligned}$$

THEOREM 5.2. *Let H be a Hilbert space and $I : H \rightarrow \mathbf{R}$ a functional of class C^1 and r be as in (29). If*

(PS) *every sequence $\{u_j\}$ in H such that $\{I(u_j)\}$ is bounded and $\lim_{j \rightarrow \infty} \nabla I(u_j) = 0$ has a converging subsequence,*

then

$$(30) \quad c = \inf_{x \in \Sigma} \max_{\|x\|=r} I(s(x)),$$

is a critical value, where $\Sigma = \{s : \{x \in X; \|x\| \leq r\} \rightarrow H; s \text{ is continuous, and } s(x) = x \text{ for } \|x\| = r\}$.

To derive Theorem 5.1 from the Saddle Point Principle we let X and Y be as in (15). From the first inequality in (15) it is easily seen that

$$(31) \quad J(u) \rightarrow -\infty \text{ as } \|u\| \rightarrow \infty,$$

for $u \in X_1$. Similarly, $J(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$ for $u \in Y$. Thus applying the known as Palais-Smale assumption (PS) implies that J satisfies the reader is referred to [52].

For extending these methods, and to complete the reader's knowledge, we refer to [52].

The term "in resonance" refers to cases where the frequency of the nonlinearity coincides with the frequency of the linear part.

In order to state the **nonlinear resonance condition**, we recall that

$$(32) \quad -\infty < g(-\infty) < g(u) < g(+\infty) < +\infty$$

We recall that $\phi_1(x) > 0$ for $x \in \Omega$ and $G. Prodi$ proved that (32) is a necessary condition for the solvability of (2).

THEOREM 5.3. *for each $q \in L^q(\Omega)$ that*

$$(33) \quad \int_{\Omega} q(x) \phi_1(x) dx < \int_{\Omega} p(x) \phi_1(x) dx$$

has no solutions other than the two solutions $u = 0$ and $u = \phi_1$.

This result is a consequence of the following hypotheses. Hypothesis (H) assume that $u \in X$ by ϕ_1 and integrate by parts

$$(34) \quad \lambda_1 \int_{\Omega} u \phi_1 dx = \int_{\Omega} q(x) \phi_1(x) dx$$

for $u \in X_1 \equiv \text{span}\{\phi_1, \dots, \phi_{k-1}\}$. Here we use strongly that $\lambda_{k-1} < \lambda_k$. Similarly, $J(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$ for $u \in Y$, and J is bounded below on Y . Thus applying Theorem 5.2 only requires checking the hypothesis (PS), known as Palais-Smale condition, and that (31) is satisfied in all of X . The assumption (28) implies that $J(c\phi_k) \rightarrow -\infty$ as $|c| \rightarrow \infty$. This and (31) imply that $J(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$ for $x \in X$. Thus (29) is satisfied. The reader is referred to [52] or [53] for the detail of that part of the proof.

For extensions of the saddle point principle, and other critical point methods, and their applications to the solvability of boundary value problems the reader is referred to [55].

6. Jumping nonlinearities

The term *jumping nonlinearity* was introduced by S. Fucik in [36] to refer to cases where

$$\frac{\partial f}{\partial u}(x, -\infty) \neq \frac{\partial f}{\partial u}(x, +\infty).$$

In order to state the seminal result concerning problems with **jumping nonlinearities** let us assume that $f(x, u) = g(u) + p(x)$ with

$$(32) \quad -\infty < g'(-\infty) \equiv \lim_{u \rightarrow -\infty} g'(u) < \lambda_1 < \lim_{u \rightarrow \infty} g'(u) \equiv g'(+\infty) < \lambda_2.$$

We recall that the eigenvalue λ_1 is simple and that eigenfunctions corresponding to λ_1 do not change sign (see Lemma 2.1). Thus from now on we denote by ϕ_1 an eigenfunction corresponding to λ_1 and we assume that $\phi_1(x) > 0$ for all $x \in \Omega$, and that $\int_{\Omega} (\phi_1(x))^2 dx = 1$. In [3] A. Ambrosetti and G. Prodi proved the following fundamental result.

THEOREM 6.1. *Let g' be a nondecreasing function. If (32) holds, then for each $q \in L^2(\Omega)$ such that $\int_{\Omega} q(x)\phi_1(x)dx = 0$ there exists $C(q) \in \mathbf{R}$ such that*

$$(33) \quad \begin{cases} -\Delta u = g(u) + q(x) - c\phi_1(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has no solution if $c < C(q)$, exactly one solution if $c = C(q)$, and exactly two solutions if $c > C(q)$.

This result motivated intense study of problems with jumping nonlinearities. Hypothesis (32) implies that $g(u) - \lambda_1 u$ is bounded below. Hence if we assume that $u \in H$ is a solution to $-\Delta u = g(u) + q + c\phi_1$, then multiplying by ϕ_1 and integrating on Ω one has

$$(34) \quad \begin{aligned} \lambda_1 \int_{\Omega} u \phi_1 dx &= \int_{\Omega} u (-\Delta \phi_1) dx = \int_{\Omega} (-\Delta u) \phi_1 dx \\ &= \int_{\Omega} (g(u) + q(x) + c\phi_1(x)) \phi_1(x) dx \\ &\geq \int_{\Omega} (\lambda_1 u + M + q(x) - c\phi_1(x)) \phi_1(x) dx, \end{aligned}$$

where M is a lower bound for $g(u) - \lambda_1 u$. Hence only if $c \geq M|\Omega|$ may (33) have a solution. This explains the nonexistence of solutions in Theorem 6.1. That (32) can have at most two solutions is strongly related to the nodal properties of eigenfunctions corresponding to the eigenvalue λ_2 (see Lemma 2.1).

If we replace in (32) the assumption $g'(+\infty) \in (\lambda_1, \lambda_2)$ by $g'(+\infty) \in (\lambda_k, \lambda_{k+1})$, one is tempted to conjecture that for large $c > 0$ the equation (33) has $2k$ solutions. This conjecture turns out to be correct when Ω is one dimensional, and even when Ω is a ball in $\mathbf{R}^N$ and one considers only radial solutions (see [17]) and [28]). E. N. Dancer in [33] provides counter examples to this general conjecture. To date, in general, all we can claim is that if $g'(-\infty) < \lambda_1 < \lambda_k < g'(+\infty) < \lambda_{k+1}$ with $k \geq 2$, then for $c > 0$ the problem (33) has at least four solutions. For early results on jumping nonlinearities the reader is referred to [31], [40], [41], [42], [36], among others. For a survey on problems with jumping nonlinearities and their applications to the design of suspension bridges see [43]. For recent results in problems with jumping nonlinearities see [34], [50], and [56].

In [36], S. Fucik noticed the relevance of understanding the set

$$(35) \quad \{(a, b); \text{ for some } u \in H - \{0\}, -\Delta u = au + bu_-\}$$

in the solvability of (33). This set is known as the Fucik spectrum of $-\Delta$ in Ω . For additional information on the Fucik spectrum see [31] and [56]. Full understanding of the Fucik spectrum remains open. For the one dimensional case, a full characterization of the Fucik spectrum may be found in [36] for the case of Dirichlet boundary condition or periodic solutions.

After thirty years of Theorem 6.1 there remains the conjecture that, for $c > 0$ large, the number of solutions to (33) is large when k is large and $\lambda_k < g'(+\infty) < \lambda_{k+1}$.

7. Superlinear problems

In this section we consider the *superlinear* case. That is when

$$(36) \quad \lim_{|u| \rightarrow \infty} \frac{\partial f}{\partial u}(x, u) = \infty.$$

This occurs, for example, if $f(x, u) \equiv g(u) = |u|^\alpha u$ and $\alpha > 0$. Because the Sobolev space H considered in Section 3 is not always contained in $L^\infty(\Omega)$, the functional J need not be well defined in H . However, since H is imbedded in $L^p(\Omega)$ for any $p \in (1, 2^*)$ with $2^* = (N+2)/(N-2)$ for $N > 2$, and $2^* = \infty$ for $N \leq 2$ (see [1]), for J to be well defined it is sufficient that

$$(37) \quad |f(x, u)| \leq A(|u|^p + 1),$$

for some $p \in (1, 2^* - 1)$. When (37) is satisfied we say that f has *subcritical growth*, or simply that f is *subcritical*. Similarly we will say that f is *supercritical* if there exist $p > 2^*$ and $A > 0$ such that

$$(38) \quad |f(x, u)| \geq A(|u|^p - 1) \quad \text{for all } (x, u) \in \Omega \times \mathbf{R}.$$

Finally we will

(39)

This guarantees that the problem is odd in the sense of Liusternik. To the Liusternik solutions if, in

(40)

This condition is the Palais-Smale condition. The Palais-Smale condition also has infinitely many solutions in the first variable (the variable u). It is shown that (2) these structure solutions may be general f 's problems has and P. L. Lions a perturbation $f(x, u) = |u|^{p-2}u$

$$(41) \quad |h(x)| \leq |H(x)|$$

where $H(x, t)$ if $N \leq 2$), $r = \beta' < 2N/(N - 2)$

THEOREM 1. Let $\alpha \in (0, 1)$ and $t \in (0, 1)$ such that $\alpha t < 1$.

One notes that $|u|^\beta u + p(x)$ for it apply if $h \equiv 0$ some of the cases when f is subcritical

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Finally we will say that f has *critical growth* if

$$(39) \quad \lim_{|u| \rightarrow \infty} \frac{|f(x, u)|}{|u|^{2^*}} \in (0, +\infty).$$

This guarantees that the functional J is of class C^1 (see [53]). When f is odd in the second variable, $f(x, u) = -f(x, -u)$, arguments that go back to the Liusternik-Schnirelmann theory show that (2) has infinitely many solutions if, in addition, there exist $\theta > 2$ and $\tau > 0$ such that

$$(40) \quad uf(x, u) \geq \theta F(x, u) \text{ for } |u| \geq \tau, x \in \Omega.$$

This condition plays a central role in proving that the functional J satisfies the Palais-Smale condition. For f not necessarily odd and $N = 1$, (2) also has infinitely many solutions. If Ω is a ball in $\mathbf{R}^N$, f is radial in the first variable ($f(x, u) = f(y, u)$ for $\|x\| = \|y\|$) but not necessarily odd in the variable u , under additional *structure conditions* on f it may also be shown that (2) has infinitely many solutions (see [18]). For the role of these structure conditions see Theorem 7.1 below. In fact, infinitely many solutions may be found even when f exceeds the growth condition (37). For general f 's proving the existence of infinitely many solutions for superlinear problems has remained elusive. The following theorem, due to A. Bahri and P. L. Lions (see [5]), obtains existence of solutions to (2) when f is a perturbation of $f_1(u) \equiv |u|^{p-1}u$ with $p \in (1, 2^*)$. More precisely, let $f(x, u) = |u|^{p-1}u - h(x, u)$ with

$$(41) \quad \begin{aligned} |h(x, t)| &\leq p(x) + C|t|^q \text{ for some } C \geq 0, p \in L_+^1(\Omega) \\ |H(x, t)| &\leq a(x) + b(x)|t|^\alpha \text{ for some } \alpha \in (0, 2), \end{aligned}$$

where $H(x, t) = \int_0^t h(x, s)ds$, $a \in L_+^1(\Omega)$, $q = (N+2)/(N-2)$ ($q < \infty$ if $N \leq 2$), $r = 2N/(N+2)$ ($r > 1$ if $N \leq 2$), $b \in L_+^\beta(\Omega)$ with $\beta > 1$, $\beta' < 2N/(N-2)(1/\alpha)$ ($\beta > 1$ if $N \leq 2$).

THEOREM 7.1. *Let $f(x, u) = |u|^{p-1}u + h(x, u)$. If $p \in (1, (N+2(1-\alpha))/(N-2))$, then (2) has infinitely many solutions.*

One notes that, for example, if $f(x, u) = |u|^\alpha u + p(x)$ for $u \geq 0$, $f(u) = |u|^\beta u + p(x)$ for $u \leq 0$ and $\alpha \neq \beta$ Theorem 7.1 does not apply. Neither does it apply if $h \equiv 0$ and $p \in (N/(N-2), (N+2)/(N-2))$. These are just some of the cases that remain open in the case of superlinear problems, even when f is subcritical and (40) holds.

8. Critical exponents

In [51], S. Pohozaev noted that if u is a solution to

$$(42) \quad \begin{cases} -\Delta u = g(u) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases}$$

then

$$(43) \quad \frac{1}{2} \int_{\partial\Omega} \|\nabla u\|^2 (x \cdot \nu) ds = N \int_{\Omega} G(u) dx - \frac{N-2}{2} \int_{\Omega} u g(u) dx.$$

In the latter equations ν denotes the outward unit normal, and ds the volume element on $\partial\Omega$. This identity, known as *Pohozaev's identity*, is particularly useful. For example, when Ω is a starlike region with respect to 0 the quantity $x \cdot \nu$ is positive, which forces $2NG(t) - (N-2)tg(t)$ to be nonnegative somewhere on the range of any solution u . Since, for example, $2NG(t) - (N-2)tg(t) < 0$ for $t \neq 0$ when $g(t) = |t|^{p-1}t$ with $p > 2^*$, we see that for such g 's the equation (42) has no nontrivial solution. With the help of the maximum principle one can also see that (42) has no nontrivial solution on starlike regions when $p = 2^*$. This is in contrast with the case $p \in (1, 2^*)$ where the problem has infinitely many solutions.

Despite the negative result of [51], in [9] H. Brezis and L. Nirenberg discovered the existence of solutions to (42) when $g(t) = \lambda t + |u|^{2^*-1}u$, and $N \geq 4$. In fact they proved that if $\lambda \in (0, \lambda_1)$, the problem (42) has a positive solution. These results in turn led for further understanding of the solvability of semilinear equations with critical growth (see (39)). The reader is referred to [7], [19], [20], and [24] for further developments in this respect. It must be noted that most of these results pertain to radially symmetric solutions. For general bounded regions the solvability of (42) is yet to be adequately explored.

9. Qualitative properties of solutions

While the above developments only state the existence, or multiplicity of solutions to semilinear problems, the methods of proof carry information on the nature of the solutions. For example when f is nonnegative, owing to *sub-supersolution methods* one may determine that solutions to (2) are actually positive functions. The reader is referred to [45] for extensive information on this topic. Positive solutions may also exist when $f(x, 0) < 0$ and $f(x, u)$ is positive for $u > 0$ large. Equations of this type are known as *semipositone equations* (see [23]).

The study of solutions that change sign is relatively new. Motivated by the results of [57] the following was established in [14].

THEOREM 9.1. *If $g(0) = 0$, $g'(0) < \lambda_1$, $tg'(t) > g(t)$ for $t \neq 0$, g satisfies (40), and for some $p \in (1, (N+2)/(N-2))$, $A \in (0, \infty)$*

$$(44) \quad |g'(t)| \leq A(|t|^{p-1} + 1) \text{ for all } t \in \mathbf{R},$$

then (42) has a solutions that changes sign exactly once (that is, $\{x \in \Omega; u(x) > 0\}$ is connected and $\{x \in \Omega; u(x) < 0\}$ is connected.)

The existence of the solution in this theorem includes a variational characterization. Such a characterization has been skillfully used by J. M. Neuberger (see [47] and [27]) to approximate such solution. The existence of

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solution that changes sign has also been used to establish the existence of other sign changing solutions (see [16]) and exact number of solutions that change sign (see [15]). For additional results in this direction, the reader is referred to [6] and [24].

While the assumption that g grow subcritically is essential in the proof of Theorem [14], results for regions where g grows at the critical rate have been recently found for regions with some symmetry (see [12] and [37]).

In case g grows subcritically, say satisfying the assumption of Theorem 7.1, and Ω is a ball one may conclude the existence of infinitely many solutions either from the variational arguments of Theorem 7.1 or from one-dimensional arguments as in [18]. In [2] it is shown that infinitely many of the solutions provided by Theorem 7.1 are nonradial. The proof is based on the analysis of the growth rate of *energy levels* of radial solutions that mimics the asymptotic expansion of radial versus nonradial eigenvalues.

Motivated by equations coming from, among others, biological models, extensive work has been devoted to finding nonnegative solutions to the Neumann problem (see [48])

$$(45) \quad \begin{cases} -\epsilon^2 \Delta u = g(u) & x \in \Omega \\ \frac{\partial u}{\partial \eta}(x) = 0 & x \in \partial\Omega. \end{cases}$$

Here a key issue is the number and location of *spikes* of the solutions, and their behavior as ϵ tends to zero.

10. Extensions and Applications

The above results beg for extensions to cases where, for example, L is non-selfadjoint, L is quasilinear, or L is given by a system of linear operators. Little is known when the operator $-\Delta$ is replaced in the above by a second order non-selfadjoint operator. Since most of the results have been derived not from the potential J but rather from the vector field generated by the operator $u \rightarrow -\Delta u - f(x, u)$ on H , new ideas are needed to deal with the nonselfadjoint case. At this point one finds no reason to believe that, for example, if $-\Delta$ is perturbed by a first order operator then the above theorems extend to the new equation. That (5) has no nontrivial solution suggests, for example, that Theorem 7.1 does not extend to nonselfadjoint problems. For extensions to quasilinear problems the reader is referred to [35] and [54] and references therein.

The equation (1) models multiple phenomena. In fact the motivating examples may be derived from physical examples that go back to classical mathematical physics. For more recent applicability of semilinear models, the reader is referred to [25], [43], and [49].

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