1-1-2004

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Recommended Citation

Castro, A., C. Maya and R. Shivaji. "An existence result for a class of sublinear semipositone systems", Dynamics of Continuous, Discrete and Impulsive Systems, Volume 7, Number 4 (2000), pp. 533-540.

Dynamics of Continuous, Discrete and Impulsive Systems **7 (2000)** 533-540

AN EXISTENCE RESULT FOR A CLASS OF SUBLINEAR SEMIPOSITONE SYSTEMS

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Abstract. We consider the existence of positive solutions for the system

$$\begin{aligned} -\Delta u_i &= \lambda[f_i(u_1, u_2, \ldots u_m) - h_i]; \quad \Omega \\ u_i &= 0; \quad \partial\Omega \end{aligned}$$

where $\lambda > 0$ is a parameter, Δ is the Laplacian operator, Ω is a bounded domain in $\mathbf{R}^n$; $n \geq 1$ with a smooth boundary $\partial\Omega$, f_i are C^1 functions satisfying $f_i(0,0,\ldots,0) = 0$, $\lim_{z\to\infty} f_i(z,z,\ldots,z) = \infty$ and $\lim_{z\to\infty} \frac{f_i(z,z,\ldots,z)}{z} = 0$, and h_i are nonnegative continuous functions in Ω for $i = 1,2,\ldots,m$. In this paper for λ large we discuss the existence of a positive solution $u := (u_1, u_2, \ldots, u_m)$. We establish our results by using the method of sub-super solutions.

AMS (MOS) subject classification: 35J55

1. Introduction

In this chapter we consider the existence of positive solutions to the system

$$-\Delta u_i = \lambda[f_i(u_1, u_2, \ldots u_m) - h_i]; \quad \Omega \tag{1.1}$$
$$u_i = 0; \quad \partial\Omega \tag{1.2}$$

where $\lambda > 0$ is a parameter, Δ is the Laplacian operator, Ω is a bounded domain in $\mathbf{R}^n$; $n \geq 1$ with a smooth boundary $\partial\Omega$, h_i are nonnegative continuous functions in Ω and $f_i : \underbrace{[0,\infty) \times [0,\infty) \times \ldots \times [0,\infty)}_{\text{m times}} \to \mathbf{R}$ are C^1 functions for $i = 1,2,\ldots,m$. Further, we assume that for each $i = 1,2,\ldots,m$, we have

$$f_i(0,0,\ldots,0) = 0$$

$$\frac{\partial f_i}{\partial u_j}(z_1, z_2, \ldots, z_m) \geq 0, \quad i \neq j, \quad z_1, z_2, \ldots, z_m \in \mathbf{R}$$

$$\frac{\partial f_i}{\partial u_i}(z, z, \ldots, z) \geq 0, \quad \forall z \in \mathbf{R}$$

$$\sum_{j=1}^{m} \frac{\partial f_i}{\partial u_j}(0,\dots,0) > 0$$

$$\lim_{z\to\infty} \frac{f_i(z,\dots,z)}{z} = 0;$$

and

$$\lim_{z\to\infty} f_i(z,\dots,z) = \infty.$$

In this paper, for λ large, we discuss the existence of a positive solution $(u_1, u_2, \dots, u_m)$. Namely we prove:

Theorem 1.1. *There exists* $\tilde{\lambda} > 0$ *such that for* $\lambda > \tilde{\lambda}$, *the system* (1.1) – (1.2) *has a positive solution* $(u_1, u_2, \dots, u_m)$. *Further,* $\frac{u_i(x)}{dist(x,\partial\Omega)} = O(\lambda)$ *as* $\lambda \to \infty$ *for* $i = 1, 2, \dots, m$.

The fact that the reaction terms in (1.1) – (1.2) could be negative somewhere in Ω (semipositone problems) makes it a harder problem to show the existence of a positive solution. See [13] for an excellent review on positone single equation problems, that is problems with positive and monotone reaction terms. It was remarked in [13] that the case when the reaction term is neagtive near the origin is mathematically challenging and, from the point of view of applications, interesting particularly for positive solutions.

Semipositone problems for the single equation case have been developed during the last ten years (see [1]-[12]). See in particular [4] and [8] where the single equation case of our problem was studied using the method of sub-super solutions. Sub-super solutions are in general hard to apply in the semipositone case since it is hard to construct a nonnegative sub-solution. In fact, in [4] and [8], a non-trivial existence result proved in [11] for a class of semipositone problem with reaction term having "falling zeros", played a crucial role in the construction of the nonnegative sub solution.

However in this paper, we will provide a direct method of constructing sub-super solutions. To our knowledge, semipositone systems have been studied in the past only in [5] and [12]. In [5] the coupling was weak so that one could use existence results from the study of single equations case in the construction of the nonnegative sub solution. In [12] the region considered was restricted to an annulus and radial solutions were established via fixed point methods.

Further, in view of applications, with harvesting in mind, assuming $h_i(x) = 0$ for $x \in \partial\Omega$; $i = 1, 2, \dots, m$, one is interested in solutions which are component wise larger than the harvesting efforts (a constrained semipositone system). In fact from Theorem 1.1 we easily obtain:

Corollary 1.2. *Let the hypothesis of Theorem 1.1 hold and let* $h_i(x) = 0$; $i = 1, 2, \dots, m$ *for* $x \in \partial\Omega$. *Then there exists* $\lambda^* > 0$ *such that for* $\lambda > \lambda^*$, *the system* (1.1) – (1.2) *has a positive solution* $(u_1, u_2, \dots, u_m)$ *with* $u_i(x) \geq h_i(x) \quad \forall x \in \bar{\Omega}$ *and* $i = 1, 2, \dots, m$.

To prove our results we use the method of sub-super solutions (see [14]). To do so, we now define sub and super solutions for systems.
A pair of functions $\bar{u} := (\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_m)$, $\underline{u} := (\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m)$ in $C^2(\bar{\Omega})$ are called ordered super solution and sub solution to

$$\begin{aligned} -\Delta u_i(x) &= \tilde{f}_i(x, u_1, u_2, \ldots, u_m); \quad \Omega \quad i = 1, 2, \ldots, m \qquad (1.3) \\ u_i &= 0; \quad \partial\Omega; \quad i = 1, 2, \ldots, m \qquad (1.4) \end{aligned}$$

if $\overline{u} \geq \underline{u}$ and for $i = 1, 2, \ldots, m$

$$\begin{aligned} -\Delta \overline{u}_i &\geq \tilde{f}_i(x, \overline{u}_1, \overline{u}_2, \ldots, \overline{u}_m); \quad \Omega \\ -\Delta \underline{u}_i &\leq \tilde{f}_i(x, \underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m); \quad \Omega \\ \overline{u}_i &\geq 0 \geq \underline{u}_i; \quad \partial\Omega. \end{aligned}$$

Then the following existence result for cooperative systems holds:

Theorem A (see [14]) *Let $\overline{u}$, $\underline{u}$ be ordered pairs of super and sub solutions of (1.3) – (1.4) respectively. Suppose that $\frac{\partial \tilde{f}_i}{\partial u_j} \geq 0$ for $i \neq j$ on $\bar{\Omega} \times D$ (cooperative system), where*

$$D := \left\{(u_1, u_2, \ldots, u_m) : \underline{u_i} < u_i < \overline{u_i},\ i = 1, 2, \ldots, m\right\}.$$

Then (1.3) – (1.4) has a solution u such that $\underline{u} \leq u \leq \overline{u}$ on Ω.

For simplicity we consider the case $m = 2$ but the proof easily extends for any finite m. We use the method of sub-super solutions to prove Theorem 1.1. We discuss the proof of Theorem 1.1, Corollary 1.2 and an example in section 2. We use Theorem A to prove our result.

2. Proof of main results

As mentioned earlier, we consider the existence of positive solutions to the system

$$\begin{aligned} -\Delta u &= \lambda[f(u, v) - h_1]; \quad \Omega \qquad (2.1) \\ -\Delta v &= \lambda[g(u, v) - h_2]; \quad \Omega \qquad (2.2) \\ u &= 0 = v; \quad \partial\Omega \qquad (2.3) \end{aligned}$$

where h_i are nonnegative continuous functions in Ω for $i = 1, 2$ and f, g are C^1 functions satisfying:

(A1) $\quad f(0, 0) = 0 = g(0, 0)$,

(A2) $\forall z, \tilde{z} \geq 0$

$$(a)\ \frac{\partial f}{\partial u}(z,z) \geq 0, \qquad (b)\ \frac{\partial f}{\partial v}(z,\tilde{z}) \geq 0,$$

$$(c)\ \frac{\partial g}{\partial u}(z,\tilde{z}) \geq 0, \qquad (d)\ \frac{\partial g}{\partial v}(z,z) \geq 0,$$

$$(e)\ \frac{\partial f}{\partial u}(0,0)+\frac{\partial f}{\partial v}(0,0) > 0 \quad \text{and} \quad (f)\ \frac{\partial g}{\partial u}(0,0)+\frac{\partial g}{\partial v}(0,0) > 0,$$

(A3) $\lim_{z\to\infty} \frac{f(z,z)}{z} = 0 = \lim_{z\to\infty} \frac{g(z,z)}{z}$,

and

(A4) $\lim_{z\to\infty} f(z,z) = \infty = \lim_{z\to\infty} g(z,z)$.

Let ψ_i ; $i = 1, 2$ be the unique positive solutions to

$$\begin{aligned} -\Delta\psi_i &= h_i; \quad \Omega \\ \psi_i &= 0; \quad \partial\Omega \end{aligned}$$

and

$$u_1(x) := u(x) + \lambda\psi_1(x), \quad v_1(x) := v(x) + \lambda\psi_2(x).$$

Then (2.1) – (2.3) can be rewritten as

$$-\Delta u_1 = \lambda f(u_1 - \lambda\psi_1, v_1 - \lambda\psi_2); \quad \Omega \tag{2.4}$$

$$-\Delta v_1 = \lambda g(u_1 - \lambda\psi_1, v_1 - \lambda\psi_2); \quad \Omega \tag{2.5}$$

$$u_1 = 0 = v_1; \quad \partial\Omega. \tag{2.6}$$

Then we wish to find a solution (u_1, v_1) of (2.4) – (2.6) such that $u_1 > \lambda\psi_1$ and $v_1 > \lambda\psi_2$ for $x \in \Omega$.

To find a sub solution, define $\underline{u}_1 = \lambda\phi_1$ and $\underline{v}_1 = \lambda\phi_1$, where $\phi_1 > 0$ is an eigen function of $-\Delta$ with Dirichlet boundary conditions corresponding to the first eigenvalue λ_1 such that

$$\frac{1}{2}\phi_1(x) > \max\{\psi_1(x), \psi_2(x)\} \quad \forall x \in \Omega \tag{2.7}$$

(this is possible since by the Maximum principle $\frac{\partial\phi_1}{\partial\nu} < 0$ for $x \in \partial\Omega$ where ν denotes the outward normal). By (2.7) we see that $\underline{u}_1 > \lambda\psi_1$ and $\underline{v}_1 > \lambda\psi_2$ for $x \in \Omega$. We now prove:

Lemma 2.1. *$(\underline{u}_1, \underline{v}_1)$ is a sub solution of (2.4) – (2.6) for λ sufficiently large.*

Proof: Let

$$\overline{\lambda} := \max\left\{ \frac{4\lambda_1}{\frac{\partial f}{\partial u}(0,0) + \frac{\partial f}{\partial v}(0,0)}, \frac{4\lambda_1}{\frac{\partial g}{\partial u}(0,0) + \frac{\partial f}{\partial v}(0,0)} \right\}.$$

Let $H(\lambda, z) := \lambda_1 z - f\left(\frac{\lambda z}{2}, \frac{\lambda z}{2}\right)$ and $\tilde{H}(\lambda, z) := \lambda_1 z - g\left(\frac{\lambda z}{2}, \frac{\lambda z}{2}\right)$ for $z \in [0, r]$ where $r = \|\phi_1\|_\infty$. We will first show that H and $\tilde{H}$ satisfy

$$H(\lambda, z) \leq 0, \quad \tilde{H}(\lambda, z) \leq 0 \quad \forall z \in [0, r] \tag{2.8}$$

for λ sufficiently large. First consider $H(\bar{\lambda}, z)$. Then $H(\overline{\lambda}, 0) = 0$ since $f(0,0) = 0$ and $H_z(\overline{\lambda}, 0) = \lambda_1 - [\frac{\partial f}{\partial u}(0,0) + \frac{\partial f}{\partial v}(0,0)]\frac{\overline{\lambda}}{2} < 0$ by the choice of $\overline{\lambda}$. Hence there exists $\delta_0 > 0$ such that $H(\overline{\lambda}, z) < 0$ for $z \in (0, \delta_0)$. But then by (A2) we have $H(\lambda, z) < 0$ for $z \in (0, \delta_0)$ $\forall \lambda \geq \overline{\lambda}$. If $\delta_0 > r$ then we have established that $H(\lambda, z) \leq 0$ $\forall z \in [0, r]$ when $\lambda \geq \overline{\lambda}$. If $\delta_0 \leq r$ then for any $z \in [\delta_0, r]$ we have

$$\begin{aligned} H(\lambda, z) &= \lambda_1 z - f\left(\frac{\lambda z}{2}, \frac{\lambda z}{2}\right) \\ &\leq \lambda_1 r - f\left(\frac{\lambda \delta_0}{2}, \frac{\lambda \delta_0}{2}\right), \quad \text{by (A2)} \end{aligned}$$

and hence by (A4) there exists $\tilde{\lambda}(\geq \overline{\lambda})$ such that for all $\lambda \geq \tilde{\lambda}$ we have $H(\lambda, z) < 0$ $\forall z \in [\delta_0, r]$. In fact $\forall \lambda \geq \tilde{\lambda}$ it follows that $H(\lambda, z) \leq 0$ $\forall z \in [0, r]$. Similarly one can prove that $\tilde{H}(\lambda, z) \leq 0$ $\forall z \in [0, r]$ where λ is sufficiently large and thus (2.8) is established.

Now assume that λ is sufficiently large so that (2.8) holds. By (2.7), (2.8) and $(A2)$

$$\begin{aligned} -\Delta \underline{u}_1 &= \lambda(-\Delta \phi_1) \\ &= \lambda \lambda_1 \phi_1 \\ &\leq \lambda f\left(\frac{\lambda \phi_1}{2}, \frac{\lambda \phi_1}{2}\right) \\ &\leq \lambda f(\lambda \phi_1 - \lambda \psi_1, \lambda \phi_1 - \lambda \psi_2) \\ &= \lambda f(\underline{u}_1 - \lambda \psi_1, \underline{v}_1 - \lambda \psi_2) \quad \text{for } x \in \Omega, \end{aligned}$$

and similarly

$$-\Delta \underline{v}_1 \leq \lambda g(\underline{u}_1 - \lambda \psi_1, \underline{v}_1 - \lambda \psi_2) \quad \text{for } x \in \Omega.$$

But $\underline{u}_1 = 0 = \underline{v}_1$ for $x \in \partial\Omega$ and hence Lemma 2.1 is proven.
Next to find a super solution, define $\overline{u}_1 = Mw$ and $\overline{v}_1 = Mw$ where w is the unique solution of

$$\begin{aligned} -\Delta w &= 1; \quad \Omega \\ w &= 0; \quad \partial\Omega \end{aligned}$$

and $M = M(\lambda) > 0$ is sufficiently large so that the following hold:

$$Mw \geq \lambda \phi_1, \quad x \in \overline{\Omega} \tag{2.9}$$

and

$$\max\left\{\frac{\lambda f(M\|w\|_\infty, M\|w\|_\infty)}{M}, \frac{\lambda g(M\|w\|_\infty, M\|w\|_\infty)}{M}\right\} \leq 1. \qquad (2.10)$$

Note that (2.9) is possible since by the Maximum principle $\frac{\partial w}{\partial \nu} < 0$ and (2.10) is possible by (A3). Also (2.9) implies that $\overline{u}_1 = Mw > \underline{u}_1 > \lambda\psi_1$ and $\overline{v}_1 = Mw \geq \underline{v}_1 > \lambda\psi_2$. We now prove:

Lemma 2.2. $(\overline{u}_1, \overline{v}_1)$ *is a super solution of* (2.4) – (2.6).

Proof: Using (2.10) and $(A2)$, we have

$$\begin{aligned} -\Delta\overline{u}_1 &= M(-\Delta w) \\ &= M \\ &\geq \lambda f(M\|w\|_\infty, M\|w\|_\infty) \\ &\geq \lambda f(Mw - \lambda\psi_1, Mw - \lambda\psi_2) \\ &= \lambda f(\overline{u}_1 - \lambda\psi_1, \overline{v}_1 - \lambda\psi_2) \quad \text{for } x \in \Omega. \end{aligned}$$

Similarly $-\Delta\overline{v}_1 \geq \lambda g(\overline{u}_1 - \lambda\psi_1, \overline{v}_1 - \lambda\psi_2)$ for $x \in \Omega$. But $\overline{u}_1 = 0 = \overline{v}_1$ for $x \in \partial\Omega$ and hence Lemma 2.2 is proven.

Now combining Lemma 2.1 and Lemma 2.2 for λ sufficiently large, by Theorem A for $m = 2$, (2.4) – (2.6) has a solution (u_1, v_1) such that $\underline{u}_1 \leq u_1 \leq \overline{u}_1$ and $\underline{v}_1 \leq v_1 \leq \overline{v}_1$ in Ω. Here clearly $u_1 > \lambda\psi_1$ (since $\underline{u}_1 \geq \lambda\psi_1$) and $v_1 \geq \lambda\psi_2$ (since $\underline{v}_1 \geq \lambda\psi_2$). Thus (2.1) – (2.3) has a positive solution (u, v) for λ sufficiently large given by

$$\begin{aligned} u &= u_1 - \lambda\psi_1 \\ v &= v_1 - \lambda\psi_2. \end{aligned}$$

Further, since $\phi_1 > 0$ for $x \in \Omega$ and $\frac{\partial\phi_1}{\partial\nu} < 0$ for $x \in \partial\Omega$,

$$\frac{u(x)}{dist(x, \partial\Omega)} = O(\lambda) \text{ and } \frac{v(x)}{dist(x, \partial\Omega)} = O(\lambda) \text{ as } \lambda \to \infty.$$

Hence Theorem 1.1 is proven.

Proof of Corollary 1.2: We need only to show that the solution (u, v) obtained in Theorem 1.1 satisfy $u(x) \geq h_1(x)$ and $v(x) \geq h_2(x)$ for $x \in \Omega$. Now $u = u_1 - \lambda\psi_1 \geq \underline{u}_1 - \lambda\psi_1 = \lambda\phi_1 - \lambda\psi_1 \geq \lambda\frac{\phi_1}{2}$ $\forall x \in \Omega$. Similarly $v \geq \lambda\frac{\phi_1}{2}$ $\forall x \in \Omega$. Thus clearly $u \geq h_1$, $v \geq h_2$ in Ω for λ sufficiently large. This follows from the fact that $\frac{\partial\phi_1}{\partial\nu} < 0$ on $\partial\Omega$ and $h_i(x) = 0$ for $x \in \partial\Omega$, $i = 1, 2$.

Example: Let

$$f(u, v) = (u+1)^{p_1}(v+1)^{q_1} + (u+1)^{r_1} + (v+1)^{s_1} - 3$$

and

$$g(u,v) = (u+1)^{p_2}(v+1)^{q_2} + (u+1)^{r_2} + (v+1)^{s_2} - 3$$

where $0 < p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2 < 1$ satisfy $0 < p_1 + q_1 < 1$ and $0 < p_2 + q_2 < 1$. Clearly f and g satisfy (A1). Now

$$\frac{\partial f}{\partial u}(u,v) = \frac{p_1(v+1)^{q_1}}{(u+1)^{1-p_1}} + \frac{r_1}{(u+1)^{1-r_1}} > 0 \quad \text{for } u,\, v \geq 0$$

and

$$\frac{\partial f}{\partial v}(u,v) = \frac{q_1(u+1)^{p_1}}{(v+1)^{1-q_1}} + \frac{s_1}{(v+1)^{1-s_1}} > 0 \quad \text{for } u,\, v \geq 0.$$

This implies $\frac{\partial f}{\partial u}(u,u) > 0$ for $u \geq 0$ and $\frac{\partial f}{\partial v}(u,v) > 0$ for $u, v \geq 0$ and thus (a), (b) and (e) of (A2) are satisfied. Similarly g satisfies (c), (d) and (f) conditions of (A2). Clearly (A3) and (A4) are satisfied. Thus f and g satisfy the hypotheses of Theorem 1.1 .

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Received January 1999; revised May 2000.