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Maths Living in Social Arenas, From Practice to Foundations

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Synopsis

Maths comes to life in human interaction. This has consequences for the mathematics itself. This paper discusses how this "coming to life" of mathematics in different social arenas influences the foundations of maths. We will argue that this influence is profound, to the extent that it is hard to upkeep the idea that there is or should be one foundation on which all mathematics can be built.

1. Introduction

During the first semester of my university maths education, it used to bother me that different courses used different definitions for the same concept. I had to work with two "different" sine functions, both again different from the definition of the sine I remembered from secondary school. I wanted *the* definition, founded firmly on easy concepts. But I had to accept that, depending on the course or what one wanted to do with it, multiple foundations for mathematical concepts were possible.

Philosophy of mathematics had a similar evolution. Early to mid 20^{th} century philosophy of mathematics was concerned with finding *the* foundation of mathematics. But this turned out to be very hard indeed. However, we will argue, the tendency to look into foundations is a fundamental aspect of mathematics, even that what makes mathematics mathematics. The problem with foundationalist philosophy of maths starts when one does not take the wide variety of mathematical practices and its social organisation into consideration, which has an influence on the maths itself, even on its foundations.

2. From Foundations to Practices

The first half of 20th century philosophy of mathematics was marked by the foundational crisis. From this crisis, entire "schools" ("-isms") developed. The details of these different positions are not relevant for this discussion and can be found in any introduction to philosophy of mathematics. Rather than what their differences are, we are interested in what they share. Since they emerged as programmes to deal with the crisis of mathematical foundations, what they shared, to a certain extent, is that they were all foundational: from metaphysical or ontological positions of what the foundations of mathematics are or should be, the building of mathematics was (re)constructed.

To this day, the matter is not settled. All programmes and positions have their merits, but all fail to account for *all* of mathematics and philosophical problems. As no clear dominant position emerged, the focus shifted "upwards"; from the foundations to the mathematical practices.[1] In recent decades, this evolved into a fully-fledged "practice turn" in the philosophy of mathematics, although maybe less profound, later, and more fragmentary than the practice turn in the philosophy of science. Since then, the social organisation and human aspect of that funny enterprise called mathematics has received more and more attention and recognition. Some topical issues in the philosophy of mathematical practices (PMP for short) are the role of cultural factors, social organisation of mathematics, historical considerations, heuristics in the context of mathematical discovery, the role of mistakes, educational links, explanation and many more.

However, should the two necessarily be in opposition? Can philosophy of mathematical foundations and philosophy of mathematical practices cross-fertilise each other? In a specific way (we will argue), discussion of foundations is unavoidable, and the reason for this is to be found in the characteristics of mathematics itself. We have to deal with foundations.

Here we enter of course the realm of the "classical" problems of "traditional" philosophy of mathematics, and risk to get bogged down in these "old" discussions. Hence, an account of PMP should be compatible with different metaphysical positions one might hold. To put it crudely: an answer to the question about whether or not vector spaces existed when dinosaurs roamed

[1] Or we should say the focus *expanded*. But in contemporary foundational debates too, elements of mathematical practice are taken more and more into consideration.

the earth is a matter of extreme indifference *to me*. But whether one answers "yes", "no", or "that question has no meaning to me", it should in theory be compatible with an account of the core of PMP.

What all mathematical practices share, independently of philosophical accounts of the deeper ontological status of their concepts, is the fact their mathematics comes to life in some kind of human or social interaction. This is the aspect we are interested in: a living mathematics, as shared, presented, discussed, taught, thought, communicated etc. We will call the societal spaces in which this interaction brings mathematics to live *social arenas*. We choose the term exactly because its connotation of action and debate.[2] It might be a difficult operation to *exactly* identify *specific* arenas, but a relatively easy criterion could be: there has to exist a *label* for it, i.e. a name or even a certain official or institutional character: a classroom, an undergraduate algebra course, a PhD defence, a mathematics conference, a university research group, a TV documentary, the office of a mathematician[3], an academic journal's readership, etc. These arenas are of course linked via individual mathematicians or groups of mathematicians, who may be active in different arenas. Note that the "same" piece of mathematics will potentially live quite differently in different arenas.

Up to this point, the proposed analysis seems trivial. Yes, mathematics is done and communicated differently in different ranks of society. If there is "a mathematics" projected differently onto the different arenas where it comes to live, this indeed does not add much. But here the fact that we take the practices seriously kicks in. As we will argue, the influence of the social arenas onto the mathematics itself reaches deeper than this, all the way to the foundations. To such an extent that it is hard to uphold that there is or should be one foundation that speaks for all of mathematical practices.

[2]The Online Oxford Dictionaries lists as a second definition for arena: "A place or scene of activity, debate, or conflict".

[3]The solitary "singleton mathematician" proving in her office fits this bill as well. She is a little society of her own, embedded in different social arenas (research groups, courses she teaches, university culture, society at large...). Also, there is a form of self-communication through internal monologue or scribbling. In [4], an insight of C.J. Keyser is cited: "As early as 1905, Keyser recognized that concepts and proofs in mathematics are essentially social affairs: "They must be intelligible to at least two minds, *or, what is tantamount, to one person at least twice*"." (our emphasis) So even if we were to isolate the singleton mathematician, we regard the way in which she "explains the mathematics to herself" as a form of communication.

We will present two case studies that show that the requirements of different social arenas have a severe impact on what one considers to be the foundation of a mathematical concept.

3. Case 1: The Sine Function

What is the sine function? It turns out this is a more difficult question than one might suspect. Undoubtedly, a periodic sine wave picture leaps to mind, but how to "properly" define this concept? It is remarkable that there are so many versions of this quite essential mathematical notion. Here is a non-exhaustive list of four commonly found definitions.

- the geometric definition, namely as the generalisation of the sine in a right-angled triangle: $\sin x :=$ the ordinate of the intersection point between the unit circle and the half-line through the origin, corresponding to an arclength x (which is equal to an angle of x radians);
- a (Taylor) series: $\sin x := \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$;
- the 2π-periodic continuation of the inverse of the arcsin function, itself defined as an integral of an algebaic expression;
- the complex definition: [5] shows that, once the exponential function $\exp x$ has been defined, one can define $\sin x := \frac{1}{2i} \exp(ix) - \exp(-ix)$.

The first definition is the approach one is most likely to meet in secondary school education. The reason is obvious: the function can be easily linked to the right-angled triangle and goniometric circle, which is needed to show interesting properties (extrema, zeroes, derivative etc.). The link with the graph of a function (and hence geometric interpretation) is essential in this arena. Another characteristic is the fact that this approach neatly follows the school curriculum[4]: the sine in a right-angled trangle is introduced in the third year of secondary school, the generalisation to the goniometric circle in the fourth, functions and the sine-seen-as-function in the fifth.

The second approach is arguably the most "economical", as only the concept of infinite series is required. However, geometric interpretation and deriving the desirable properties become tedious. If, however, there is a link with

[4]In the Dutch-speaking Belgian school system.

computational side of things[5], this becomes an interesting option, as it puts forward the approximation method as fundamental. In other approaches, the series is a property rather than a definition, as an instance of Taylor's theorem.

The third and fourth definitions start from heavy (respectively real and complex) analysis. The third defines $\arcsin x = \int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^2}}$ with $(|t| < 1)$ and consequently defines the sine function as the periodic continuation of its inverse. The integral, inverses and all properties can then be derived through the familiar $\epsilon - \delta$ definitions, cementing this approach in the topology of $\mathbb{R}$. The forth is similar, but using machinery from complex analysis. These approaches might be considered the most "rigorous", as "it should be stressed that we derived the basic properties of the trigonometric functions [...] without any appeal to the geometric notion of angle." ([5]) Interestingly enough, after having defined the sine and cosine, [5] continues to argue that "[his definitions] coincide with the functions $\cos x$ and $\sin x$, whose definition is usually based on geometric considerations." So to show that his definition is adequate for the sine function, he shows that the rigorous definition gives rise to all the properties we expect of a sine based on geometric reasoning.But to derive these properties of the "geometric" sine, a certain amount of trust needs to be put into the geometric considerations we started from. This seems circular, but an instance of a "virtuous circle": provide more solid foundations for a concept we already know.

Of course, it is possible to transform one representation in the other. What is a definition in one arena will be a property in the other and vice versa. Payoffs need to be weighed against another. Yet, the situation is not symmetrical. By favouring one definition over the other, one commits to what one deems to be more essential as a basis for the concept. The choice depends on the requirements of the social arena in which the sine function will be used.

So what *is* the sine function? It seems that this question is impossible to answer. And this very fact isn't a problem, rather on the contrary. It is remarkable and fruitful that these motley crew of practices has so many approaches to a concept so familiar as the sine function. Note that this observation is compatible with most "classical" metaphysical positions in philosophy of mathematics. The social constructivist is obviously happy, the constructivist has (at least one) constructively acceptable version, there is an embedding in $\mathbb{R}$ and (hence) set-theoretic structures, the Platonist may claim all are projections of the ideal entity of the sine function.

[5] As e.g. in Bishop and Bridges's constructivist analysis, see [1]

There is, however, one position that proves to be untenable in view of these observations. Namely the view that there should be but one foundation of mathematics, whatever metaphysical stance one might have about this foundation, because we saw that the foundation of the mathematical concept is relative to the requirements of the social arena in which the concept is used.

4. Case 2: The Line

What is a (straight) line? This is another very familiar mathematical notion that turns out to be hard to define. Again, we will present a few approaches.

A first is the very pragmatic approach to accept that a line is a line and simply get over it. We have a basic understanding of what "straightness" is or should be. Be it from encounters with it in nature (a lightbeam or the horizon), a conceit of civilisation (a ruler or train track) or our innate human proneness to regularity, uniformity, harmony and symmetry. In a lot of arenas, it is therefore not problematic to simply live with a loosely defined concept of "straight line", founded on this acknowledged notion of straightness.

But even in arenas where formalisation and abstraction is imperative, such as axiomatic geometry, one of the easiest approaches is to accept a line as a line an no further. But in these arenas, the aim is different: lines are treated as a *primitive concept*, tamed through axioms that stipulate how lines (should) behave, hence making the connection with our intuitions of what an abstracted straight line is. The concept "line" is not further analysable, and determined by how it interacts with other notions, such as points. For example, Hilbert in [3] defines a line as an object completely determined by two points, indefinitely extendible etc.

Another possibility is to define a line in a coordinate grid system as an object satisfying a linear equation of the form $ax + by + c = 0$. Here, geometric interpretation is secondary, but the line-concept is founded on the real-number concept, which easily harvests the powers of analytic geometry.

Here again, very different approaches were possible. Be it pragmatic, axiomatic or analytic, the angle of attack reveals what is deemed to be a (part of the) foundation for mathematics, within a specific social arena.

5. Practices as Foundations

We have discussed how the foundations of mathematical concepts differ in various social arenas. It is therefore dangerous to put forward general, unifying principles that serve as a banner to put all of mathematics under. As soon as one hears claims along the lines of "a fundamental characteristic of all of mathematics is [...]", one will be able to come up with something that is clearly to be considered as a piece of mathematics, but which doesn't satisfy the characteristic. However, we will here boldly propose exactly such a characteristic, maybe tempering the claim to apply to "most of mathematics". It is a more delicate version of the classical claim that the notion of *proof* lies at the heart of mathematics.

Claim: Mathematics connects new concepts to previously defined and studied concepts.

The latter are deemed to be the deeper, more intuitive, more easily graspable, more fundamental concepts from which to start exploring the to-be-defined concepts. This leads to a practice of connecting mathematical entities to another. As concepts are built on one another, the natural question arises of what the deepest concepts or assumptions are on which we are building. Therefore there is, in all mathematical practices, a fundamental foundationalist tendency present. But what these foundations are, their solid basis deemed unproblematic, intuitive, given, may be very different in the different social arenas.

The foundationalist tendency is *mathematical* and only turns philosophically problematic if one considers the foundational requirements of one social arena to be *the* foundations for all arenas. Hence, the problem with foundationalist philosophy of mathematics is not the foundationalism *per se*, but rather the fact they take the foundations of pure mathematics (in-principle-traceability-to-axioms-or-set-theory, formal proof requirement, preference for rigour over intuition etc.[6]) to speak for all social arenas in which maths is performed.

This has consequences for the notion of proof. Traditionally, the maxim goes that a piece of mathematics is in principle reducible to a set theoretic definition and proofs work formally from a small set of axioms. As we have seen, this

[6] It is not claimed that e.g. intuition or informal proof are not part of this arena. Here, however, we discuss *foundational requirements*.

cannot account for the different proof concepts we encountered in different arenas. Even within academic mathematics, this picture does not hold, as Leitgeb in [2, page 271] remarks:

> "[...] group theorists, topologists, and probability theorists usually do not even know these set-theoretic axioms — the set theory they use is most likely a version of naive set theory — and it is not conceptually essential to their proofs that they rely on an axiomatic system of set theory."

6. Conclusion

In this paper, we discussed two examples that show how requirements of different social arenas are reflected in the definition of mathematical concepts. This reveals how different arenas have different ideas about what is to be considered as foundational concepts. We have argued that looking into foundations is an essential aspect of all mathematical endeavours, however, seen the diversity in foundational requirements, one cannot expect the foundations of maths in one social arena to speak for all of mathematics. The fact that mathematics "lives" in different arenas has profound consequences, all the way down to its foundations.

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References

[1] E. Bishop and D. Bridges, *Constructive Analysis*, Springer-Verlag, Berlin, 1985.

[2] O. Bueno and Ø. Linnebo, *New Waves in Philosophy of Mathematics*, Palgrave Macmillan, Basingstoke, 2009.

[3] D. Hilbert, *The Foundations of Geometry*, The Open Court Publishing Company, La Salle, IL, 1950.

[4] S. Restivo, *Mathematics in Society and History - Sociological Inquiries*, volume 20 of *Episteme*. Kluwer Academic Publishers, Dordrecht, 1992.

[5] W. Rudin, *Principles of Mathematical Analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, 3rd edition, New York, 1976.