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Ducci's Four-Number Game: Making Sense of a Classic Problem Using Mobile Simulation

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Synopsis

Ducci's Four-Number Game is a classic mathematical puzzle appealing to a wide audience for its procedural simplicity, mathematical richness, and aesthetic values. This article first describes a few activities appropriate for school students and mathematics teachers to make sense of the intriguing behavior of the game. Then, using a mobile simulation, we delve into the lengths of the Four-Number Game and the corresponding probability distribution. The Ducci number game is playful, engaging, and full of mathematical surprises.

Keywords: *Ducci, four-number game, mobile simulation*

1. Introduction

Games and puzzles are key themes in recreational mathematics, blending problem solving with various humanistic dimensions of mathematics, including history, culture, and aesthetics. While games and puzzles are frequently used in extracurricular settings such as homes, math clubs, or science centers, they have also been recognized as pedagogically powerful for stimulating mathematical reasoning and cultivating a healthy mindset among children in formal schooling [3]. In a playful setting, games and puzzles invite students to take intellectual excursions into the world of mathematical thinking, where they create their own ideas, change directions, make unexpected findings, feel their emotional ups and downs, and, more often than not, pose new problems [20]. For the curious minds, there are a host of resources that are accessible to students at all levels (e.g., [9]).

I have personally enjoyed all kinds of mathematical games and puzzles. Meanwhile, I have tried to bring a few of them to my mathematics classes for young children and K-12 preservice and inservice mathematics teachers. One day, when I was talking about puzzles and early mathematics education, a preservice teacher shared with the class a number game that she really likes — the Four-Number Game. She remembered the structure of the game and was able to illustrate it with an example. She said that she learned the number game from a former mathematics teacher and enjoyed the problem and its surprising results. I have since read more about the Four-Number Game, including its history, its variants, and its numerous connections to elementary and advanced mathematics. Along my own recreational journey with the problem, I have used it with some young children and mathematics teachers, wondering all the time how much more mathematics I could get out of this beautiful number game in a K–12 setting. In this article, I share a few activities appropriate for K–12 students and math teachers, taking advantage of the new technologies such as concept mapping and mobile programming.

2. Ducci's Four-Number Game

The Four-Number Game, also known as the Four-Number Problem or Four-Number Puzzle, was first defined and studied by E. Ducci of Italy in the 1880s [7] and has since been extensively reviewed and investigated in recreational and professional mathematical circles from 1940s to 2010s [1, 5, 8, 10, 11, 12, 14, 21, 22, 23]. The number game can also be played over real numbers. For our present purposes, however, we will focus on nonnegative integers.

To begin with, we pick four numbers for a quadruple $v_0 = (x_1, x_2, x_3, x_4)$, and subsequently calculate the distances between adjacent numbers to obtain the next quadruple $v_1 = (|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_4|, |x_4 - x_1|)$, where the fourth number of the original quadruple is compared with the first. The process is iterated on v_1 and subsequent quadruples until a pattern emerges and becomes stable. The resulting sequence of quadruples is often called a *Ducci sequence*.

A formal definition, if necessary, goes as follows: A *Ducci four-number sequence over nonnegative integers* is a mapping $T : \mathbb{Z}^4 \to \mathbb{Z}^4$ such that for $v = (x_1, x_2, x_3, x_4)$,

$$T(v) = (|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_4|, |x_4 - x_1|).$$

Thus, $T^n(v)$, with $n \geq 0$, gives the nth iteration of the Ducci mapping. Interestingly, given any specific quadruple v over nonnegative integers, the Ducci mapping will eventually arrive at $T^m(v) = (0, 0, 0, 0)$, where the sequence becomes stable; such an m, upon its first occurrence, is called the *length* or *order* of the quadruple.

Ducci's Four-Number Game has been, over a century, discovered and rediscovered a few times [22] and revisited often for its educational and recreational values [1, 2, 16], demonstrating its mathematical and pedagogical appeal among professional mathematicians and mathematics educators. Indeed, the Four-Number Game is quite accessible and enjoyable even for primary school children using basic whole number arithmetic. It can also be vertically adapted to provide an engaging mathematical journey for secondary and college students of mathematics through abstract algebra, vector space, and probability theories [14]. In the 1980s, the Four-Number Game was introduced to K–12 mathematics teachers by professor Janita Copley of the University of Houston [2] and has since spread far and wide among mathematics educators, including myself.

For a preliminary discussion or for use in elementary grades, the Four-Number Game can be played on a geometric scaffold, which is usually a square but can also be a rectangle or even a circle [10]. If a square is used, the initial numbers are placed at its vertices, and the subsequent numbers (i.e., distances between adjacent numbers) are placed at the midpoints of each side for a new square which leads to yet another iteration. Figure 1 shows the Ducci sequence with $v_0 = (40, 25, 33, 52)$, which converges to $(0, 0, 0, 0)$ after four steps, hence $Len(v_0) = 4$. The squares are sometimes called *difference boxes* or *diffy boxes* [2]. When used with young children, colored markers can be used at each iteration to keep track of the progress.

With its simplicity, beauty, and mathematical richness, the Four-Number Game is truly one of the classic mathematical curiosities, deserving a place on the list of mathematical treasures both at school and outside the classroom. However, beyond a basic introduction, how far can a K–12 student or mathematics teacher travel on the Ducci Map?

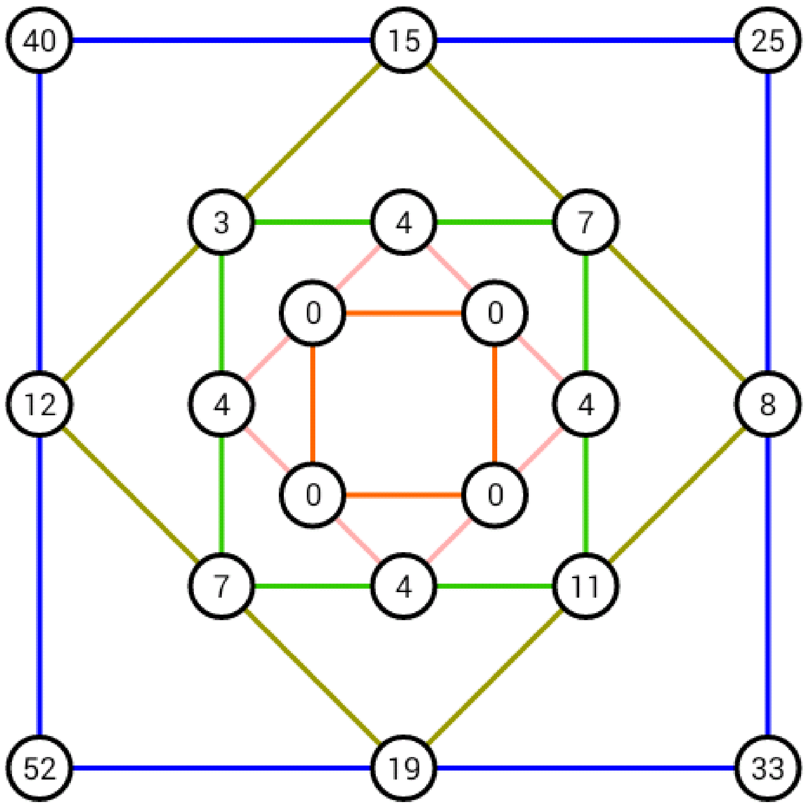


Figure 1: A Ducci sequence of squares starting with $v_0 = (40, 25, 33, 52)$, which has a length of 4.

3. Initial Explorations Using Concrete Numbers

For young children or for a preliminary exploration, Ducci's Four-Number Game can be played on integers between 0 and 100 (or any other grade-appropriate upper bound) in order for students to understand the structure and the rules of the game.

First, in small groups or individually, students can pick four random numbers between 0 and 100, and proceed with the game until they reach a stable pattern or the all-zero state. Pedagogically, it will be more motivating for students to discover the end state of the game while they compare their numbers with those of their peers. Students should be encouraged to try more quadruples, using either random numbers or numbers of their own choice. A few questions will naturally emerge regarding the length of the game and the conditions for longer games.

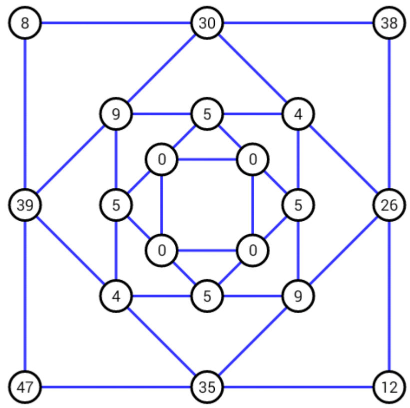
Second, as students play the Four-Number Game, they can record the length of their numbers on a whole-class bar graph for a big picture of game lengths. In a class of 25 students, if everyone plays the game four times, there will be 100 instances, whose distribution is interesting and invites conjectures or more experiments.

Third, when the square box is used, it is tempting to prompt students to ask a few *what-if* questions [6] in terms of geometric transformations. A square has both rotational and reflectional symmetry, which allows for further problem posing. What if I rotate the square 90, 180, or 270 degrees? What if I flip the square about a diagonal of the square or horizontally or vertically? These operations all have implications at a higher level [4]. For young students, however, it is surprisingly appealing to see how such operations have no effect on the overall structure of the game. If necessary, some physical square cards can be used for students to act on the four numbers around the square. The consequence of such reflections on the subsequent iterations is interesting, as shown in Figure 2. Surprisingly, such reflections as well as the obvious cases of rotation have no effect on the length of a given quadruple.

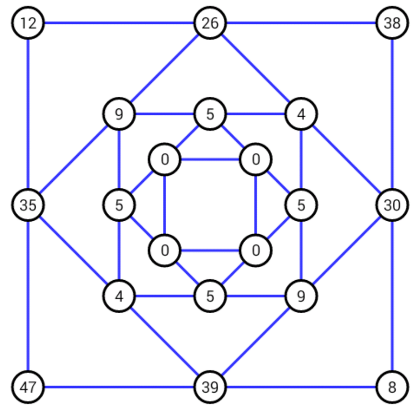
These activities are within the reach of elementary students, allowing them to experience the playfulness and structural appeal of mathematics. Certainly, there is no need to mention dihedral groups at this level, which are otherwise worthwhile at higher levels.

4. Exploring the Even-Odd Dynamics in a Quadruple

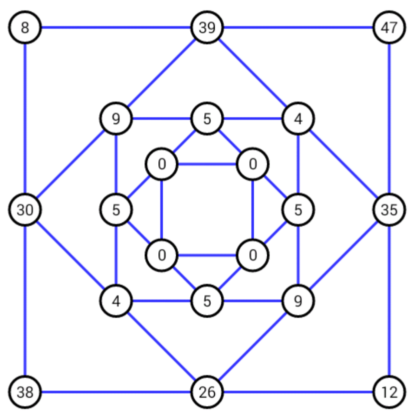
Playing Ducci's Four-Number Game with specific whole numbers is no doubt an engaging activity for young children. In addition to whole-number subtraction, they have rich opportunities to think about the numeric patterns and the surprisingly simple result. In the upper grades, students can explore the same patterns using large numbers, for example, between 0 and 1000, using calculators if necessary. They will be surprised at the fact that large numbers do not quite affect the length of a quadruple as they might have initially expected. One of the properties of the Four-Number Game is



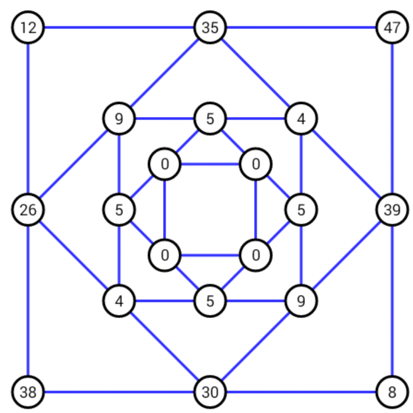
(a) Original quadruple.



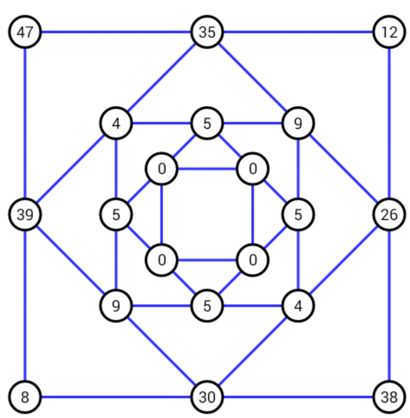
(b) Reflected about the 38-47 diagonal.



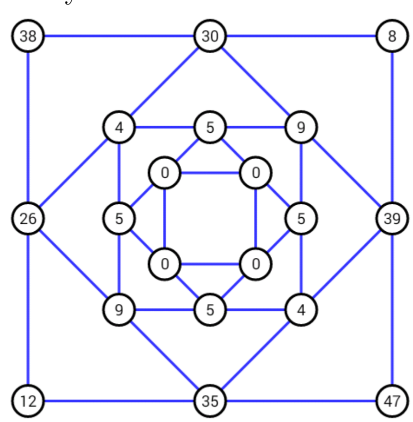
(c) Reflected about the 8-12 diagonal.



(d) Reflected about both diagonals or both horizontally and vertically.



(e) Reflected vertically.



(f) Reflected horizontally.

Figure 2: Exploring the effects of geometric reflections on the Ducci sequence with $v_0 = (8, 38, 12, 47)$.

appropriate and accessible for middle and secondary students — the parity (evenness or oddness) of the four numbers along the iterations. If students have not noticed the even or odd numbers in their previous activities, they may be guided to attend to the even or odd numbers in a Ducci sequence. Do all the four numbers become even after a certain number of iterations? What may be the largest number of steps for all the four numbers to become even? And why?

The discussion about even and odd numbers can be launched with specific numbers. Then, a more abstract approach could be used for students to develop insight into the problem dynamics. Paper-based exploration is adequate, but technology-enabled concept mapping allows for flexible manipulations. Before setting out to explore the even-odd properties of the four numbers, students could be presented with the following questions:

1. If two numbers are both even, is the difference between the two numbers even or odd?
2. If two numbers are both odd, is the difference between the two numbers even or odd?
3. If one number is even and the other is odd, is the difference between the two numbers even or odd?
4. Given four numbers in the game, in how many different ways can they be even or odd?

The first three questions are simple but are interesting for students in the middle grades to think over. A formal proof is reasonable in this case and could be sought, if necessary, using the definitions of even and odd numbers. An integer is even if it can be written as $2k$ for some integer k. An integer is odd if it can be written as $2m+1$ for some integer m. From these definitions it is possible to show that the distance between two even numbers or two odd numbers is even, and the distance between an odd number and an even number is odd. These facts, once established, allow for the even-odd tracking of the four numbers along the iterative process.

The fourth question is equally interesting. Given a quadruple of four numbers, each number can be either even or odd. There are altogether $2^4 = 16$ cases. Some students may intuitively think that there are 8 cases, which can be readily addressed when they are asked to list all the different cases. Indeed, some of the 16 cases are related or, rather, equivalent to others [23].

However, for school students, they can just start with all 16 cases and explore their connections. Pedagogically, all the 16 cases can be recorded as nodes in a concept map on paper or on the computer screen. Then, each of the cases can be assigned to an individual student or a small group, who will figure out the even-odd pattern in the next iteration and relate it to an existing node. Incidentally, the class could discuss why there is always an existing case that matches their next-step pattern. Through such systematic analysis, students will be able to create a map that looks like the one in Figure 3.

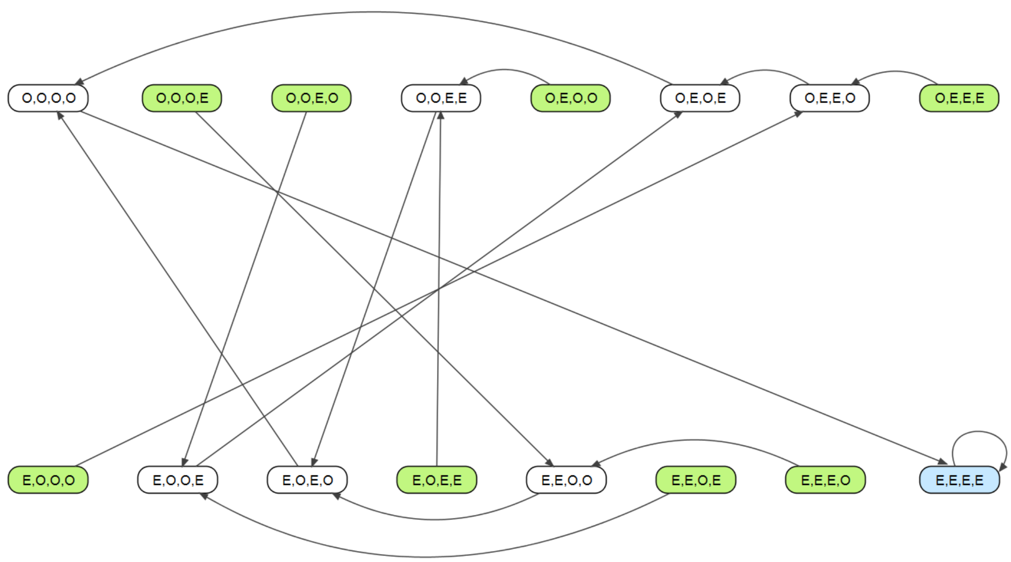


Figure 3: The even-odd map of the Ducci Four-Number sequence.

Starting from any possible even-odd state, we can follow the arrow directions and arrive at the final stable state, (E, E, E, E), after at most 4 steps. To simplify the map in Figure 3, we can regroup the nodes according to lengths and rotational symmetry, which is easy to accomplish if the initial map is created using some concept mapping software.

Figure 4 shows the regrouped map with six distinct clusters. There are eight cases that take four steps to arrive at the end case of all even numbers. Along the way, they travel through three other states. It is interesting, for example, that a quadruple, if not all odd, will become all odd before becoming all even. Also, if a quadruple has three odd or three even numbers, it will become two consecutive even and two consecutive odd numbers, then alternating even, odd, even, odd, and then all odd, before arriving at all even. The process is like a metamorphosis, once recognized, appealing to the curious mind!

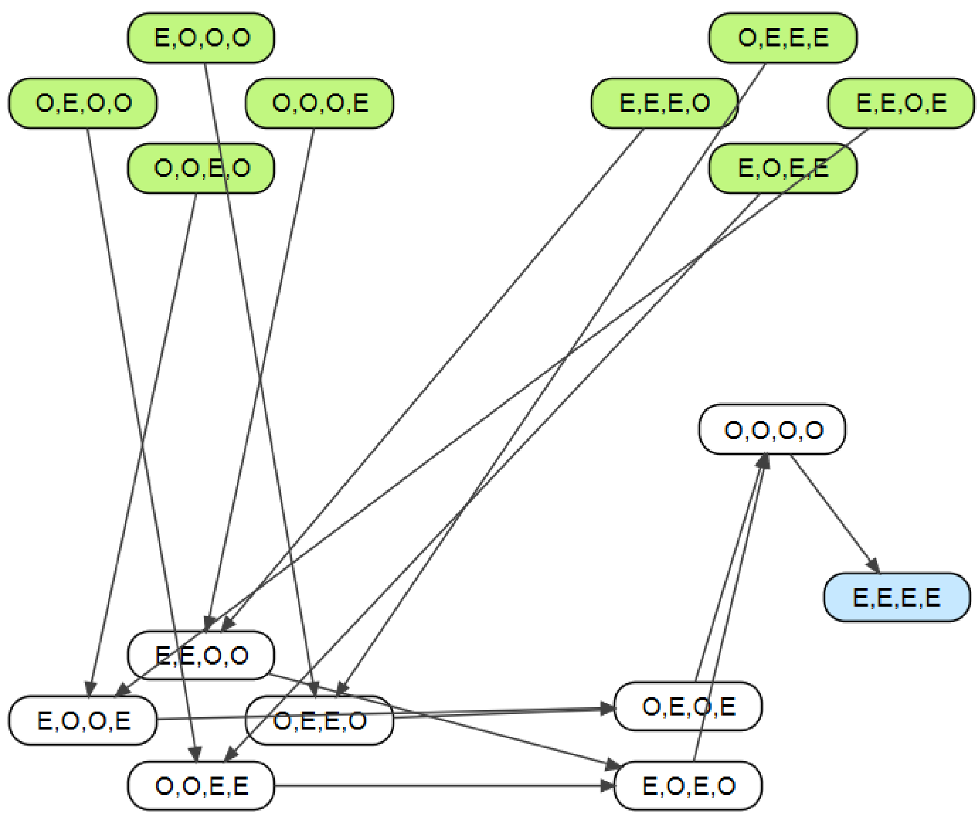


Figure 4: A clustered even-odd map that identifies six equivalent classes among the even-odd states.

The odd-even observation further serves as the foundation for proving that the Four-Number Game on integers converges to the case of all zeros after a certain (finite) number of iterations. If we start with $v_0 = (x_1, x_2, x_3, x_4)$, then, after at most 4 steps, all the four numbers will become even, at which point we could factor 2 out and continue with the process. Then, after at most 4 more steps, we have another quadruple of all even numbers, from which another 2 can be factored out. The process goes on. Since x_1, x_2, x_3, x_4 are given to be positive integers, the process cannot go on indefinitely without converging to all zeros.

5. Simulation for Further Ducci Explorations

The Ducci Four-Number Game has far more treasures to be discovered than what is presented above, most of which, however, require an understanding of advanced mathematics beyond the scope of K–12 mathematics education. To fill the gap, I turned to mobile learning technologies and built an Android app for the Ducci game using the MIT AI2 ® platform (http://ai2.appinventor.mit.edu). A similar simulator can certainly be built for the computer and other platforms. The goal is to provide accessible opportunities for K–12 students and educators to explore the space of the Ducci game and observe patterns in the mathematical behavior of the game such as lengths and probability distribution.

5.1. The Mobile Four-Number Game App

The mobile Ducci app and its source code are both freely available at the Google Play® store and the MIT AI2 ® gallery, respectively.[1] The app has two main modules: a Four-Number Game that takes user inputs and a probability explorer that simulates the probability distribution. The Four-Number Game allows users to ask numerous *what-if* questions, using their own chosen numbers or randomly-generated numbers. It is clearly more efficient in terms of calculation and aesthetically inviting in light of the geometric structure and colorful presentation (see Appendix A). The probability simulator (see Appendix B) allows users to change the range of the four numbers to explore the effect of the number range (or the lack thereof) on the probability distribution of quadruple lengths. Obviously, simulations are not mathematical proofs. They, however, serve as cognitive bridges that help students and teachers get to appreciate the beauty of the problem and the rich mathematics underneath. Perhaps they will continue to think over the problem as they proceed to advanced mathematics.

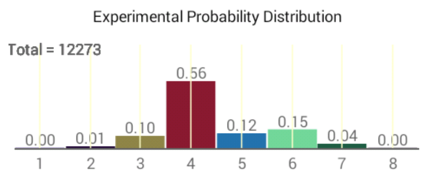
[1]The Google Play web site for the game is https://play.google.com/store/apps/details?id=appinventor.ai_moodle_bu.FourNumberProblem, last accessed on July 10, 2019.

5.2. Probability Distribution

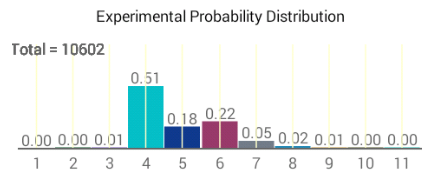
The length or order of a quadruple is among the most intriguing properties of the Four-Number Game. Through examples, students will be able to conjecture that most of the quadruples have a length between 4 and 8 if the game is played on integers between 0 and 100. One wonders naturally how to find quadruples that have a longer length. In fact, over the domain of integers, it has been proven that it is possible to construct a quadruple of an arbitrary length [8, 14]. Although some upper bounds have been established [8], there does not seem to be a formula for the maximal length of a Four-Number Game played, for example, on integers between 0 and 100. Sally and Sally [14] have a comprehensive discussion of the probabilities behind the Four-Number Game on real numbers, based on the previous work of Ullman [21]. While the exposition is fascinating, [21] is hardly accessible to K–12 students and mathematics educators. Nonetheless, the probability distribution of the quadruple lengths is tantalizing. We may for instance wonder how likely it is to find a random quadruple of length 12, say, if the game is played on integers between 0 and 100. The mobile app allows users to simulate the Four-Number Game on integers and establish an experimental probability distribution, which helps make sense of the rareness of quadruples of length 9 or more.

Figure 5 summarizes the experimental probability distributions of quadruple lengths over integers from 0 to 10, 100, 1000, and $10,000$, respectively. The following are a few observations that might make worthwhile discussion topics for school students and mathematics teachers.

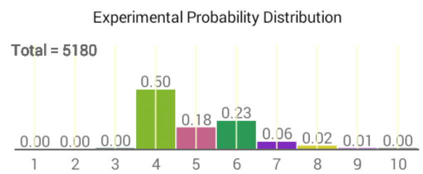
As shown in Figure 5, the integer range or the size of the numbers does not have much impact on the overall probability distribution of quadruple lengths. Regardless of the number sizes, there is about a 50% chance that a random quadruple has length 4, for example. Indeed, it has been established that a random quadruple over real numbers has a 50% chance of having length 4, independent of the number sizes [14, 21]. If one picks a random quadruple, there is a 99% probability that it has a length less than 9, independent of the number sizes. It is worth noting that the distributions in Figure 5 all have a fixed integer range and therefore do not exactly reflect the probability distribution of quadruple lengths over real numbers, although they are very close because of the nature of the problem situation.



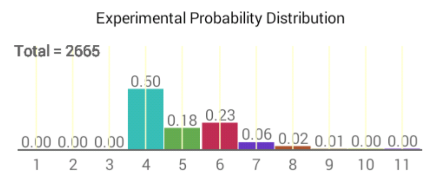
(a) Over integers between 0 and 10.



(b) Over integers between 0 and 100.



(c) Over integers between 0 and 1000.



(d) Over integers between 0 and 10,000.

Figure 5: Experimental probability distributions of Ducci lengths for quadruples on various ranges.

Further, in the case of small ranges, it is possible to conduct an exhaustive enumeration of all the cases and calculate the exact probability of a certain length. Over integers between 0 and 10, for example, there are a total of $14,641$ cases, among which 8320 have length 4. Therefore, the probability of getting a quadruple of length 4 is $8320/14641$, or about 56.8267%.

However, quadruples with longer lengths do occur in the case of large numbers, albeit at an extremely low probability. For example, the quadruple $(0, 653, 1854, 4063)$ has a length of 23, excluding the initial step [24]. Table 1 shows a list of quadruples that have lengths longer than or equal to 14, found by the mobile app. Of course, their rotations or reflections, with respect to the square, all have the same length. From a pedagogical perspective, it is a rewarding and exciting experience for students to discover quadruples of long lengths and share with each other, aside from the frustration of a long wait.

5.3. Properties of Quadruples with Long Lengths

Experimental probability distributions allow students to make sense of the abundance of quadruples with lengths between 4 and 8. Additionally, stu-

Table 1: Quadruples with Length ≥ 14 Found by the Mobile App.

x_1	x_2	x_3	x_4	Length
812	549	64	956	16
158	15	915	425	15
680	992	418	511	14
683	321	990	881	14
280	893	792	612	14
725	752	801	891	14
865	84	211	442	14
449	672	796	39	14
308	130	32	634	14
23	533	450	300	14

dents can be guided to examine quadruples of long lengths and make observations or pose questions. A few lower and upper bounds have been established for special quadruples [5], which are interesting at higher levels of mathematics education. For school students and teachers, it is appropriate to look into the order (increasing or decreasing) of numbers in a quadruple in a cyclic manner. By definition, a quadruple $v = (x_1, x_2, x_3, x_4)$ is *cyclically monotone* if $(x_1 > x_2 > x_3 > x_4)$ or $(x_1 < x_2 < x_3 < x_4)$, as it is or after a rotation. Accordingly, a cycle can start from any one of the four numbers clockwise or counterclockwise. For example, $(9, 10, 1, 6)$ is cyclically monotone because it is monotonically increasing from 1 to 6, 9, and 10. Using the mobile app, students can find interesting quadruples of long lengths and examine their monotonicity or the lack of monotonicity.

Table 2 shows some examples, their lengths, and monotonicity. It is interesting that all quadruples of length 7 or more are cyclically monotone and all those of length 4 or less are not cyclically monotone. Quadruples of length 5 or 6 are even more interesting — some are cyclically monotone, others are not, as shown by the examples. These observations allow students to appreciate the complexity and patterns of the Four-Number Game. As a matter of fact, Brown and Merzel [5] proved a lemma in this regard: If a quadruple v is not cyclically monotone, then its length is at most 6. If it is cyclically monotone, its length is at least 5. Therefore, lengths 5 and 6 interact with a quadruple's cyclic monotonicity.

Table 2: The relationship between quadruple lengths and cyclic monotonicity.

x_1	x_2	x_3	x_4	Length	Cyclically Monotone?
10	1	2	5	8	Yes
9	10	1	6	8	Yes
9	8	5	0	8	Yes
9	6	1	10	8	Yes
8	0	3	4	7	Yes
8	0	2	4	7	Yes
7	10	4	6	7	Yes
6	4	0	7	7	Yes
3	10	0	1	6	Yes
7	1	2	4	6	Yes
5	9	6	4	6	No
9	2	2	4	6	No
4	1	2	3	5	Yes
7	1	3	5	5	Yes
6	3	10	9	5	No
2	8	7	3	5	No

5.4. Unexpected Discoveries

Unexpected discoveries or surprises are an integral part of the aesthetic experience of mathematical exploration and have for long been recognized for their motivational roles in mathematical teaching and learning [13, 15, 17, 18, 19]. As described in the previous sections, Ducci's Four-Number Game is indeed full of pleasant surprises, ranging from the even-odd dynamics to its symmetry and probability distribution. Using the mobile simulation, students are inclined to explore large numbers, only to find that it does not make much difference to the overall distribution of quadruple lengths. That is a worthwhile finding. On the other hand, the Four-Number Game can be played on a small range such as integers between 0 and 5 or those between 0 and 1. In addition to random simulations, an exhaustive iteration can be used to map out the length distribution. Either way, there are surprises to be found.

For example, over integers between 0 and 5, there 32 quadruples of length 7, 128 quadruples of length 6, and 88 quadruples of length 5, 768 quadruples of length 4. For many students, it is surprising to find that a quadruple may have a length that is greater than the maximum of the underlying set of numbers (i.e., 5 in the current case). Equally interesting are quadruples with only one non-zero number that have a length of 4. Seeing a few such cases might lead one to ask if they imply that a quadruple such as $(0, 0, 0, N)$, where N is any non-zero integer, always has a length of 4.

That is indeed true and can be readily proven by running the game manually. After three steps, all the four numbers will become N and the fourth step yields a quadruple of all zeros. Table 3 shows four independent quadruples of length 7 and five independent three-zero quadruples of length 4.

Table 3: Four independent quadruples that have a length of 7 and five independent three-zero quadruples of length 4 over integers between 0 and 5.

x_1	x_2	x_3	x_4	Length
5	4	3	1	7
5	3	2	1	7
4	3	2	0	7
4	0	1	2	7
0	0	0	5	4
0	0	0	4	4
0	0	0	3	4
0	0	0	2	4
0	0	0	1	4

What if the Four-Number Game is played on integers from 0 to 1? What length does a randomly selected quadruple may have? Numerous what-if questions can be posed for manual calculation or computer simulation that may lead to surprises and shareable moments among students.

6. Conclusion

Ducci's Four-Number Game is truly a mathematical curiosity, for its simplicity, aesthetic appeal, and rich mathematics. During the past three decades, there have been persistent efforts to bring parts of the game into K–12 mathematics education in professional development and recreational mathematics [2, 16]. In a playful manner, students have many opportunities to explore numeric and geometric patterns and wonder about its convergence and its dynamic behavior. Using concept mapping tools, students can investigate the even-odd progression of the four numbers and understand the rationale for its fast convergence to zeros. The mobile Ducci Four-Number App allows students to experiment with their ideas without the tedious calculation and further establish an empirical probability distribution. Quadruples of long lengths, albeit extremely rare, give students a strong sense of discovery especially in a social setting where students play, share, and discuss their mathematical discoveries [20].

Building on existing literature, this article presented a few engaging activities for school students and teachers of mathematics. It is my hope that the mobile four-number app will bring more mathematics behind Ducci's Four-Number Game to K–12 mathematics education both informally and formally. Perhaps, some young students will continue to play the Ducci game in more general cases and address the open problems related to Ducci sequences.

7. Acknowledgement

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A. User Interface for the Ducci Four-Number Game App

The mobile game is available at https://play.google.com/store/apps/details?id=appinventor.ai_moodle_bu.FourNumberProblem. See Figure 6 below for the user interface of the game app.

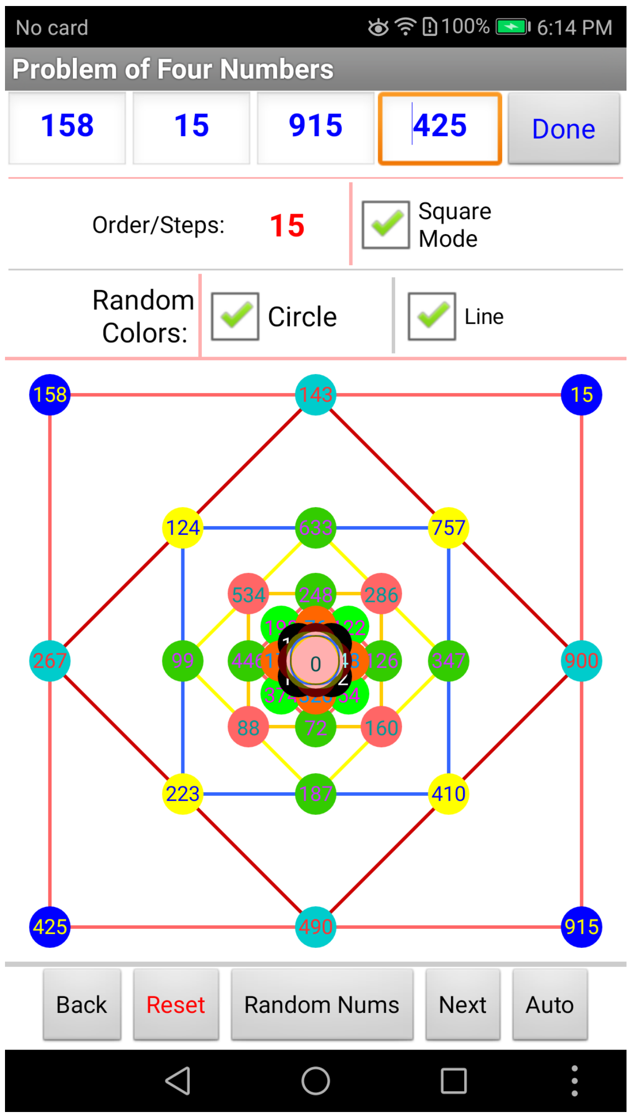

Figure 6: User Interface for the Ducci Four-Number Game.

B. User Interface for the Ducci Probability Distribution Simulator

See Figure 7 for the user interface of the probability distribution simulator.

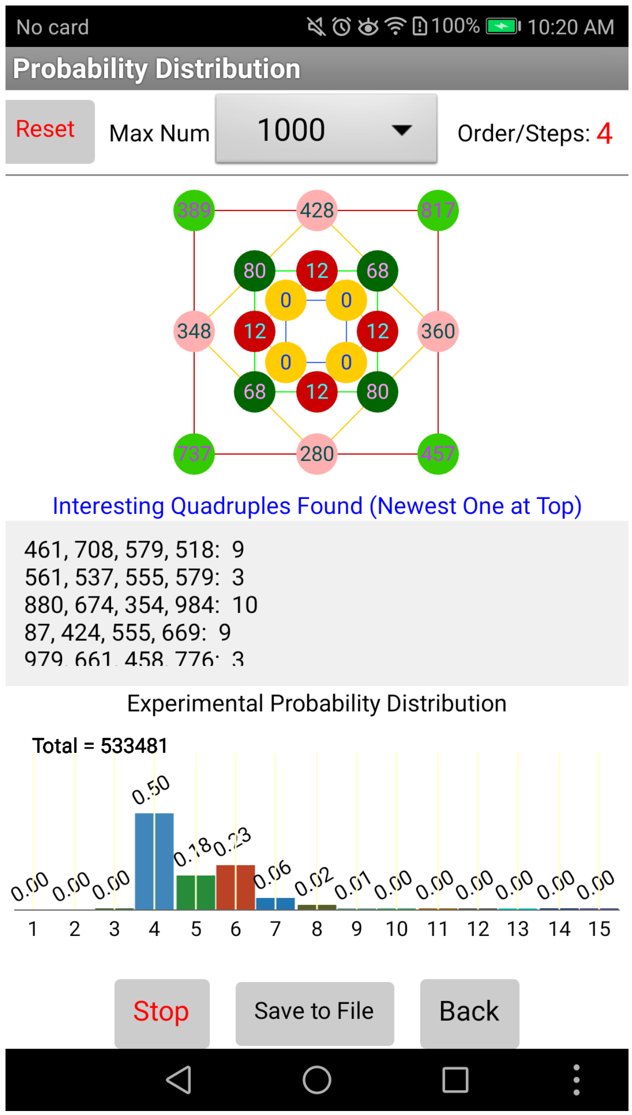

Figure 7: User Interface for the Ducci Probability Distribution.