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Wyatte C. Hooper
California State University, Fullerton

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Cover Page Footnote

I would like to acknowledge Dr. Martin Bonsangue, Professor, Dept. Chair at California State University, Fullerton, for providing inspiration and guidance for this article.

Archimedes of Syracuse and Sir Isaac Newton: On the Quadrature of a Parabola

Wyatte C. Hooper

Department of Mathematics, California State University, Fullerton, USA
w.hooper1995@gmail.com

Synopsis

Good mathematics stands the test of time. As culture changes, we often ask different questions, bringing new perspectives, but modern mathematics stands on ancient discoveries. Isaac Newton's discovery of calculus (along with Leibniz) may seem old but is predated by Archimedes' findings. Current mathematics students should be familiar with parabolas and simple curves; in our introductory calculus courses, we teach them to compute the areas under such curves. Our modern approach derives its roots from Newton's work; however, we have filled in many of the gaps in the pursuit of mathematical rigor. What many students may not know is that Archimedes solved the area problem for parabolas long before the use of algebraic expressions became mainstream. Archimedes used the geometry of the ancient Greeks, which gave him a vastly different perspective. In this paper, we provide both Archimedes' and Newton's proofs involving the quadrature of the parabola, trying to remain true to their original texts as much as feasible.

1. Introduction

Good mathematics stands the test of time. As culture changes, we often ask different questions, bringing new perspectives and introducing new methods, but modern mathematics stands on ancient discoveries. Isaac Newton's discovery of calculus (along with Leibniz) may seem old to mathematicians of the twenty-first century, but much in it is predated by Archimedes' findings. Current mathematics students should be familiar with parabolas and simple curves of the form $f(x) = ax^{\frac{m}{n}}$, where a, m, and n are constants.

In our introductory calculus courses, we teach our students to solve for the areas under or between such curves. Our modern approach derives its roots from Newton's work; however, we have filled in many of the gaps in the pursuit of mathematical rigor. What many students may not know is that Archimedes solved the area problem for parabolas long before the use of algebraic expressions became mainstream.

More specifically, Archimedes used the geometry of the ancient Greeks to solve for the area bounded by a parabola and a chord. When this area is added to the area under the parabola, which we know how to compute thanks to Newton, we get a trapezoid. Therefore if both Archimedes and Newton are correct, then their results must be compatible and we should be able to add the area under a parabola given by an expression computed via Newton's methods and the area between that same parabola and a chord given by an expression computed via Archimedes', and get the area of the resultant trapezoid. This is exactly what we do in Section 4. Before we get to that point however, we shall first look at Archimedes' proof for his method from his letter to Dositheus (in Section 2) and Isaac Newton's proof in his paper *Two Treatises Of The Quadrature Of Curves And Analysis By Equations Of An Infinite Number Of Terms* (in Section 3).

2. Archimedes of Syracuse (287-212 BCE) and the Parabola

Archimedes was born in Syracuse, Sicily, around 287 BCE. He was known as both a mathematician and an engineer. As a young man, Archimedes studied with Euclid's successors at Alexandria. It is clear that he was well versed in Euclidean geometry, and shared many letters with the other notable mathematicians of the era. Most people know him for his mechanical inventions that won him a place in many mythical legends. A few of these inventions are still used today, such as the Archimedean Screw. However, he seemed to always prefer the unambiguous nature of pure math to the rough reality of mechanics. Some of his notable contributions include bounding π within $(3\frac{10}{71}, 3\frac{1}{7})$ [2, page 98] and discovering the volume of a sphere.

Archimedes was known for his fixation on difficult problems. For example, there are many stories of him being dragged to the bath after working nonstop on a problem for days or being oblivious to the perils around him due to his preoccupation with his work. It is believed that he died as a result of this preoccupation during the aftermath of a Roman invasion [2, pages 84–89].

His work titled *Quadrature of the Parabola* was contained in one of his first letters to Dositheus.[1] It included 24 propositions and gave both a physical and geometric proof of his result [1, page 527].

Archimedes' approach to the areas related to a parabola [1, pages 527–537] is much different from Isaac Newton's, which we will look at in a bit, and may seem strange to modern readers. The Greeks did not have a solid number system, and most of their math was based in geometry. When modern readers see an expression such as 16^2, they may think of the number 256, but if confronted with the phrase "the square of 16" Archimedes would most likely have thought of a square with a side length of 16. In his works, or more truthfully, in the modern translations of his works, and in what follows, we are to understand the notation $\overline{AB}^2$ to refer to the square on the line segment $\overline{AB}$.[2]

Similarly, Archimedes did not think of a parabola as a curve or as the graph of a curve. Apollonius of Perga had discovered that the intersection of a cone and a plane parallel to the side of the cone is the shape we call a parabola [1, page 615]. This development led to much interest in what became known as conic sections and their areas. See Figure 1.

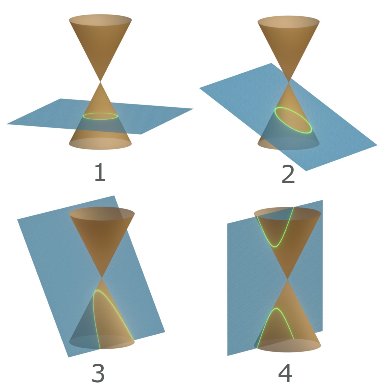

Figure 1: Types of conic sections. Image by JensVyff, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=99610156.

[1] The term "quadrature" comes from the attempt to create a square with an equal area to a given shape, using only a compass and an unmarked straightedge.

[2] We will be using anachronistic notation in this paper as the squaring notation was not developed until much later.

Archimedes gives his proof of the quadrature of the parabola in his letter to Dositheus. Here he tells Dositheus, he "first discovered [this method] by means of mechanics and then exhibited by means of geometry" [1, page 527]. Being both an engineer and a mathematician, he began to tackle this problem by creating a physical scale to measure the area. This mechanical approach gave him the result to aim for, but this was not sufficient for his proof. He felt it necessary to dive into the geometry to show that the area contained by a parabolic segment was four-thirds the area of the largest triangle bounded inside of it.

Let us go into some of the details. Here we follow [1]; also see [4]. Archimedes started by seting up his scale as a lever that balanced out the areas of a triangle with one vertex tied to the end of the lever and another vertex tied to the fulcrum with squares attached to the opposite end of the lever. He drew a parabola that crossed two of the vertices of the triangle and then split the area of the triangle into infinitely small pieces to find the ratio of the triangle to the area of a parabolic segment. See Figure 2 [1, page 532].

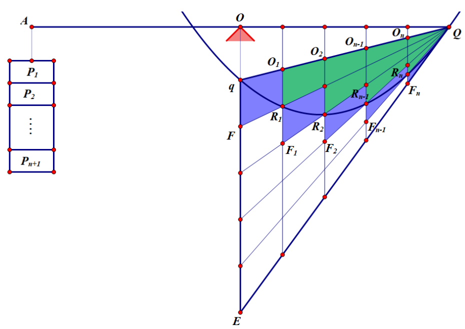

Figure 2: Archimedes' lever in his quadrature argument.

In two separate propositions he showed that the area of the big triangle is four times larger than the area of the largest triangle bounded inside of the parabolic segment (let P be the third vertex of this triangle), and the area of the large triangle is three times larger than the area of the parabolic segment [1, page 534]. That is,

$$\Delta QqE = 4\Delta QqP$$

and

$$\Delta QqE = 3 \times \text{the parabolic segment } QqP.$$

Therefore,

$$\text{the parabolic egment } QqP = \frac{4}{3}\Delta QqP.$$

This approach gave Archimedes the right answer, but he was not satisfied with it as a proof. Wanting to get away from physics, he began working with Euclidean Geometry. Archimedes' plan of attack was to place the largest triangle possible into the parabolic segment. This would create two new smaller parabolic segment in which he would put two more triangles. Afterward, he could continue in an infinite process to give the complete area of the first parabolic segment as the sum of all of the triangles. To make this a solvable problem, he would then show that each iteration of triangles has an area of $\frac{1}{4}$ the iteration before it. His method turned the area of the parabola problem into an infinite sum of triangles and is known as the method of exhaustion.

In Proposition 21 of his letter to Dositheus, he claims, if $\overline{Qq}$ is a line crossing a parabola at two points, V is the midpoint of the segment $\overline{Qq}$, and M is the midpoint of $\overline{QV}$, where $\overline{VP}$ and $\overline{MR}$ are parallel to the axis, such that P and R are on the parabola (see Figure 3), then

$$\Delta PQq = 8\Delta PRQ.$$

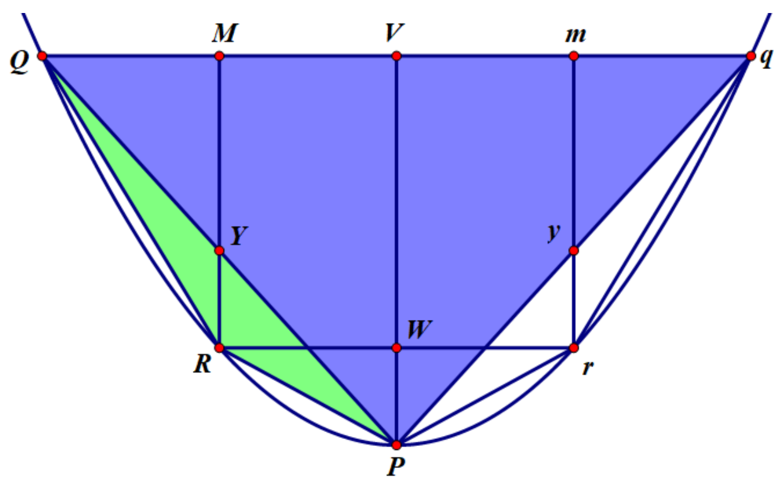

Figure 3: Proposition 21 and Step One in its proof.

Proof of Proposition 21. Let Y be the intersection between $\overline{QP}$ and $\overline{RM}$ and let $\overline{Rr}$ be parallel to $\overline{Qq}$ where W is the intersection between $\overline{Rr}$ and $\overline{PV}$.

Archimedes' first step was to claim that

$$\overline{PV} \text{ is to } \overline{PW} \text{ as } \overline{QV}^2 \text{ is to } \overline{RW}^2,$$

by Proposition 3 in his letter. He never gave the proof of this, claiming it was given in *Elements of Conics* [1, page 528].

We can be convinced of the veracity of this statement by letting P be the origin of our graph, and then realizing that our parabola has to have the equation $f(x) = ax^2$, for some constant a. Therefore,

$$ax_1^2 \text{ is to } ax_2^2 \text{ as } x_1^2 \text{ is to } x_2^2.$$

Since M is the midpoint of $\overline{QV}$ and $MVWR$ is a rectangle, $\overline{QV} = 2\overline{RW}$, and we have $\overline{QV}^2 = 4\overline{RW}^2$. So,

$$\overline{PV} \text{ is to } \overline{PW} \text{ as } \overline{QV}^2 \text{ is to } \overline{RW}^2,$$

so that

$$\overline{PV} \text{ is to } \overline{PW} \text{ as } 4\overline{RW}^2 \text{ is to } \overline{RW}^2,$$

or $\overline{PV} = 4\overline{PW}$.

From Figure 3, we can see that $\overline{PV} = \overline{PW} + \overline{RM}$. Thus $\overline{PV} = 4\overline{PW}$ gives us,

$$\overline{PV} = \frac{4}{3}\overline{RM}.$$

Noting that $\Delta PQV \sim \Delta YQM$ and $\overline{QV} = 2\overline{QM}$, we have $\overline{PV} = 2\overline{YM}$. So we have

$$\overline{YM} = \frac{2}{3}\overline{RM}$$

or equivalently

$$\overline{YM} = 2\overline{RY}.$$

Now we are ready to begin looking at some triangles. We know

$$2\Delta QMY = \Delta QMP \text{ since } \overline{PV} = 2\overline{YM},$$

and

$$\Delta PRQ = 2\Delta YRQ \text{ since } \overline{QM} = \overline{MV},$$

as well as

$2\Delta YRQ = \Delta QMY$ since they live in the same parallel and $\overline{YM} = 2\overline{RY}$.

Therefore, $\Delta PQM = 2\Delta PRQ$ gives us

$$\Delta PQV = 4\Delta PRQ,$$

which leads to

$$\Delta PQq = 8\Delta PRQ,$$

which is what Archimedes wanted to show [1, pages 535–536]. □

Archimedes was careful in the way he set the last proposition up. We should notice that everything we did can be carried over symmetrically to the other side, that is, in Figure 3, $\Delta PRQ = \Delta Pqr$, and this result is accurate for any line crossing the parabola in two points. Archimedes exploits this to produce an infinite sum that, as he will eventually show, equals the area of the parabolic segment; see Figure 4.

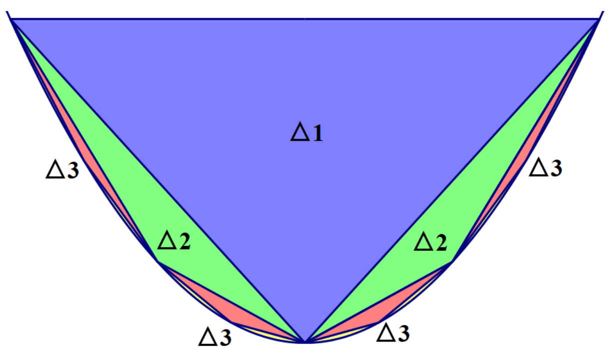

Figure 4: Archimedes' Method of Exhaustion using smaller and smaller triangular segments.

Here is how we would write this in modern notation:

$$\Delta 1 + 2\Delta 2 + 4\Delta 3 + 8\Delta 4 + 16\Delta 5 + 32\Delta 6 + \cdots = \text{The area of the parabolic segment.}$$

By Proposition 21, we have $8\Delta(N+1) = \Delta N$, which is equivalent to $\frac{1}{4}\Delta N = 2\Delta(N+1)$. So, our summation ends up being equivalent to

$$\Delta 1 + \frac{1}{4}\Delta 1 + \frac{1}{16}\Delta 1 + \frac{1}{64}\Delta 1 + \frac{1}{256}\Delta 1 + + \cdots = \text{The area of the parabolic segment.}$$

Archimedes claims that this summation to the Nth term must be less than the area of our segment as any such partial sum will be adding up to give the area of a polygon inscribed within the parabolic segment (this is more or less his Proposition 22). So the next step is to evaluate the summation in its entirety.

Archimedes starts doing this in Proposition 23 where he claims that if we define a (finite) sequence $A_1, A_2, \cdots A_N$, such that A_1 is the largest, and each consecutive A_n is a quarter of A_{n-1}, as in Figure 5, then the sum of all the areas is $\frac{4}{3}A_1 - \frac{1}{3}A_N$.

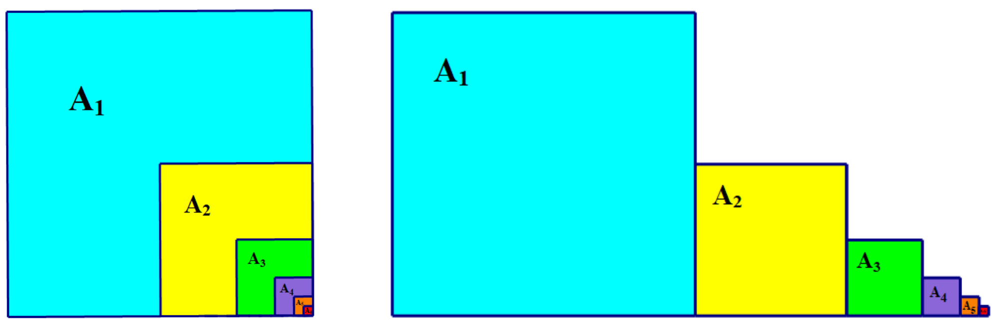

Figure 5: Two depictions of the sequence of squares in Proposition 23.

Proof of Proposition 23. Archimedes starts his argument by defining $a_2 = \frac{1}{3}A_2$, $a_3 = \frac{1}{3}A_3$, $a_4 = \frac{1}{3}A_4$, ..., and finally, $a_N = \frac{1}{3}A_N$. But we have $\frac{1}{4}A_1 = A_2$ by definition of our sequence, and so $\frac{1}{4}A_1 + a_2 = A_2 + a_2$. This gives us

$$\frac{1}{3}A_1 = A_2 + a_2.$$

More generally, to get $\frac{1}{3}A_n = A_{n+1} + a_{n+1}$, we can start analogously with $\frac{1}{4}A_n = A_{n+1}$. Thus,

$$\frac{1}{4}A_n + \frac{1}{12}A_n = A_{n+1} + \frac{1}{12}A_n \qquad \text{or} \qquad \frac{1}{3}A_n = A_{n+1} + a_{n+1}.$$

From here follows

$$A_2 + a_2 + A_3 + a_3 + A_4 + a_4 + ... + A_N + a_N = \frac{1}{3}(A_1 + A_2 + A_3 + ...A_{N-1}).$$

If we cancel out equal parts, we are left with

$$A_2 + A_3 + A_4 + ... + A_N + a_N = \frac{1}{3}A_1,$$

and by adding A_1 to both sides we get

$$A_1 + A_2 + A_3 + A_4 + ... + A_N + \frac{1}{3}A_N = \frac{4}{3}A_1,$$

which is what Archimedes intended to show [1, page 536]. □

Archimedes is going to make use of this fact in the context of the triangular areas in the parabolic segment, so let us rewrite it in that format as well:

$$\Delta 1 + 2\Delta 2 + 4\Delta 3 + 8\Delta 4 + \cdots + 2^{N-1}\Delta N = \frac{1}{3}(4\Delta 1 - 2^{N-1}\Delta N) \qquad (1)$$

Now Archimedes seems to have everything he needs to compute his final sum and prove his final proposition (Proposition 24, which asserts that "every segment bounded by a parabola and a chord is equal to four-thirds of the triangle which has the same base as the segment and equal height" [1, page 537]), but he is not done yet. Archimedes has a sense of how tricky infinite sums can be. So, as a precaution, he claims at the start of his proof of this final proposition that the parabolic segment is either less than $\frac{4}{3}\Delta QqP$, greater than $\frac{4}{3}\Delta QqP$, or equal to $\frac{4}{3}\Delta QqP$.

Let $K = \frac{4}{3}\Delta QqP$ and assume the area of the parabolic segment is more than K. Recall that the triangular segments introduced in each consecutive step of the Method of Exhaustion from Figure 4 are getting smaller and smaller, and contributing smaller and smaller areas to the total sum. But more importantly for our argument here, as these triangles are added, the remaining parabolic segments are getting smaller and smaller as well. Therefore at some point, these leftover parabolic segments will have to be smaller than the difference between the area of the original parabolic segment and K. Let us assume that this happens at step N. But then adding all the triangular pieces up to that point

$$\Delta 1 + 2\Delta 2 + 4\Delta 3 + 8\Delta 4 + 16\Delta 5 + 32\Delta 6 + \cdots + 2^{N-1}\Delta N,$$

we should get a number that is greater than K. This contradicts Proposition 23 (or more precisely, Equation (1)), so it cannot be true.

If next we assume the area of the parabolic segment is less than K, then once again, there will be a contradiction: at some point, the area added by new triangular segments will have to be smaller than the difference between K and the area of the parabolic segment; that is, for some N we will have:

$$\text{the area added at step } N = 2^{N-1}\Delta N < K - \text{ the area of the segment.}$$

For this N consider now the partial sum

$$\Delta 1 + 2\Delta 2 + 4\Delta 3 + 8\Delta 4 + 16\Delta 5 + 32\Delta 6 + \cdots + 2^{N-1}\Delta N,$$

which, by Equation (1), will differ from K by a third of $2^{N-1}\Delta N$. Since K exceeds this partial sum by an area less than $2^{N-1}\Delta N$, and the area of the initial parabolic segment by an area greater than $2^{N-1}\Delta N$, the area of the segment has to be greater than K, which contradicts our assumption.

So the only option left is that the area of the parabolic segment has to equal $K = \frac{4}{3}\Delta QqP$, which is Archimedes' final result [1, page 537]. Q.E.D.

Archimedes was thus able to find a relation between the area of a parabola and a simple triangle. As mentioned earlier in Footnote 1, the term quadrature comes from the attempt to create a square with an equal area to a given shape, using only a compass and an unmarked straightedge. This was a significant movement in Greek mathematics, and curved shapes were notoriously tricky. Greeks mathematicians had a good understanding of how to increase lengths by a ratio, and how to square a triangle. All Archimedes had to do was show that the area of a parabolic segment was a ration of the area of a triangle, and they could create the square from there.

Here, Archimedes had considerable success, but he also battled with the squaring of a circle without such luck. He did, however, make significant progress towards evaluating π. His method for finding the area of a parabola is not widely taught, but there are several familiar aspects to his work. Archimedes masterfully works the geometry to prove the relation he wants to find, and the method of exhaustion his work crucially depends on is the first hint towards calculus. We will see how Isaac Newton uses a similar summation to solve the area under a generalized simple curve.

3. Sir Isaac Newton (1642-1727) and the Parabola

Isaac Newton is much closer to modern times than he is to Archimedes. He was born on Christmas day in 1642, and we know a lot more about his personal life than we know about Archimedes. For instance, we know that his father died a few months before he was born, and his mother got remarried and left Newton with his grandmother while he was still young. Isaac Newton was an introvert, spending more time reading books and carrying out his experiments rather than spending time with his peers. He studied at Cambridge College during a time where the political atmosphere of the college was less than ideal. Most of Newton's learning was done on his own in the library. Isaac Newton was known for his ability to focus on a single problem for days on end. He shared this attribute with Archimedes. One of Newton's significant contributions was his discovery of Calculus. This discovery was covered in controversy, because Gottfried Leibniz made similar discoveries before Newton published his findings. While the courts of the time sided with Newton, most people today believe that both findings were made independently. Newton was also known for the binomial expansion theorem which he makes use of in the proof we present below [2, pages 162–165].

Isaac Newton's work should look familiar to modern readers. In his book, *Analysis by Equations of an Infinite Number of Terms*, he gives the first proof for the power rule. That being said, there are several gaps in his proof. Most of these gaps deal with infinitely small units he calls fluxions. Newton starts by stating his rule to be,

"If $ax^{\frac{m}{n}} = y$; it shall be $\frac{an}{m+n}x^{\frac{m+n}{n}} = \text{Area } ABD$."

Here ABD refers to the parabolic segment given in Figure 6.

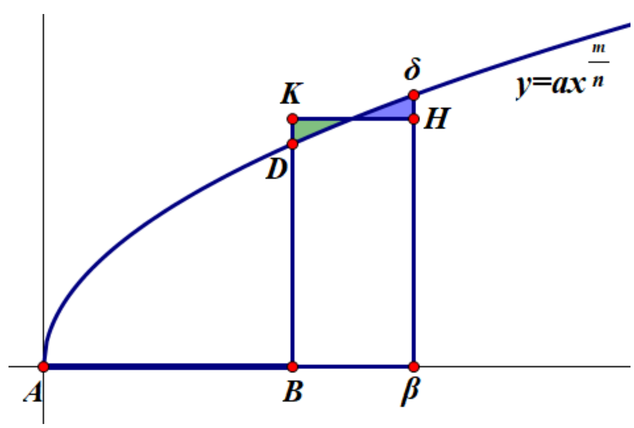

Figure 6: Newton's curve.

When Isaac Newton first gives this rule, he claims, "the thing will be evident by an example", but waits until the end of his paper to provide his proof. One should note that Isaac Newton is approaching the problem with a broader scope, but his result covers a parabola.

As shown in Figure 6, Newton lets $AD\delta$ be any curve whose base is $\overline{AB} = x_1$, where $\overline{BD} = y$ and the area $ABD = z_1$, and he lets the area under the curve up to x be the output of a function $z(x)$.

We are going to let $\overline{B\beta} = o$ and $\overline{BK} = v$, and assume that the area of the rectangle $B\beta HK$ equals the area of the vertical slab $B\beta\delta D$.

Therefore $\overline{A\beta} = x_1 + o$, and the area of the segment $A\delta\beta$ is equal to $z_1 + ov$. Evaluating our function $z(x)$ at $x_1 + o$ gives,

$$z(x_1 + o) = z_1 + ov.$$

At this point, Newton gives a few examples that lead him to the general case. For the general case, assume

$$z(x) = \frac{n}{m+n} a x^{\frac{m+n}{n}}.$$

For notational ease, let $c = \frac{an}{m+n}$ and $p = m + n$. Then, our assumption is

$$z(x) = c x^{\frac{p}{n}}.$$

When we evaluate this at $x_1 + o$, we get

$$z(x_1 + o) = c[x_1 + o]^{\frac{p}{n}},$$

which is equivalent to

$$(z(x_1 + o))^n = c^n (x_1 + o)^p.$$

Since $z(x_1 + o) = z_1 + ov$, we have

$$(z_1 + ov)^n = c^n (x_1 + o)^p.$$

Now, Newton uses his binomial expansion theorem to both sides to get

$$(z_1 + ov)^n = z_1^n + n z_1^{n-1} ov + \frac{n(n-1)}{2} z_1^{n-2} (ov)^2 + \dots$$

on the left, and

$$c^n\left(x_1+o\right)^p=c^nx_1^p+c^npx_1^{p-1}o+c^n\frac{n(n-1)}{2}x_1^{n-2}o^2+\dots$$

on the right. Since these are to be equal, and $z_1^n=c^nx_1^p$, we have

$$nz_1^{n-1}ov+\frac{n(n-1)}{2}z_1^{n-2}\left(ov\right)^2+\dots=c^npx_1^{p-1}o+c^n\frac{n(n-1)}{2}x_1^{n-2}o^2+\dots,$$

Therefore, dividing everything by o gives

$$nz_1^{n-1}v+\frac{n(n-1)}{2}z_1^{n-2}ov^2+\dots=c^npx_1^{p-1}+c^n\frac{n(n-1)}{2}x_1^{n-2}o+\dots$$

Now we look at this equation as o moves towards 0. We should note here that limits were not well-defined when Newton did his work, and there was some argument about the validity of this approach. Indeed it might seem as if this involves dividing by zero in the last step if then right after we let $o=0$. However, he does take this limit, so all terms on both sides except the first ones go to zero, and v goes to y. So we get,

$$nz_1^{n-1}y=c^npx_1^{p-1}.$$

Now comes a series of ingenious algebraic moves: we first write

$$nz_1^{n-1}y=c^npx_1^{p-1},$$

as:

$$\frac{nz_1^ny}{z_1}=\frac{c^nx_1^pp}{x_1}.$$

Thus, using $z_1^n=c^nx_1^p$ once again, we obtain:

$$\frac{ny}{cx_1^{\frac{p}{n}}}=\frac{p}{x_1},$$

and we get

$$ny=pcx_1^{\frac{p}{n}-1},$$

which, in our original terms, gives

$$ny=(m+n)\left(\frac{an}{m+n}\right)x_1^{\frac{m+n-n}{n}},$$

and finally $y = ax^{\frac{m}{n}}$.

Here Newton concludes that if the area under a curve is $z(x) = \frac{n}{m+n}ax^{\frac{m+n}{m}}$, then the curve has to be $y = ax^{\frac{m}{n}}$. Newton then claims if this is true, the converse must also be true. So the area under a curve of the form $y = ax^{\frac{m}{n}}$ must be $z(x) = \frac{n}{m+n}ax^{\frac{m+n}{n}}$. Q.E.D.

This is where Newton leaves his proof, and it may seem incomplete. In reality, he proved the converse of his claim and used that to justify the assumption. His claim is true however, and we still use this rule to this day. As long as we know the formula for the curve, we can enter the input for the section of the graph we are interested in, and Newton's formula will output the area under the curve for that section.

4. Archimedes + Newton = Trapezoid!

Now we are going to solve an example problem. We will start with a parabola and a line that cuts through it at two points. Then, we will be able to use Archimedes' approach from Section 2 to find the area inside the parabolic segment, and Newton's rule ftrom Section 3 to find the area underneath the curve. From Figure 7, we can see that the two areas added together give us a trapezoid. So we will also calculate the total area using the formula for the area of a trapezoid and see if it is equal to the sum of its parts.

Here is the statement of the problem. Figure 7 shows the setup.

Let $f(x) = \frac{x^2}{3}$.

(a) Use Archimedes' method to find the area between $f(x)$ *and the line that passes through* $Q = (-3, 3)$, *and* $q = (6, 12)$.

(b) Use Newton's method to find the area under the curve $y = f(x)$ *from* $x = -3$ *to* $x = 6$.

(c) Find the sum of the results from (a) and (b), then check to see if it equals the area of the trapezoid formed.

Part (a). To use Archimedes' method from Section 2, we need to find the area of ΔQqP. To do this, we will start by finding the midpoint V of $\overline{Qq}$.

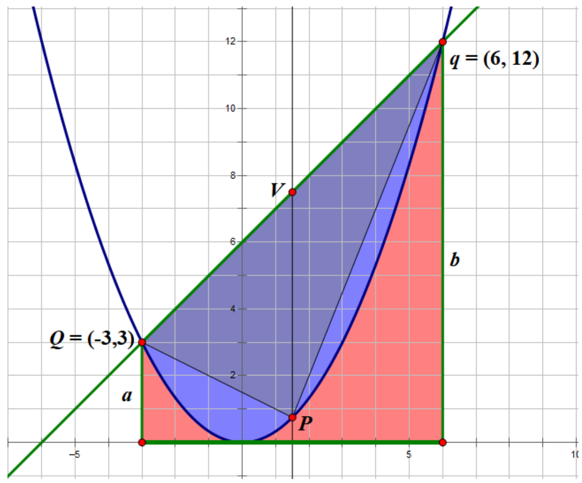

Figure 7: The sum of the two areas yields a trapezoid!

By applying the midpoint formula, we get

$$V = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(1\tfrac{1}{2}, 7\tfrac{1}{2}\right).$$

The next step is to find the tip, P, of the triangle that we need:

$$P = \left(1\tfrac{1}{2}, f(1\tfrac{1}{2})\right) = \left(1\tfrac{1}{2}, \tfrac{3}{4}\right).$$

Since ΔQVP, and ΔqVP have the same height and the same base length ($\overline{QV} = \overline{qV}$), we know that the areas of the triangles are equal. Therefore, the area of ΔQqP is twice that of ΔQVP.

To find the area of ΔQVP, we need to find the length of $\overline{VP}$ and the height of the triangle between $\overline{VP}$ and Q. The length of $\overline{VP}$ can be calculated by subtracting the y-value of P from that of V;

$$\overline{VP} = 7\tfrac{1}{2} - \tfrac{3}{4} = 6\tfrac{3}{4}.$$

The height between $\overline{VP}$ and Q can be calculated by the difference between the x-values of V and Q:

$$\text{the height of the triangle } = 1\tfrac{1}{2} - -3 = 4\tfrac{1}{2}.$$

Therefore,

$$\text{the area of } \Delta QVP = \frac{1}{2} \cdot (6\tfrac{3}{4}) \cdot (4\tfrac{1}{2}) = 15\tfrac{3}{16}$$

and since $\Delta QqV = 2\Delta QVP$,

$$\Delta QqV = 2(15\tfrac{3}{16}) = 30\tfrac{3}{8}.$$

Now that we have the area of ΔQqP, we are ready to apply Archimedes' result ("every segment bounded by a parabola and a chord is equal to four-thirds of the triangle which has the same base as the segment and equal height" [1, page 537]) to find the area bound between the parabola and $\overline{Qq}$. We have:

$$\text{the area of the parabolic segment } = \frac{4}{3} \cdot \Delta Qqv = \frac{4}{3} \cdot (30\tfrac{3}{8}) = 40\tfrac{1}{2}.$$

By Archimedes' result, we have found that the area bound between the parabola and the chord $\overline{Qq}$ is 40.5.

Part (b). Now we are going to apply Newton's result from Section 3 to find the area under the curve $y = f(x)$ from $x = -3$ to $x = 6$.

Recall that to use Newton's result, we need our curve in the form $y = ax^{\frac{m}{n}}$. For our curve $f(x) = \frac{x^2}{3}$, we get $a = \frac{1}{3}$, $m = 2$, and $n = 1$. Now by Newton's rule, we get that

$$z(x) = \frac{n}{m+n}\, ax^{\frac{m+n}{n}} = \frac{1}{9}x^3$$

is a function that outputs the area under our curve from the y-axis to x.

Now, all we have left to do is evaluate $z(x)$ at the two end points $x = -3$ and $x = 6$ and compute the difference.

$$z(6) - z(-3) = \frac{1}{9}(6)^3 - \frac{1}{9}(-3)^3 = 27$$

Therefore, by using Newton's result, we have found that the area under the curve $y = f(x)$ from $x = -3$ to $x = 6$ is 27.

Part (c). Now, to make sure that these two results support each other, we want to use the standard formula for the area of a trapezoid to find the area of the region obtained by putting together the two areas and check to make sure it equals the sum of its parts we computed in parts (a) and (b).

Recall that the area of a trapezoid is computed as $\frac{1}{2}(b_1 + b_2)h$, where b_1, b_2 are the two base lengths, and h is the height. So, for the trapezoid in Figure 7, we compute

$$\frac{1}{2} \cdot (3 + 12) \cdot 9 = 67.5.$$

We check to make sure that the sum of the results of parts (a) and (b) is equal to 67.5:

$$40.5 + 27 = 67.5.$$

Because this is true, we know that the results of Archimedes and Newton support one another.

5. Conclusion

As the previous section shows, the findings of Archimedes and Newton complement one another. The area of the parabolic segment found by Archimedes' method added to the area under the curve given by Newton's rule equals the area of the trapezoid formed. This is a good reminder, for us and for our students, that often in mathematics, there is more than one approach that will get the answer, and often a new perspective can reveal some hidden relations.

Archimedes masterfully worked the geometry to bring out several relations that were not obvious. He worked these relations to transform the question he is asking into a sum of infinite parts. Being the brilliant mathematician he was, he constructed a method of solving this long before these problems were being commonly studied. His approach here can be viewed as the first step towards calculus and impacted the ideas that would eventually go into our textbooks.

Newton's work laid the framework of modern calculus. His result could be generalized over all simple curves, which made it more useful. He was able to define a function that gives the area under a curve by assuming it exists and working backward to get the formula of the original curve. He made use of his binomial expansion theorem and started the idea that would lead to our current notion of limits. His mathematics has influenced the course of history and is still studied.

While their approaches were different, Archimedes and Newton shared a common theme of infinitely small particles, and some tough summations. Both of these topics are covered in our calculus classes today and have their roots in these ancient texts. As has been stated before, we stand on the shoulders of giants.

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