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# Mathematics Students as Artists: Broadening the Mathematics Curriculum

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## Synopsis

Mathematics has often been referred to as an art. For some it is "the purest of the arts", where the mathematicians' art is "asking simple and elegant questions about our imaginary creations, and crafting satisfying and beautiful explanations". Yet with classroom time given primarily to "covering the curriculum", testing, and practicing problem-solving procedures, students' opportunities to appreciate the aesthetic dimension of mathematics are often limited. To promote a responsive environment in an effort to enable students to become artists of their own mathematics experience, I consider in this paper two facets of the mathematics classroom. Content-wise I make the argument that students need to see problem-clarifying strategies in conjunction with problem-solving techniques, as the former are essential for making progress when engaging a mathematics problem where an explicit solution is not apparent. The other facet aiming to promote student agency is providing them opportunities to work with their own practical/professional concerns as students so as to become more creative and productive artists of their mathematics experience.

## 1. The Underlying Problem

Mathematics is often referred to as an art. Halmos [11] locates the "art of mathematics" with its "counterpart in painting"; Hardy [13] with poetry; while Russell [22] sees its "austere beauty" most closely related to sculpture. For Lockhart it is "the purest of the arts", where the mathematicians' art is "asking simple and elegant questions about our imaginary creations, and crafting satisfying and beautiful explanations" [15, page 4]. And Neale believes most all students of mathematics can appreciate mathematics' aesthetic dimension.

She shares that "school students can have just the same experience [as university students]: when they're given the opportunity to engage with rich questions, when they can play with mathematical ideas, when they have the chance to experience multiple strategies to the same question rather than just getting the answer in the back of the textbook and moving on" [19, page 1].

Yet with mathematics culture committed to an aesthetic of concision ("less is more", as Neale notes), mathematics textbooks tend to be filled with techniques, definitions, and proofs, and often lack discussion regarding the rationales for decisions made. This cultural practice tends to preclude textbooks from promoting students' "play[ing] with mathematical ideas" or having the "experience [of applying] multiple strategies to the same problem". The austere view extends to the classroom when teachers follow the textbook with dedication, and that practice may well begin quite early in students' mathematics experience. For example, "findings [of students ages 6 to 13] suggest that children do mathematics without much thought or opportunity to discuss what it actually might be" [28, page 588]. And as is well recognized, the experience of mathematics for many is often quite a struggle, often lacking any aesthetic rewards, as students try to make sense of a discipline having associated with it a literal phobia and a commitment to brevity in its expression.

Research suggests that textbooks are not as helpful as one would hope. "Among the obstacles for learning are mathematical texts, especially the economical style of presenting mathematical content in introductory textbooks, which abandons the context of discovery in favour of the context of justification, as well as the very common focus on a correct solution" [20, page 354]. Such a limited and limiting view negatively impacts student experience. For "how textbooks are designed provides a window into the nature of mathematics that students are expected to learn. They characterise not only the content but also advocate what students are to be able to do with that content — what mathematics and behaviours are to be encouraged" [23, page 143]. The extent of the problem is considerable. "Studies have shown that the majority of mathematics teachers in secondary schools follow the prescribed textbook when planning and implementing their mathematics programs" [24, page 183].

Yet what is recommended by the NCTM could not be more different:

> "Students should engage in making conjectures, proposing approaches and solutions to problems, and arguing about the validity of particular claims ... they should be the audience for one another's comments ... discourse should be focused on making sense of mathematical ideas ... and solving problems" [18, page 45].

To promote a responsive environment in an effort to enable students to become artists of their own mathematics experience, I consider in this paper two facets of the mathematics experience. Content-wise, I make the argument that mathematics textbooks need to provide greater discussion of and opportunities for students to work with problem-clarifying strategies, heuristics, along with the traditional problem-solving techniques. Problem-solving techniques tend to fill mathematics textbooks providing procedures for ready solution regarding classes of problems (word problems, systems of equations, series convergence, etc.), while problem-clarifying strategies such as "taking the problem apart", "tinkering", and "generalizing" often tend to go unmentioned even though these are productive means for gaining perspective about making progress with problems when it is not clear how to proceed directly to solution.

It is of course valuable for mathematics students to learn problem-solving techniques, including algorithms and theorems, so as to build a collection of productive means for resolving problematic mathematical situations. But it is often not an appreciative understanding that mathematics students have secured but a collection of limited techniques they've memorized for solving particular classes and cases of problems. How to factor, solve a system of equations, determine whether an infinite series converges or not, etc., are valuable procedures. Yet what the mathematician and educator Alfred North Whitehead lamented eighty years ago in his *An Introduction to Mathematics*, still seems to be the case: "The unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception . . . . Here lies the road to pedantry" [27, page 8].

More recently, Fan and Zhou in examining mathematics textbooks for lower secondary school found "the majority of problems in the US books . . . were routine, traditional, and moreover of single-step" [8, page 69]. But the problem is not isolated. Hussain, via questionnaires and group interviews in 112 Bahraini primary government schools, came to find that "the most dominant teaching style [in mathematics classrooms] is explaining-then-drilling. The authoritative voices of the teachers dominate what is going on . . . the mathematics teacher is the ultimate source of the knowledge which is involved in mathematics textbooks and students' participation is constrained from learning together or to gain active epistemic roles in knowledge construction" [14, page 287]. The problem seems pervasive. Gravemeijer and collaborators share that "Since Dutch teachers follow their textbooks very conscientiously and typically do not transcend what textbooks offer, the result [on a national exam] was that

the 12$^{\text{th}}$ grade students did not reach proficiency on the level of more advanced conceptual understandings" [10, page 39]. The authors then make the point that the outcome was consistent with the goals of the mathematics textbook "which showed a lack of attention for more advanced conceptual mathematical goals and a tendency to aim for algorithms as potential end points" [10, page 40].

We can readily recognize the distinction when mathematics students, having learned some technique(s), are presented with a problem that requires more than their blanket application. In an effort to motivate their students, the mathematics teacher may well urge them to "think!", but their faces make clear that all they can think about is the stress they are experiencing, as they have no idea as how to proceed toward problem resolution. In Section 2 I will consider further how to repair and transcend this problematic and disturbing reality so that a growing number of mathematics students are actually more capable of dealing with complexity, of being able to engage in essence a blank canvas more creatively. The distinction to be made more explicit is between "experience guided by intelligent habit, rather than merely routine habit — [the former] is experience characterized by resourcefulness, inventiveness, ingenuity, tenacity, efficiency ..." [5, page 256].

The other issue that deserves consideration in helping students have a more rewarding mathematics classroom experience is creating opportunities for them to uncover and work with their own concerns, that is, the development of a behavioral curriculum, with regard to their mathematics development. Along with broadening their mathematical understanding by including problem-clarifying strategies, we need to help students become more capable of working with personal (and essential) qualities of practice that shape their mathematics experience (such as becoming more patient; reflecting more on their work; catching errors of hasty judgement). We need to provide students psychological and practical means of dealing more effectively with the challenge of learning mathematics, and enable them to be more connected to the creative effort of doing mathematics. For many students it is often only the terror of the "terror and triumph" [21] Polya recognized in engaging mathematics that is experienced. Helping students develop their art and craft through their own thoughtful decision-making would naturally require their personal/professional development as they have an opportunity to make their mathematical experience more likely to be productive and appreciated. Our discussion of this issue will follow the first, the intentional focus on problem-clarifying strategies with the aim to create a more productive, more satisfying mathematical experience.

## 2. Developing Students' Aesthetic Sensibilities Through Heuristics

As noted, problem-solving techniques have value. What I argue here is that were mathematics textbooks also to promote and elucidate problem-clarifying strategies, including valuable mental actions such as "visualizing", "looking for a pattern", and "using simpler numbers", there would naturally be less student confusion regarding how to make progress while engaging with a mathematics problem when a direct path is not apparent. Given the pressures associated with mathematics teachers' concern to "cover the curriculum" and with mathematics textbooks illogically eschewing dialogue with readers, the focus tends to be on techniques being presented and logical arguments being demonstrated, not strategies being discussed. Yet were we to appreciate the observation of Keith Devlin that making the problem simpler is "how we do mathematics",then heuristics would be an essential element in both mathematics classrooms and textbooks.

When attempting to work on relatively challenging mathematics problems (with no technique coming readily to mind) it is problem-clarifying strategies that enable one to find a way out of the darkness into the light. And the confusion may well not be one requiring sophisticated mathematics for resolution, but just effective means to simplify the situation so students would be able to see and think more clearly. Here is an example problem from a community college remedial mathematics course on which I worked with students as a tutor in the college learning center.

> "A road 7/8 mile long is under repair. The crew working on the problem can repair 1/56 of a mile in an hour. How many hours would it take to repair the road?"

The associated quantities apparently made students uncomfortable, so much so that their stress left them quite confused as how to proceed. When the conversation turned to making the problem simpler, we resolved to "choose easier numbers", and the problem became available for their solution. That is, when we revisited the problem with the road being 20 miles long, and the crew rate of repair 2 miles an hour, the students readily saw what they had to do. The problem-clarifying strategy transformed the problem into their seeing that dividing a fraction by a fraction was required.

Visualizing is another valuable problem-clarifying strategy (recall Bhaskara's "Behold!") that could well help students engage with some problem(s) more readily. It would be a good idea for mathematics textbooks to demonstrate this problem-clarifying strategy and to offer opportunities for application.

For example, consider the context of finding relationships between infinite sets. Consider an instructor drawing the Cartesian coordinate plane on the board along with the line $y = x$, and then informing students that they can now claim the set of all the real numbers represented by the closed interval $[0, 1]$ is equipotent to those in the interval $[1, \infty)$. The question students will naturally come upon is: Why is the line $y = x$ so special? They can recognize that that line actually divides the first quadrant in half, as the angle associated with the line is 45°. This awareness implies that all the lines passing through the origin starting at the line $y = 0x$ (the $x$-axis) and rotating to the line $y = 1x$ have coefficients from 0 to 1, and so constitute all the real numbers from 0 to 1. And the other half of the first quadrant would contain all the lines from $y = 1x$ to the $y$-axis, which represents a line with infinite slope.

A commitment to include problem-clarifying strategies as an integral element of students' mathematica development could impact exams as well, where the focus would be broader than determining the extent to which students have command of the relevant problem-solving techniques. Both problem-solving techniques and problem-solving strategies could be incorporated into a single question if one were to be intentional about things. Here's one instance. Standard questions involving linear equations include variations of:

> "If $2x + 3y = 15$, determine the $x$- and $y$-intercepts, and use your findings to draw the given line."

Figure 1 would be a demonstration that a student could solve that problem.

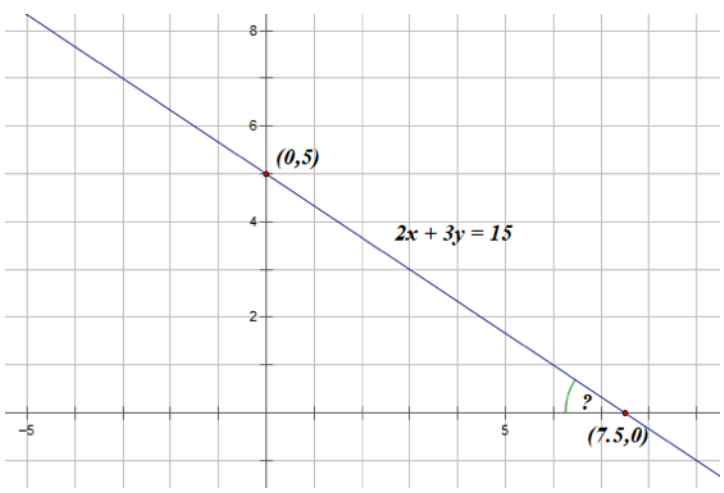


Figure 1: Sample solution for the given linear equation.

For students who have been engaged in learning and applying problem-clarifying strategies, a second part to the problem can be given:

> "What could you say about the measure of the smaller angle the line makes with the $x$-axis? Explain your findings."

When confronted with this problem, students who had come to appreciate the idea of making the problem simpler, have drawn a right triangle with the vertical and horizontal segments 5 units in length, along with the angle of 45° (see Figure 2).

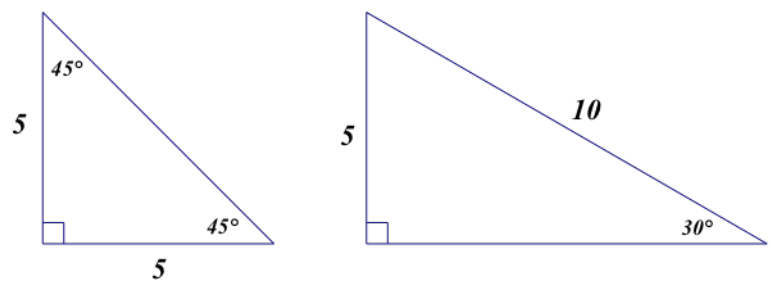


Figure 2: Making the problem simpler by comparing right triangles.

Noting the horizontal segment determined by the given line was greater than 5 units (7.5), the thinking was that the angle had to be smaller than 45°. Other students working in groups and with the idea of making the problem simpler found other pathways. A number of students drew a 30-60-90 degree triangle with the vertical segment still 5; but the hypotenuse was included, being 10. Realizing that would mean the horizontal segment would be $5\sqrt{3}$, they determined that that length was greater than 8 units. So, the angle had to be between 30 and 45 degrees. Some of the students left their answer as a value bounded by those two degree measures. A few others (having made a hasty decision) found the average. Others working in small groups realized that as 7.5 was closer to $5\sqrt{3}$ than to 5, decided that the angle measure would have to be closer to 30° than to 37.5°. They then chose various values closer to 30° than to 37.5°, given their imaginative sizing up of the situation. As part of the concluding conversation, a natural consequence of their working on the problem was students wondering aloud if there was a way to find what exactly the angle measure was. Here finally I began a discussion of inverse trigonometric functions — something the students wanted to know about. And when it was determined that the angle was 33.7°, to the nearest tenth, there were expressions of real satisfaction.

In the absence of problem-clarifying strategies, students' experience in the mathematics classroom often demonstrates that it is the ingenuity of the most innately talented that makes progress possible [26, page 115]. The other students often find themselves making dedicated effort to lessening the felt tension by engaging in counterproductive behaviors, such as telling themselves they can't do the problem so as to end the discomfort as soon as possible. Boaler relates: "One of the biggest mistakes students make with math problems is that they often rush in and do *something* ... without really considering what is being asked of them, whereas successful problem solvers spend some time really thinking about the problem" [3, page 186; italics in original]. And the insecurity is not ameliorated when mathematics students are presented with techniques without any explanation. Sharing their confusion, they wonder "how would anyone know to do that?", and "why does that work?"

With teachers following the explicit procedures that most mathematics textbooks focus on, it is not difficult to imagine an inverse relationship between the teacher's confidence with clearly presenting a technique, and students' confusion while making efforts to practice and memorize the procedure. We can appreciate Moise's observation that "it is simplistic to suppose that people remember what they are told, and understand the things that are explained to them clearly" [17, page 471]. And he didn't think "that we ought to feel complacent about our present lack of an adequate theory of teaching [mathematics]. One of the tasks of the coming generation is to create one" [17, page 473]. Halmos, at the same panel, shared that "mathematics is really all about solving problems" and, with Polya, stated that "our job as teachers ... is to teach how to ask questions" [12, page 467]. Both these quotes seem to promote the idea that productive questioning ought to be a primary consideration of mathematics educators and textbooks. In order to promote that possibility with greater regularity and with a growing student participation, students should be given ample opportunities to draw upon problem-clarifying strategies to inform their investigating, so as to encourage them to develop their artistry in asking good questions. Instances follow.

## 3. Toward a More Aesthetic Mathematics Engagement

Students of mathematics are well aware of textbook demonstrations to solve a quadratic equation by setting one side equal to 0. Factoring then becomes the problem focus involving a collection of similar quadratic equations. Yet, the process of setting one side equal to 0 wasn't used to solve linear equations; why now?

Students may well learn a procedure but miss something deeper. Were there discussion acknowledging that solving most quadratic equations can be quite a challenge, unlike linear equations, with the terms on different sides of the equal sign (e.g., "Find the value(s) of $x$ that satisfy $x^2 = -4x+21$"), attention would naturally turn to the compelling question: how to "make the problem simpler". From this vantage point, students can appreciate that when a product has a value of zero, one or the other(s) of the product factors must be zero; while with any product having a value other than zero, there would be an infinite number of possible values. So when one side of the quadratic equation is set equal to 0, the focus is to try to express the other side as a product of factors.

With this orientation, the practice of factoring deservedly gains appreciative understanding (though limited in application as it is). In contrast, with the focus on the technique of factoring, absent of discussion of making the problem simpler, mathematics students practice such procedures but with limited awareness of the impetus for the practice. And such an isolated effort does not help students see mathematics as an art, but instead as a collection of isolated techniques, as Whitehead noted.

Another consideration: the problem-solving technique students are usually provided with when having to divide by a fraction is "invert and multiply". Surely the procedure is efficient; it's been offered as the solution technique at least from 1850 [5, page 61]. Yet ask students why it works, and often a blank stare is the response. But ask them what would make the problem simpler; agreement is soon reached that dividing by the number 1 would be ideal, as the answer would be the numerator. So, with that goal in mind, students can more appreciate the problem-clarifying process.

To divide by 1/4, for instance, students could consider adding 3/4, or multiplying by 4/1. Trying their first idea, including adding the same value to the numerator, they would find it is not effective, as the result wouldn't make sense. (For example, when dividing 10 by 1/4, what is being determined is how many quarters in 10; but adding 3/4 to both the numerator and denominator results in the answer being $10\frac{3}{4}$.) But when multiplying by the reciprocal, both in the numerator and denominator, the answer does make sense. (Students regularly suggest that what is done to the denominator should be done to the numerator, their intuition acknowledging their aesthetic sensibility for "balance".) The associated discussion makes clear that multiplying both the numerator and denominator by the reciprocal, which makes the problem simpler, is actually "inverting and multiplying".

But here the solution procedure is not presented in its pristine expression to be practiced, but as a meaningful expression of their resourceful thinking. Seen in the light of that awareness it will more likely be appreciated by students. And the positive energies as a direct consequence can most likely be further drawn upon to foster other creative efforts. With similar experiences they can become more the artists of their mathematics experience. And with enough repetition, attempts at making the problem simpler when faced with mathematical situations experienced as complicated would become a valued practice for students, and so "pertinent questioning" and artistic resolution could well become more the way things are in mathematics classrooms.

Yet it is not just procedures absent of explanation that cause students difficulty. Mathematics textbooks and teacher telling as the primary mode of discourse rather than student exploration leading to the discovery of valid and valued mathematics relationships keeps students at an unnecessary distance from their mathematical experience.

Consider a situation where students are given the opportunity, for example, to explore and determine how they would distinguish four-sided figures from each other based on their investigations, rather than being presented with the standard collection of quadrilaterals with their distinctive qualities. The former would more likely lead to students developing a more investigative and so informed mathematical sensibility, as part of their growing imagination regarding the different figures and the characteristics that they find distinguishes one from the other. Here the focus is on an essential aspect of creative thinking — an intuition becoming increasingly informed. And this will likely not happen in a classroom where students' naiveté is seen as a shortcoming. As Dewey noted, we can think of immaturity not as an expression of ignorance but one of open-mindedness [6]. And that perspective, concretized in classroom practice, could well help promote students' developing aesthetic as productive problem solvers.

What if mathematics textbooks acknowledged students' naive concerns, for example, regarding why it is that a circle has 360°? Is it that that number of degree units happen to fit and fill the circle? This seems to me to be a perfectly legitimate question, demonstrating thoughtful consideration and deserving acknowledgement. In that pedagogical direction, it would seem reasonable, and hopeful, were students to ask why $m$ represents the slope of a line; doesn't $s$ make more sense? And, what makes a right angle "right" — can't it point to the left? [9]

Common mathematical statements such as these which tend to be presented as givens to be memorized, and others similar to them (such as how would anyone come up with the method of false position? or, with the idea of showing that the harmonic series diverges by considering a different series?) may leave students with an underlying state of confusion, especially for those who reflect on the given material! Assuming it is acceptable in mathematics textbooks to make such statements without discussion would seem to promote an alienating distancing rather than an inviting engagement.

I list these instances to suggest that students need to be provided the opportunity to express their naive but thoughtful perspective when engaging with mathematics, including the most basic claims. Otherwise, there is little if any opportunity for their developing aesthetic sensibilities, for in essence their thinking has been denied, albeit often unconsciously.

Surely a circle can be divided into a collection of equal units to a different number than 360 partitions. (The great mathematician Laplace wanted there to be 400 such divisions.) As noted earlier by more than one mathematician, the art of doing mathematics depends on asking good questions, and that includes those from a naive perspective. This suggests it would be worthwhile were mathematics textbooks and teachers to model that (naive, inquiring) behavior. In doing so, students may well gain confidence in the realization that the questions they have are legitimated by their teacher and textbook.

Additionally, efforts to be more able to stay with a challenging situation by applying productive mental actions, considerations such as how to be more patient, resilient, observant, productively self-critical, and taking other emotionally productive actions deserve classroom attention. (Einstein claimed that he wasn't brilliant; rather his successful efforts were due to his patience.) The psychological dimension deserves consideration in mathematics classrooms and textbooks as well if the object is to help foster mathematics students' greater positive emotional energy. For in its absence, there is little chance of the requisite resilience and patience needed to deal well with complex situations. How that can be promoted is the focus of the following section.

## 4. Mathematics Students' Self-Development as Curriculum

Research reports that "the pupil's voice is seen as increasingly important in understanding schooling" [28, page 282]. And recent educational literature given to mindfulness, growth mindset, and group projects, for example, makes the argument that group projects offer exciting opportunities for mathematics students to become more aware of the nature and focus of their own thinking.

Taking advantage of these ideas might require more explicit attention. The Assessment Standards for School Mathematics [18] make clear that students' assessment of their own mathematical efforts is crucial to their development.

Such a focus could have a profound effect on students' lives. How willing one is to persist at challenging tasks and how well one can plan ahead, pay attention, remember and follow instructions, and control impulses and emotions involve the so-called executive functions. Also known collectively as self-regulation or self-control, these have long been considered a key life skill. In that regard, efforts toward developing student self-awareness as means to promote their reflection so as to better cope with stress is gaining attention [3, 15]. More specifically, having students reflect on their behavior in engaging with problems and having the opportunity to make needed changes from a constructive perspective could have profound gains both in the classroom and in their future as citizens. Such considerations seem essential to a productive engagement with any creative experience.

To give that effort direction and support in the mathematics classroom, it would be valuable to give students the time to uncover what goal(s) they believe would enable them to be more capable mathematics students [9]. With those in mind there is the opportunity for discussions regarding what would be effective means for their becoming more the mathematics student they want to be. Such considerations clearly have psychological and social value. And their sharing allows students to appreciate that their peers have the same difficulties they have, and could well promote a dedicated collective effort, providing additional support.

As Dewey noted, "social conditions obstruct the development of judgment and insight or effectively promote it" [2, page 285]. That's to say, how seriously mathematics students take their development, in terms of becoming more capable, confident, and creative, is a function of how seriously their mathematics teachers and textbooks do.

Our students need to appreciate that they, as all of us, are malleable material, something that can be worked with. This truth can become clear to them when they consider the fundamental role habits have played and do play in their lives. Not only did habits make possible their being able to read, write, and walk, they also shape presently how they think, do things, and feel [7, page 25]. Once they come to realize the power of habit and appreciate the possibility of change, students have the opportunity to ask themselves what new habit(s) they want to secure, or what old habit(s) they want to discard.

Through building new productive habits and dropping old unproductive ones, they can become more capable and confident artists of their experience; they themselves can make their experiences with mathematics much more authentic and rewarding. We can and should help them by providing opportunities to work at securing/omitting specific practices, in essence giving legitimacy to their own emotional/behavioral curriculum relevant to their mathematics experience. Without such opportunities there's little chance for many to change so as to become more able and self-appreciative mathematics students.

To support my students' developing greater self-awareness and creative/critical expression, I gave class time at the beginning of the school year for them to think more about what changes in their practice might create opportunities for them to engage with mathematics more satisfactorily and so enhance their mathematics experience. Discussion included making clear that a goal without an awareness of sign-posts of progress would likely have as much consequence as making a wish. Wishing precludes volition, and so it isn't generally a good problem-solving strategy, except perhaps with regard to special days of the year. Similarly, mathematics teachers telling students, for example, to "be patient" may work for some, but likely not if it isn't clear to them what signs of being patient look like.

Making note of signs of progress is most important as they are the concrete expression that thought and action are coming together, that growth is taking place. Until those sign-post behaviors can be made clear and then integrated into their practice as mathematics students, it is difficult to imagine much gain. So, with regard to a student's developing aesthetic of becoming, for example, more patient, resilient, thoughtful, and so on, when working on mathematics problems, discussion of explicit practices, in textbooks and in the classroom, can well serve as a guide and gauge for them as to how things are developing. (For most students, such an effort might deserve as much recognition as a test score recorded on a report card.)

It is here we find an opportunity for mathematics to work for the public good. With students developing productive behaviors and valuable attitudes and dispositions as a consequence of their learning mathematics, the public is clearly "getting their money's worth". For such development represents personal and professional growth. Students increase their awareness of what they can do to make a problematic situation more open to investigation and successful resolution; in short they learn to think and act more constructively. The mathematics classroom, where imagination, experimentation, and reflection are essential components, is a perfect setting for their developing aesthetic.

The possessor of those life-enriching behaviors would be in a relatively ideal position to gain a valuable mathematics education and be more successful in general. This is to say, enabling mathematics students to get in touch with what really concerns them with regard to their more complete development in dealing with challenging situations in mathematics, and what they really want to be able to do as part of their mathematichlal experience, is of significant value, and as such, deserves the consideration of mathematics teachers and textbooks.

Just as an artist steps back and decides what is needed to make their dedicated effort more satisfying, more cohesive, students need time and opportunity to take seriously their developing artistry in shaping their mathematics experience. To support and codify that effort beyond providing emotional support, I asked my students to think about and write down three behaviors that would serve as sign-posts, signs demonstrating that they are making progress toward a new practice they wish to have or an old practice they want to rid themselves of. This provided them a personal/professional checklist to keep in mind, something concrete to hold onto that both supported their effort and desire. This kind of activity is especially important for those students who are trying to change some counter-productive reactions they've developed in response to prior stressful experiences in mathematics classrooms. For students who bring old feelings of discomfort when engaging with new problems can find themselves falling into old counterproductive behaviors. At the extremes, such behavior might involve either freezing mentally, being so insecure about how to proceed, or acting frenetically and claiming whatever came to mind first is the answer to alleviate the stress. But such defensive responses are surely not what they as students or we as their teachers hope would actually be the learning outcome. So having those sign-post behaviors close by when studying can serve as gentle reminders to keep their eyes on the prize. And the more they can see or be helped to see signs of their aesthetic development of course, the better.

Student efforts along these lines should be noted in conversations and incorporated more completely into evaluations. In this way we acknowledge our students' development toward becoming the artists they want to be as formally and institutionally important, which of course it ought to be. The exclusion of such concerns regarding students' developing aesthetic sensibilities and principles from conversations about and evaluations of their mathematics experience is indeed a major omission in school practice, at least from the perspectives of the student and society. In their absence we can expect that

whatever poor habits students enter our mathematics class regarding their thinking and acting and emotional responding will most likely still be there when they graduate, and so persist as they continue to live on as adults. That absence of development represents a great loss of opportunity for them and for all of us.

Thus we should make and take the opportunity to encourage our students in these efforts. In order to promote more cohesive and dedicated effort, class discussion could include mentioning that the later we wait with regard to looking at how things are going and deciding what we would want to happen the more difficult it naturally is for valuable change to occur. To help students in that realization I found it valuable to suggest more than once that they take seriously Aristotle's understanding that "it is a matter of real importance whether our early education confirms us in one set of habits or another. It would be nearer the truth to say that it makes a very great difference indeed. In fact all the difference in the world" [1, page 1103b1-25]. Anecdotally I have often noted the unfortunate thinking of many of the relatively older students along the lines of: "Since I made it this far doing what I do, why do I need to introduce or stop anything now?" So having this conversation as early as possible is quite important.

To help students appreciate the effort needed, I have often shared reflections by significant others. Dewey's observation that "the self is not something ready-made, but something in continuous formation through choice of action" provides a good starting point for the discussion. While a number of students think, as noted, that by their "advanced" age, the way they do things is "written in stone", they need to come to see that it's not. They need to appreciate that looking at how they do things and deciding if there is something they could improve on is an essential aspect of their mathematics education and, more completely, of their developing aesthetic of who they want to be. The second quotation I refer to on how best to proceed is Socrates' observation that "the nearest way to glory — a short cut as it were — is to strive to be what you wish to be thought to be". Here the point made is that they can decide how they want others as well as themselves to think about them as a mathematics student and as a thoughtful and capable person, and then make decisions so that actual person comes more and more into focus. And here is one from the musician Miles Davis: "Sometimes you have to play a long time before you sound like yourself". That's to remind them that their aesthetic development toward doing things better takes time, and effort, and "mistakes", of course.

To finally convince them to make the effort, I explicitly point out that this isn't just another school year coming up; it's a year of their life, and time is not a renewable resource. And hopefully, they appreciate the special opportunity and effort required to be artists of their own lived experience.

With their mathematics teacher expressing concern and interest individually and collectively regarding how they are doing, students have an opportunity to appreciate that their personal and professional development is being taken seriously. In the absence of that concerted focus displayed early in the school year, students get caught up in being students, routinely doing what is being asked of them by others. In this taken-for-granted way they lose sight of what they were asking of themselves, and how very important it was. Instead, they tend to become other-directed, and so their personal aesthetic concerns tend not be given the energies they deserve. Having them write about their efforts from time to time helps to serve as gentle reminders; keeping a journal has proven valuable for promoting reflection and thoughtful evaluation, as many mathematics educators have discovered.

## 5. Concluding Remarks

As can be appreciated, working with students' developing aesthetic is not an easy activity to keep alive in the mathematics classroom as "real mathematics learning" usually means focusing on "covering a curriculum". But if students can develop themselves as creative problem solvers as they gain familiarity with problem-clarifying means for making problems simpler, and if they are able to work on those emotional behaviors that have made their mathematics learning problematic, so that they can finally become the mathematics student they truly want to be, the social and political implications suggest themselves: a more thoughtful populace open to and capable of dealing with difficult issues becomes a more realistic goal. If mathematics textbooks and classroom activities demonstrate a recognition that "the capability and willingness to assess their own progress and learning is one of the greatest gifts [mathematics] students can develop" [25, page 8], the aesthetic quality of the mathematics experience will be eminently available for the student, with positive consequences for society as well.

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