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How to Bake a Theorem

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We bake bread to satisfy our hunger, and we prove theorems to satisfy our hunger for knowledge and existence. Here I explore this analogy.

To bake bread, all you need is flour, water, yeast, and salt. Flour to shape and form, water to bind and activate, yeast to leaven, and salt to flavor.

To bake a theorem, you also need flour, water, yeast, and salt, of sorts. A mathematical theorem is a statement about some relationship between mathematical objects. Importantly, a theorem must have a proof. A proof is a series of logical steps that trace a theorem back to self-evident mathematical truths. It isn't enough to show that a theorem holds for a few or even many cases. A proof shows that a theorem is true all of the time. A theorem without a proof is just a guess, a conjecture. It is merely dough, limp and useless until baked through. It does not satisfy our hunger for certainty.

Theorems are baked with the same ingredients as bread, in a way. Mathematical flour is the raw material, the definitions of the mathematical objects. A triangle has three straight sides and three angles. If one of the angles of a triangle measures $90°$, then the triangle is a right triangle. The rules and procedures of math are the water. These rules activate the definitions in the flour and form something kneadable and workable. To find the area of a square, multiply the length of the square by the width. The logic and the flow of the mathematical argument make up the yeast, the lifting agent. If two squares have the same side length, then the two squares have equal area. And the mathematical salt is the details, the wording, the commentary on the math.

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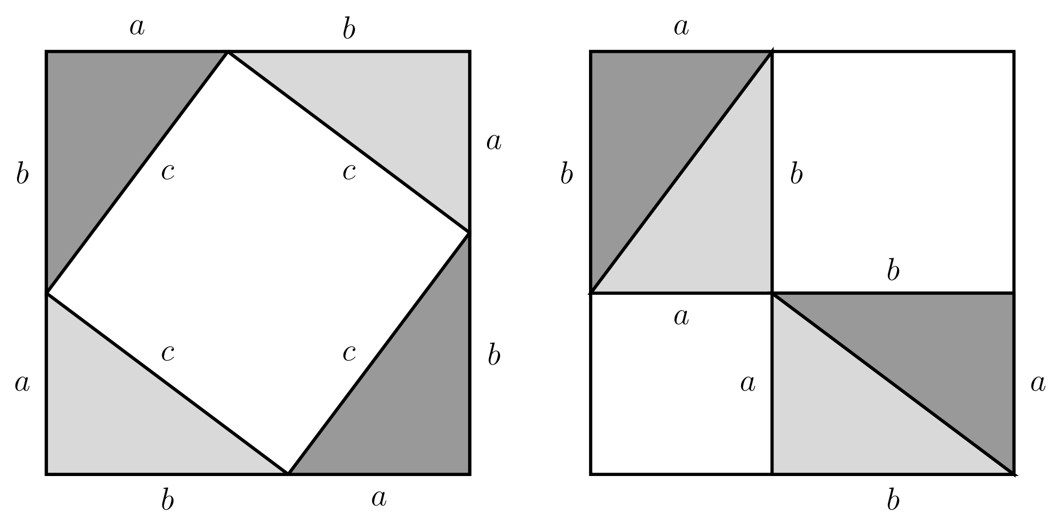
Math isn't just pushing around symbols; it is the exploration of patterns and ideas. Pointing out these patterns gives flavor to a proof.

The Pythagorean theorem is an ancient loaf, proved thousands of years ago.

Theorem. *Given a right triangle with sides of length a, b, and c, where the side of length c is opposite the right angle, then $a^2 + b^2 = c^2$.*

Before giving the proof of this theorem, a little aside about proofs. Sometimes reading a proof isn't that interesting, or even worth the time. The proof of the classification of finite simple groups is contained in hundreds of articles that total over 10,000 pages. In cases like these, only a few people actually check to make sure the proof is correct and the rest of us just skip to the statement of the theorem. The statement of the theorem at the end of the proof is sometimes preceded by a little symbol $\therefore$, three dots in a triangle that mean "therefore". It's the mathematical symbol for skipping to the end because the proof was too long, and you didn't read it.

However, if the proof of a theorem is not very long then reading it may be valuable. Proofs have a way of showing how mathematics works and explaining why the patterns are interesting. The proof of the Pythagorean theorem is not long, but if you feel so inclined you can skip to the "$\therefore$" at the end. Here's the proof, in a picture:



The square on the left contains four right triangles each with sides of lengths a, b, and c. The remaining area of the white square is c^2, since the white square has sides of length c. The square on the right is the same size and contains four identical right triangles, just rearranged. The remaining white area is formed by two squares, one with sides of length a and the other with sides of length b. The white area on the right is the sum of these two areas, which is $a^2 + b^2$.

$$\therefore a^2 + b^2 = c^2.$$

It doesn't matter what the numbers a, b, and c are. We could draw the same types of squares starting with any right triangle. That's what makes this a proof that the Pythagorean theorem holds for every right triangle. Drawing a right triangle and measuring its sides are the flour of this theorem. The rules of geometry are the water, allowing us to draw additional lines and shapes around the right triangle. The logic of rearranging the shapes without changing their areas provides the lift, the yeast that finishes the proof that $a^2 + b^2$ does in fact equal c^2. A little salt might be that this is only one of many ways to prove the Pythagorean theorem.

The mathematician Andrew Vazsonyi recounted his first meeting with another mathematician, Paul Erdős, when he was 14 and Erdős was 17 [1].

"How many proofs of the Pythagorean theorem do you know?" Erdős asked.

"One," said Vazsonyi.

"I know 37," said Erdős.

(This question was just the second thing Erdős said to Vazsonyi when they first met. The first was "Give me a four digit number," to which Vazsonyi replied "2,532." In a second, Erdős came back with, "The square of it is 6,411,024." There was no greeting, no hello, no how are you.

$\therefore$ Paul Erdős loved math and math alone.)

The Pythagorean theorem is one of many theorems with multiple proofs. Multiple proofs are, of course, not necessary for the progress of mathematics and our desire for certainty. But sometimes finding a totally novel way to prove an established theorem illuminates another aspect of the mathematical patterns. The flavor of the theorem can be enhanced by looking at it from a new perspective. Although the Pythagorean theorem has many proofs, many of them modern, the result of $a^2 + b^2 = c^2$ maintains the flavor of ancient times.

Bread is not only one of humanity's oldest creations, it remains a staple in some form or another in many cultures around the world. Bread wasn't the first food that early humans ate, not by a long shot, but it developed into a consistent system that has fed billions of humans over the millennia.

Besides simply eating to keep ourselves alive, humans feel a need to understand and know things. Mathematics may be older than bread itself. The earliest evidence of counting and number that we have found is the Ishango bone found near the Nile, which clocks in at 20,000 years old. The bone has a series of tally marks carved into it, and while the meaning of these marks is not clear, the marks indicate that some prehistoric human was attempting something we would describe as mathematics. Bread, on the other hand, didn't show up until around 14,000 years ago.

The study of mathematics has ebbed and flowed across various world cultures, but at the root of it all is the human need to organize and understand. Ancient mathematicians wanted to prove theorems in math, wanted to set in stone ideas that they felt were universally true and eternal. They ate bread to fend off starvation, and they proved theorems to fend off uncertainty, to establish a sense that human activities matter.

The ancient Greeks prized idealized geometry, but they couldn't solve all their problems as easily as the Pythagorean theorem. One of these problems was squaring the circle. The Greeks loved to do geometry with only a compass for drawing circles and a straightedge for drawing lines, but measurement of shapes was not allowed. The problem of squaring the circle started with a circle, and the goal was to draw a square with exactly the same area as the circle using only a compass and straightedge.

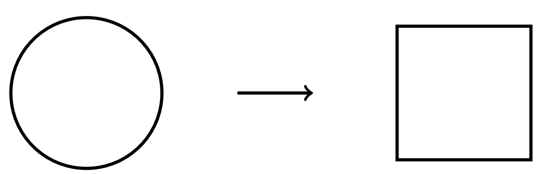


Figure 1: Taking a circle and drawing a square with the exact same area.

The ancient Greek civilization came and went without a solution to squaring the circle, but not for lack of trying. The Greeks had a word, $\tau\varepsilon\tau\rho\alpha\gamma\omega\nu\iota\zeta\varepsilon\iota\nu$, meaning to busy oneself with squaring the circle. They tried mixing a lot of mathematical dough to bake this theorem. Mathematicians kept attempting to solve the problem for centuries afterward. In this case all their efforts led to failure, because they were attempting an impossible problem.

Theorem. *Given a circle in the plane, it is impossible to construct a square with the same area as the circle using only straightedge and compass.*

The Greeks (and everyone after them) had been using the wrong sort of yeast in their pursuits. No amount of mathematical tinkering would ever raise a circle to become a square.

Curiously, the proof of this impossibility theorem did not take place in geometry but rather in the realm of algebra. Sometimes to bake a theorem, one has to harvest the flour from a different field of mathematics. Pierre Wantzel proved in 1837, centuries after the Greeks first posed the problem, that in order to square a circle the number π has to be a solution to a polynomial equation with integer coefficients. There have to be some whole numbers $a_0, a_1, a_2, ..., a_n$ such that $a_n\pi^n + a_{n-1}\pi^{n-1} + \cdots + a_2\pi^2 + a_1\pi + a_0 = 0$. Finally, in 1882, Ferdinand von Lindemann proved that no such equation existed, that π was too irrational to be the solution of an algebraic equation like this.

$\therefore$ Squaring the circle was officially impossible.

Many theorems are conjectured centuries before they are proved, slowly fermenting and taking on complex flavors. The theorem of the impossibility of squaring the circle has another dimension of flavor. Most mathematical theorems are constructive, describing patterns and processes that you can sink your teeth into. To experience the Pythagorean theorem, you can draw a real right triangle. It's odd to taste a theorem like the impossibility of squaring the circle, which is fully baked and yet not quite satisfying. Impossibility theorems show that even math has boundaries. Perhaps impossibility is an acquired taste.

Despite the potential disappointment that squaring a circle is impossible, it is comforting to know that we can put that particular problem to rest. Knowing the impossibility is better than not knowing if a mathematical problem can be solved or not. We can move on to other problems, of which there are many because math is far from complete. Theorems continue to be baked today, and some conjectures remain open, waiting for someone to come along and mix together a perfect recipe to finally prove them.

The Collatz conjecture sprung up in the 1930s and currently remains unbaked. The conjecture concerns the Collatz process. Start with any positive integer. If the integer is odd, multiply it by 3 and add 1. If it is even, divide it by 2. Keep applying these rules to the new numbers you get. For example, if we apply this process to the number 42, we eventually get down to the number 1:

$$42 \xrightarrow{\div 2} 21 \xrightarrow{\times 3,+1} 64 \xrightarrow{\div 2} 32 \xrightarrow{\div 2} 16 \xrightarrow{\div 2} 8 \xrightarrow{\div 2} 4 \xrightarrow{\div 2} 2 \xrightarrow{\div 2} 1.$$

If we try this with the number 17, then the process also leads to 1:

$$17 \xrightarrow{\times 3,+1} 52 \xrightarrow{\div 2} 26 \xrightarrow{\div 2} 13 \xrightarrow{\times 3,+1} 40 \xrightarrow{\div 2} 20 \xrightarrow{\div 2} 10 \xrightarrow{\div 2} 5$$
$$5 \xrightarrow{\times 3,+1} 16 \xrightarrow{\div 2} 8 \xrightarrow{\div 2} 4 \xrightarrow{\div 2} 2 \xrightarrow{\div 2} 1.$$

The Collatz conjecture is that no matter which positive integer we start with, this process will eventually reach 1. Despite how simple the conjecture is, no one has yet been able to prove this.

Many people have published various methods they have used to attack this conjecture. They have drawn trees of the paths that different numbers take as they collapse down to 1. They have delved into other number systems to try to gain insight on the Collatz process. Nothing has worked yet. It seems highly likely that the conjecture is in fact true; the process has been verified (by computer) for all numbers up to 100 quadrillion. In every observed case, the Collatz process reaches 1. But no one has yet come up with a real proof that it works for *every* positive integer. "Mathematics may not be ready for such problems," said Paul Erdős, who knew 37 proofs of the Pythagorean theorem at age 17 [1]. Erdős published over 1,500 mathematical papers in his lifetime; he knew a thing or two about what it takes to bake a theorem.

Erdős spent his entire life in the pursuit of mathematical theorems. He never owned a house of his own, preferring to travel the world, crashing on other mathematicians' couches in between lectures he gave at conferences and universities. He couldn't drive a car, and all of his personal belongings fit in a suitcase. To him, daily actions such as eating bread were incidental to being alive enough to do mathematics. The need to prove theorems and become certain of things was far greater than the need to do anything else.

And yet, Erdős was well aware that there would always be another math problem left to solve. He, and every other mathematician, could never stave off uncertainty permanently.

Kurt Gödel's famous Incompleteness Theorem in 1931 forever changed the way humans think about mathematics. Until that time, many people felt that math was deterministic, that for any conjecture there was always a proof of it out there somewhere. Gödel's theorem shocked the world. His theorem was a meta-math theorem, a theorem about the nature of mathematics itself. He proved, with completely correct logic, that every consistent mathematical system with the power to do basic arithmetic was incomplete. "Incomplete" means that there are mathematical conjectures that can neither be proved nor disproved using the rules of their mathematical systems.

Theorem. *In mathematics, there will always be statements whose truthfulness can never be determined.*

Impossibility may be an acquired taste, but even to those accustomed to it Gödel's Incompleteness Theorem is strong. Gödel baked the ultimate impossibility theorem. There are conjectures that can never be proved or disproved, and here's the kicker: It's also impossible to know what conjectures are like this. If someone hands you an unproven mathematical statement, you can't tell whether it is possible to prove it or not.

It may seem like Gödel brought only bad news to the mathematical community, but there is one upside to his theorem. Since mathematics cannot be complete, we can rest assured that there will always be some sort of math problem to solve. If we come across a mathematical conjecture that we can't prove, we can decide if we want it to be true or not and use it as an axiom, an assumption, instead. This will open the door to another system of mathematics that will have new conjectures to prove.

So we go forward, eating bread today even though we know it is only a temporary solution to hunger, that we will have to bake and eat some more tomorrow. Gödel showed that our pursuit of proofs of mathematical theorems is no different. We don't know what theorems we will be able to prove and which ones we won't, but we keep trying, even knowing it is a temporary solution to our hunger for certainty, that even if we prove a theorem today there will be more theorems to prove tomorrow.

$\therefore$ There is no "$\therefore$" for the whole of math.
We will be baking theorems forever.

References

[1] Paul Hoffman, *The Man Who Loved Only Numbers*, Fourth Estate Limited, London, 1998.