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The LSB Theorem Implies the KKM Lemma

Gwen Spencer and Francis Edward Su

Let S^d be the unit d -sphere, the set of all points of unit Euclidean distance from the origin in \mathbb{R}^{d+1} . Any pair of points in S^d of the form $x, -x$ is a pair of *antipodes* in S^d . Let Δ^d be the d -simplex formed by the convex hull of the standard unit vectors in \mathbb{R}^{d+1} . Equivalently, $\Delta^d = \{(x_1, \dots, x_{d+1}) : \sum_i x_i = 1, x_i \geq 0\}$. The following are two classical results about closed covers of these topological spaces (for the first see [6] or [3], for the second see [5]):

The LSB Theorem (Lusternik-Schnirelmann-Borsuk). *If S^d is covered by $d + 1$ closed sets A_1, \dots, A_{d+1} , then some A_i contains a pair of antipodes.*

The KKM Lemma (Knaster-Kuratowski-Mazurkiewicz). *If Δ^d is covered by $d + 1$ closed sets C_1, C_2, \dots, C_{d+1} such that each x in Δ^d belongs to $\cup\{C_i : x_i > 0\}$, then the sets C_i have a common intersection point (i.e., $\cap_{i=1}^{d+1} C_i$ is nonempty).*

A cover satisfying the condition in the KKM lemma is sometimes called a *KKM cover*. It can be described in an alternate way: associate labels $1, 2, \dots, d + 1$ to the vertices of Δ^d and demand that each face of Δ^d be covered by the sets that correspond to the vertices spanning that face. (Thus vertex i is covered by set C_i , the edge between i and j are covered by $C_i \cup C_j$, etc.)

Both of the foregoing set-covering results are perhaps best known in connection with their equivalent formulations in topology: the LSB theorem is equivalent to the Borsuk-Ulam theorem [3], and the KKM lemma is equivalent to the Brouwer fixed point theorem [5]. Also, the LSB theorem has found spectacular application in proofs of the Kneser conjecture in combinatorics [1], [4]. The KKM lemma has numerous applications in economics (see, for example, [2]).

Since the Brouwer fixed point theorem can be obtained as a consequence of the Borsuk-Ulam theorem [7], it is natural to ask whether there is a direct proof of the KKM lemma using the LSB theorem. The purpose of this note is to provide such a proof.

Theorem. *The LSB theorem implies the KKM lemma.*

Consider Σ^d , a d -sphere under the L^1 norm:

$$\Sigma^d := \{(x_1, \dots, x_{d+1}) : \sum_i |x_i| = 1\}.$$

Observe that the LSB theorem holds for Σ^d , since Σ^d and S^d are related by an antipode-preserving homeomorphism. Note that Σ^2 is just the boundary of a regular octahedron, while for general d , Σ^d is the boundary of the $(d + 1)$ -*crosspolytope*. It is the union of 2^{d+1} facets that are simplices, one for each orthant of \mathbb{R}^{d+1} (see Figure 1).

It will be convenient, then, to use the LSB theorem for Σ^d to prove the KKM lemma, because Δ^d is naturally embedded in Σ^d ; namely, Δ^d is the facet of Σ^d for which $\sum_i x_i = 1$. Call this facet F_{top} , the “top” facet, and call the antipodal facet the “bottom” facet F_{bot} . Let F_{mid} signify the complement of $F_{\text{top}} \cup F_{\text{bot}}$ in Σ^d , the “middle” band of

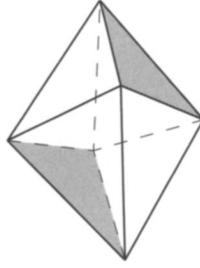


Figure 1. The 2-sphere Σ^2 in the L^1 -norm, which is the boundary of an octahedron. The “top” and “bottom” facets are shaded.

the d -sphere. The strategy of our proof is to assume for the sake of contradiction that a KKM cover of Δ^d has no common intersection point. Then we extend these sets to construct a closed cover of Σ^d whose sets contain no pair of antipodes, thereby contradicting the LSB theorem.

Part 1 of proof: construction. We first consider the case where a given KKM cover C_1, \dots, C_{d+1} of Δ^d is *nondegenerate* (i.e., for each x in Δ^d and set C_i , x is in C_i only if $x_i > 0$). In the alternate characterization of the KKM cover, this means that each face is covered *only* by the sets that correspond to the vertices spanning that face. For example, the figure at left in Figure 2 is degenerate because the white set covers a point on the bottom edge of the triangle.

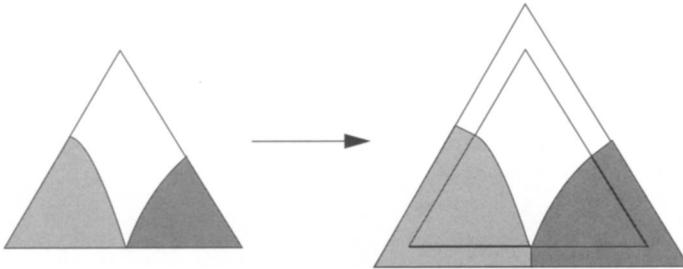


Figure 2. In these diagrams, the sets are closed and thus contain their boundaries. At left the simplex Δ^d has a degenerate KKM cover because the white (nonshaded set) covers a point on the bottom edge. At right, the same KKM cover of Δ^d has been “thickened” to form a nondegenerate KKM cover of a larger simplex Δ^d .

For the sake of contradiction, assume that there is no point common to all the sets C_1, \dots, C_{d+1} . For each i let $-C_i$ be the set in F_{bot} antipodal to C_i . Let B_i be the complement of $-C_i$ in F_{bot} . By assumption every point of F_{top} is excluded from at least one C_i . Hence the complementary sets B_i form an open cover of F_{bot} (in the relative topology). Moreover, the sets B_i satisfy a certain kind of nondegeneracy that follows from the nondegeneracy of the C_i : for x in F_{bot} , $x_i = 0$ implies that x is covered by B_i . By normality, the sets B_i can be shrunk to obtain closed subsets E_i of B_i that still cover F_{bot} and satisfy the same nondegeneracy.

Now that F_{bot} has been covered, we construct a cover of $F_{\text{top}} \cup F_{\text{mid}}$. For $x = (x_i)$ in Σ^d let $\text{pos}(x) = \sum_{x_i > 0} x_i$. Note that $\text{pos}(x) = 0$ on F_{bot} but $\text{pos}(x) > 0$ on F_{top} and F_{mid} . Define a function $f = (f_i)$ on $F_{\text{top}} \cup F_{\text{mid}}$ as follows:

$$f_i(x) = \frac{x_i + |x_i|}{2 \text{pos}(x)}.$$

Then f is a continuous function taking $F_{\text{top}} \cup F_{\text{mid}}$ to F_{top} , and it fixes F_{top} pointwise.

The set $D_i := f^{-1}(C_i)$ is a closed subset of $F_{\text{top}} \cup F_{\text{mid}}$ in the relative topology. We may think of D_i as extending the set C_i on F_{top} to cover F_{mid} . In fact, D_i extends the boundary of C_i in a linear fashion across F_{mid} (see Figure 3). We record some observations about the sets D_i :

Observation 1. *Since the C_i cover F_{top} , the D_i cover $F_{\text{top}} \cup F_{\text{mid}}$.*

Observation 2. *Since f fixes F_{top} , each D_i restricted to F_{top} is just C_i .*

Observation 3. *If x is in D_i , then $x_i > 0$.*

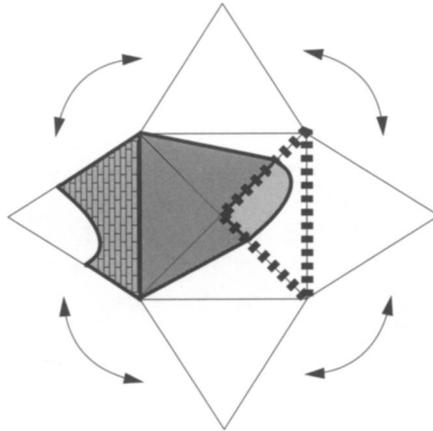


Figure 3. The octahedral 2-sphere Σ^2 unfolded, with shaded set A_i derived from a set C_i in the KKM cover. (The set A_i consists of three regions: light-shaded, dark-shaded, and bricked. The light-shaded region is C_i ; it sits in F_{top} , the triangle with dashed outline. The set D_i extends C_i and includes both the light-shaded and dark-shaded regions of A_i . The bricked region is E_i ; it sits in the facet antipodal to F_{top} . Note its relation with C_i .)

The first two observations are apparent from the definition of f , and the last follows by noting that if x is in D_i , then $f(x)$ is in C_i , so the nondegeneracy of C_i implies that $f_i(x) > 0$. But this can only occur if $x_i > 0$.

Now let $A_i = D_i \cup E_i$. We verify that the A_i cover Σ^d and are closed sets, but that no A_i contains a pair of antipodes. This verification will contradict the LSB theorem, forcing us to reject our initial assumption that the KKM cover had no common intersection point. ■

Part 2 of proof: verification. Clearly the A_i cover Σ^d . This is a consequence of Observation 1 and the fact that the E_i cover F_{bot} .

To show that A_i is closed, note that E_i is a closed subset of Σ^d and D_i is closed in $F_{\text{top}} \cup F_{\text{mid}}$ (but not necessarily Σ^d). Thus it suffices to show that any limit points of D_i in F_{bot} must lie in E_i . Observation 3 implies that a limit point x of D_i must satisfy $x_i \geq 0$, but since points in F_{bot} have no positive coordinates, a limit point of D_i in F_{bot} must satisfy $x_i = 0$. By the nondegeneracy of E_i , x must be in E_i .

To show that A_i contains no pair of antipodes, we observe that E_i cannot contain a pair of antipodes. In light of Observation 3, neither can D_i , because x_i and $-x_i$ cannot both be positive when x lies in D_i . All that remains for us to check is that there is no x in D_i such that $-x$ is in E_i . But this can only occur if x is in F_{top} . By construction C_i cannot have antipodes in E_i , so Observation 2 shows that D_i has no antipodes in

E_i . Hence the A_i form a cover of Σ^d by $d + 1$ closed sets, yet no A_i contains a pair of antipodes. This contradicts the LSB theorem. ■

Part 3 of proof: degenerate KKM covers. Finally, we consider the case where the KKM cover is degenerate. We claim that a degenerate cover of Δ^d can be made non-degenerate by “thickening” up the boundary and extending the cover in a way that introduces no new common intersection point.

Fix some $\epsilon > 0$, and let Δ'^d be a regular d -simplex that shares the same barycenter as Δ^d but is enlarged by factor $1 + \epsilon$. Note that Δ'^d is a “thickened” version of Δ^d . Given a KKM cover of Δ^d by $\{C_1, C_2, \dots, C_{d+1}\}$, we construct a KKM cover $\{C'_1, C'_2, \dots, C'_{d+1}\}$ of Δ'^d that is nondegenerate. Consider any x in $\Delta'^d \setminus \Delta^d$. The line from x to the barycenter intersects the boundary of Δ^d at a unique point, call it $r(x)$. (In fact, $r(x)$ retracts $\Delta'^d \setminus \Delta^d$ onto the boundary of Δ^d .) Let

$$C'_i = C_i \cup \{x : r(x) \in C_i \text{ and } r_i(x) > 0\},$$

where $r_i(x)$ is the i th coordinate of $r(x)$. One can check that the C'_i are closed, and by construction there are no points of $\cap C'_i$ in $\Delta'^d \setminus \Delta^d$ (see Figure 2). This “thickened,” nondegenerate cover can then be used as in the first part of this proof. ■

We remark that, although our proof of the KKM lemma is nonconstructive, the asserted KKM intersection is hiding in our construction in the following way. When we assume (falsely) that the asserted KKM intersection does not exist, we are (wrongly) led to conclude that the B_i cover the bottom facet of Σ^d . In actuality, these open sets do not cover the bottom facet; the set of points that are exposed are precisely the points whose antipodes comprise the asserted KKM intersection in the top facet.

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