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What Would the Nautilus Say? Unleashing Creativity in Mathematics!

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Synopsis

Too often mathematics is viewed by students and the general public alike as a set of formulas and techniques. However, mathematicians know that it involves so much more than banal procedures and requires deep thought and creativity. In this work we introduce an activity designed to make creative mathematical exploration accessible to young students and still interesting for those with many years of mathematical training. Popular culture often references the nautilus shell as an example of a golden spiral in nature. While many mathematicians assert this claim is false based on the formal definition, others have provided potential avenues for a relationship between the spiral created in the nautilus shell and the golden ratio. We spell out this debate and ask students to explore this question and see what patterns they can find in the nautilus shell. Might an alternative frame give the nautilus shell a golden hue or is it really just fool's gold?

Keywords: golden spiral, golden ratio, nautilus.

1. Introduction

When mathematics is presented as a set of procedures, students disengage and develop incorrect ideas about the subject and their own potential [1, 2]. An increasing body of research demonstrates that mathematics can and should be taught at lower grades in ways that appeal to students, promote equity and lead to high achievement [3, 4, 5]. Evidence from this work has shown that successful mathematics instruction promotes collaboration with students working together to solve problems, in multi-modal environments [6, 5, 7, 8]. At the cutting edge of mathematics research, success depends on whether you can make the case for your ideas through proof, argument and reasoning. The notion of right or wrong turns from a black and white concept into arguments mired in shades of gray. Creativity clearly exists in the work of research mathematicians. However, in K-12 mathematics classrooms in the US too often procedural fluency is valued over creativity to the detriment of students' mathematical growth [1]. Changing mathematics instruction to embrace creativity, problem solving, and multiple solution pathways has the potential to unlock mathematics for students [6, 1, 9].

In our work, we strive to bring creative mathematical ideas that require deep thought, conversation and reasoning to school students at a young age, so they can experience the beauty of mathematics. The question of whether the Nautilus can be defined with a golden sheen provides an illustrative example of such an opportunity. While the nautilus shell is often represented in popular culture as an example of a golden spiral, many mathematicians reject this idea [10, 11, 12, 13]. They acknowledge that it is a classic example of a logarithmic spiral but claim it does not have the growth factor Φ required to make it the special case of the logarithmic spiral traditionally deemed the golden spiral. Yet, who decides where Φ must appear for the nautilus to be 'golden?' Could an alternative frame give the nautilus shell a golden hue or is it really just fool's gold?

In this paper, we examine multiple arguments for and against classifying the nautilus as an example of a golden spiral. We also offer a semi-structured task for further investigation of this question that is appropriate for middle school students and beyond. Sometimes the most creative ideas come from those least trained in the standard practices of a discipline – recently a team of computer scientists audaciously solved the famous cake cutting problem that has eluded mathematicians for years [14]. When one is intimately familiar with a problem or topic it can be that much more difficult to see it apart from the common axioms [1], [15]. We offer this investigation as a starting point for others to unleash their own creativity as they work to understand the nautilus shell.

Our hope is that asking, "what would the nautilus say?" can allow children and adults alike to find their own answers by pushing against the walls we so often draw around mathematical questions. We believe this question can be a starting point in inspiring people to tear open the 'math box' they may be living in and see mathematics for the freeing, beautiful, creative subject that it can be.

2. So many spirals so little subtlety: Is there only one golden spiral?

If the nautilus shell has no relation to the golden spiral, as many mathematicians claim, then it is fair to ask why it remains so ubiquitous as an example of just such a spiral in popular culture? There is widespread agreement that the nautilus shell is an example of a logarithmic spiral represented by the polar equation, $r = ae^{b\theta}$, where a and b are arbitrary constants [16], [17], [18]. The golden spiral is commonly defined as a special case of a logarithmic spiral with a specific growth rate $b = \Phi$. While this appears to be the primary confusion that critics of the nautilus as a golden spiral bring up, it is worth noting that it is not the only common misstatement made with relation to the golden spiral.

The logarithmic spiral and golden spiral are sometimes treated as interchangeable terms in reputable mathematical sources [19]. Additionally, some writing about the golden spiral uses the term interchangeably with the spiral formed by Fibonacci rectangles pictured in Figure 1 [16].

While this spiral is a close approximation of the golden spiral, they are not in fact equivalent. When the actual equiangular golden spiral is laid on top of this spiral formed using the Fibonacci sequence, close inspection reveals that the two spirals are not exactly the same. The Fibonacci spiral is piecewise defined by a circular arc in each square that together form the spiral image. Figure 2 shows the actual golden spiral overlaid on the image of the Fibonacci spiral from Figure 1. For readers who are interested in digging further into this distinction, Sharp [20] provides a nice discussion and Appendix A shows an illustration of how one might construct a Fibonacci spiral.

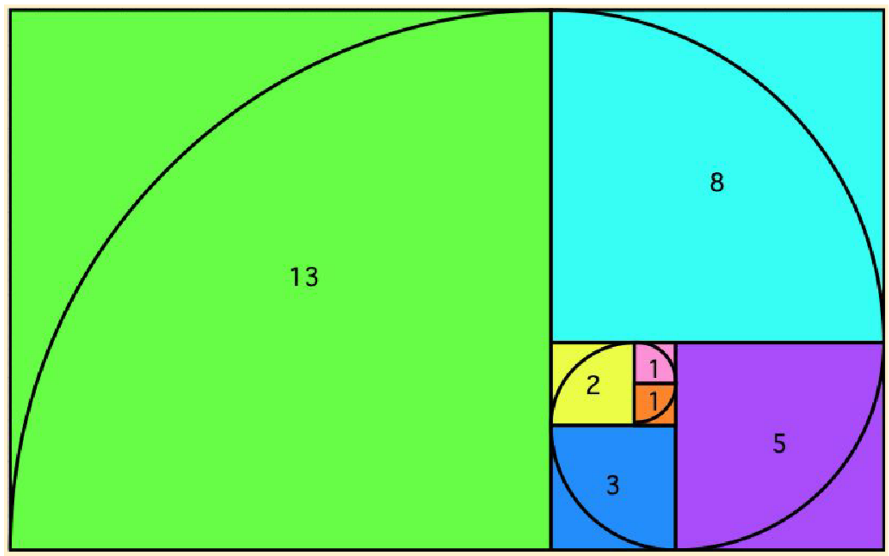

Figure 1: A common spiral formed by connecting arcs created from each square in the Fibonacci sequence.

This example of the similar spirals illustrates how we often use approximations in mathematics and sometimes generalize statements in ways that are useful if not exactly correct.

The classic form of the golden spiral, as a special case of the logarithmic spiral, is created by building out a true logarithmic spiral where the line that rotates around the origin has an exponentially increasing radius. In the logarithmic spiral deemed to be golden the growth factor is Φ.

However, other frames could be applied to spirals created with parameters related to the golden ratio. Are they also golden or does only one spiral qualify? Sharp [20] includes several alternative frames in his paper about the golden section. He mirrors the spiral to create a double barrel, inverts the curves to create sharp points, and creates curves based on golden section triangles. He even includes an example he calls the 'wobble' spiral which he initially found because of an error in the code being used to build a different image. With this plethora of spirals, he demonstrates creativity in examining different ways these spirals can be constructed. If these additional spirals might be considered golden, then is there room to consider alternative frames that could be applied to the nautilus shell?

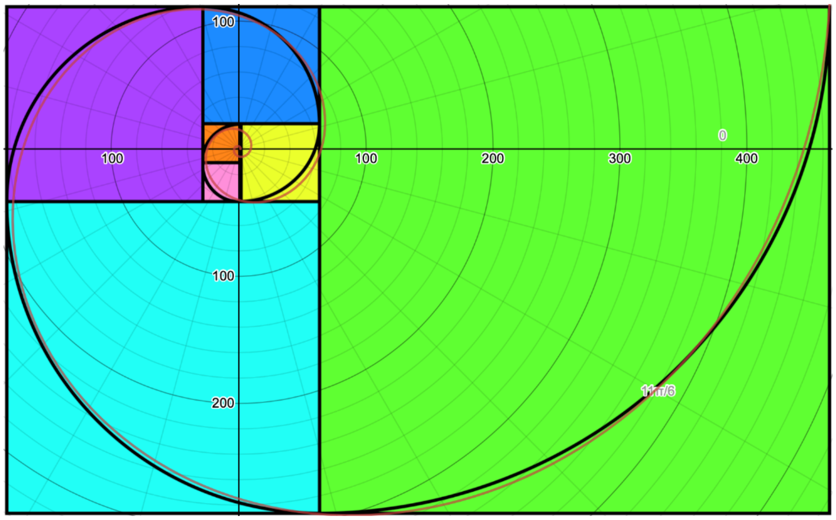

Figure 2: The Golden spiral (in red) is overlaid on an image of the Fibonacci spiral (in black).

3. Why the nautilus might be golden

Meisner [21], a strong proponent of the golden ratio, has made the case that while the nautilus shell does not fit the classic form of a golden spiral, it should not be immediately dismissed as having no relationship to the golden ratio, as some mathematicians have claimed. He used some creativity to open up the possible relationship of the golden ratio and the nautilus shell. Instead of considering a spiral in which the width of each successive chamber expands by golden ratio proportions, every 90 degrees, which is the most often used Fibonacci approximation of a golden spiral, he used 180 degrees. The spiral he created appears to be remarkably close to the spirals formed by many nautilus shells.

Inspired by Meisner's work, the second author conducted a related investigation using a shell in our possession. She measured the shell at set increments, with the tools in Figure 3, and used those measurements to build an approximate model of the growth of this shell. While not converging exactly to the golden ratio, the model at 175 degrees does appear very close to Φ as shown in Figure 4. The average ratio for the 175 degree rotation has an approximate difference of only 0.017 which is about 1% error and falls within accepted error margins. While this investigation does not prove the nautilus

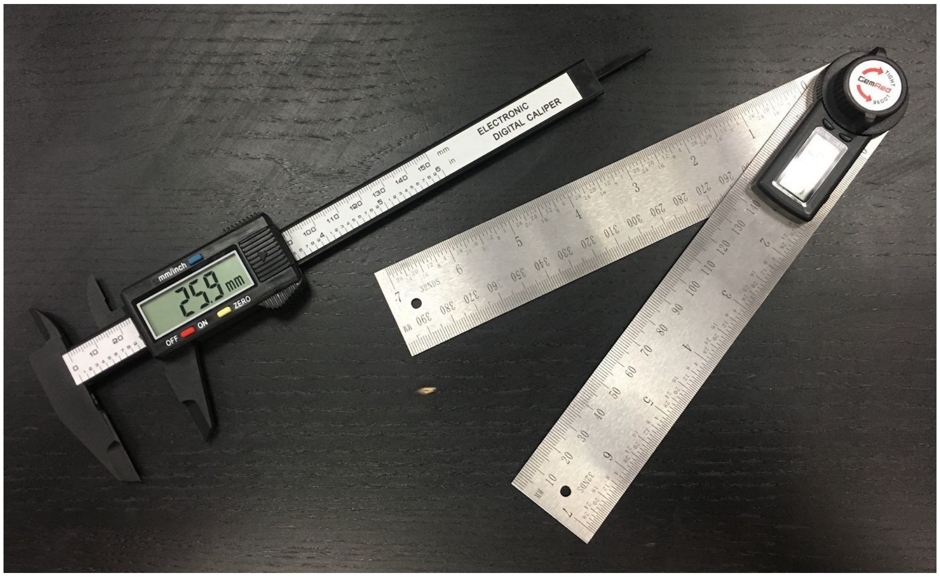

Figure 3: Digital caliper (left) and angle finder (right).

is a golden spiral it could be a starting point for discussion and is an example of the kind of investigation a student may conduct.

In the midst of examining the question of whether the golden ratio might be found in a nautilus shell, it is worth keeping in mind that these shells come from natural creatures and are not all identical. Therefore, when attempting to find math in nature one should proceed with caution, recognize variation and allow for some element of error. Among many of the mathematicians debating this idea 2% error seems to be a standard accepted metric [22], [13]. The choice, while somewhat arbitrary, is based off of accepted error margins in engineering.

One interesting case for the connection between the nautilus shell and the golden ratio was made by Barlett [22]. Using a large collection of nautilus shells he found that the mean aspect ratio of the shells was not the 4:3 ratio, around 1.333, quoted in much of the literature about the creatures but actually closer to 1.310. More relevant to the current discussion, he found that one species of nautilus was distinct from the others and had an average aspect ratio of 1.356, which is quite close to the square root of Φ and known as χ the meta-golden ratio. This species is called the crusty nautilus.

Rotation (25°)	Length of Segment (mm)	Ratio for 25° rotation	Ratio for 50° rotation	Ratio for 75° rotation	Ratio for 100° rotation	Ratio for 125° rotation	Ratio for 150° rotation	Ratio for 175° rotation
0	21.7							
1	26.2	1.20737						
2	27.9	1.06489	1.28571					
3	32.0	1.14695	1.22137	1.47465				
4	35.1	1.09688	1.25806	1.33969	1.61751			
5	37.0	1.05413	1.15625	1.32616	1.41221	1.70507		
6	38.5	1.04054	1.09687	1.20313	1.37993	1.46947	1.77419	
7	42.1	1.09351	1.13784	1.19943	1.31563	1.50896	1.60687	1.94009
8	44.0	1.04513	1.14286	1.18919	1.25356	1.37500	1.57706	1.67939
9	47.4	1.07727	1.12589	1.23117	1.28108	1.35043	1.48125	1.69892
10	51.3	1.08228	1.16591	1.21853	1.33247	1.38649	1.46154	1.60313
11	54.8	1.06823	1.15612	1.24545	1.30166	1.42338	1.48108	1.56125
12	56.3	1.02737	1.09747	1.18776	1.27955	1.33729	1.46234	1.52162
13	60.2	1.06927	1.09854	1.17349	1.27004	1.36818	1.42993	1.56364
14	65.5	1.08804	1.16341	1.19526	1.27680	1.38186	1.48864	1.55582
15	70.4	1.07481	1.16944	1.25044	1.28467	1.37232	1.48523	1.60000
16	77.1	1.09517	1.17710	1.28073	1.36945	1.40693	1.50292	1.62658
	Average	1.08324	1.16352	1.25108	1.33650	1.42378	1.52282	1.63504

Figure 4: Results of measurements and extrapolation conducted by Cathy Williams of a nautilus shell in her possession.

Clearly the most straightforward framing of a golden spiral and measurements of the nautilus shell do not match. However, when mathematicians have shown openness and creativity in their thinking, some have found potential relationships.

4. Unleashing the creativity of mathematical learners

As mathematics educators we see an opportunity for students to engage with mathematics playfully, by asking them to investigate whether the nautilus shell can be related to the golden spiral. The straightforward, primary definition most commonly used for a golden spiral clearly does not apply to the nautilus. However, as the work of others has shown, it might still possess less obvious relationships to Φ, which could be found through creative re-framing.

One of the biggest problems with many students' experiences of mathematics is the passive ways they are invited to engage. Students are taught methods to use and copy, and they are rarely invited to choose their own methods, encouraged to adapt and apply different methods, or voice their own questions and curiosities [23]. Mathematics class for many students, involves answering questions they have never asked [4]. The lack of agency students experience — with few opportunities to use their own thoughts and ideas — is a particular problem for adolescents who are entering a period of life when decision making is a basic human drive. Many students turn away from mathematics because they see no role for independent thought and no opportunity for personal agency [3, 4].

Our nautilus question (Appendix B) invites students to use their own reasoning and thinking, to make decisions and to experience agency. The task points out that there is a debate about if or how the golden ratio appears in the nautilus shell. With the questions 'do you see the golden ratio' and 'do you see other patterns' and the request to make a decision on what the nautilus says, students are given freedom to explore the nautilus shell, what ratios, patterns, and other aspects of mathematics the shell can illuminate.

The task is designed to be open, flexible, and provide ample room for creative engagement. The first page of the task includes notes for the facilitator of the task. This page is not meant for students, but to help the instructor understand the goals of the task. It also includes a recommended list of supplies to maximize the opportunities for students to work with the images provided in the task. Pages two through five are for the student: these pages include a short explanation of the debate about the nautilus, a few questions to help students get started in their investigation, and three images from nautilus shells in our collection. When doing this activity in class be sure to allocate sufficient time for play. While the exact amount of time will depend on the age of your students and the environment you teach in, creativity can not be rushed.

In Appendix C one middle school student analyzes the Nautilus 1 shell and writes the following conclusion, "We know the golden ratio is approximately 1.618034, therefore the nautilus 1 fits approximately inside the golden rectangle. (the student constructed a square around the nautilus 1 shell) However, it is not completely accurate because there are many parts that don't fit in exactly. Therefore, the Nautilus is an approximate golden ratio." The student has made a number of choices in their framing of the problem. One may not agree with the conclusion the student reaches and could challenge their definitions, but this middle school student has made choices and from this work the teacher could facilitate a discussion of ideas students had formed during their engagement with the task.

In many cases it is the experts in the field who have the hardest time looking past their own preconceived notions [24, 25]. It is challenging to think creatively in mathematics when you see mathematics as a narrow collection of rules and procedures. When mathematics is presented as rules and definitions during lecture, students are not offered the opportunity to apply their creativity. Furthermore, they are not familiar with possible, allowable tools to play with mathematical situations that are open and flexible, similar to our task. Keith Devlin, a well-respected mathematician, recently shared a list of his go-to tools for working on a mathematics task [26]. These tools are far from what students think to use.

Devlin's list shows collaboration tools like email, LinkedIn, and Twitter; calculation tools like Wolfram Alpha, Matlab, and Mathematica; and learning tools like Wikipedia, google and youtube. The last of the 13 tools that Devlin shares is a pencil and paper. This is the first tool that most of our students consider. This is another indication that the nature of mathematical work performed by mathematicians often does not reflect the opportunities students are provided to engage with the subject in school. To see Devlin explain, check out this video on the youcubed website.

As digital calipers and angle measures are common instruments found on the desks of engineers (Figure 3), there is room in mathematics classrooms for similar instruments to engage in mathematics. In addition to their first customary tool of the paper and pencil, student engagement would increase if they were encouraged to use technology and other mathematical aids to assist in problem solving. While the mathematics classroom is a distinct space, many students will eventually enter spaces where they are using tools for problem solving in mathematics, engineering, and other disciplines so they should learn the flexible thinking that such tool use requires.

5. Concluding Remarks

We have presented variations of this task to a range of audiences including students in grades 4 through 12, elementary and secondary school teachers, and college professors. We have found all audiences have engaged in different ways. With so many avenues for investigation, this task has a low floor and high ceiling [4, 1].

Investigations can start with just drawing a rectangle around the shell and examining the ratio between the sides. Some participants have focused on the angles between the shell supports. They have asked relevant mathematical questions like how to find the area of a chamber, which relates to topical subjects such as finding the area of curved spaces in calculus courses.

When possible, we have provided participants with digital calipers and angle measurement devices to assist their investigations. Some participants have used the software program Desmos to examine different spirals and as a digital measurement tool.

Included in Appendix C and D are some student work samples from middle school students who worked on the task. The work in Appendix D includes not only numbers but is colored in ways that highlight the mathematics and gives artistic flair. One student wrote a note saying that they admire the Fibonacci sequence and the ways the shell follows a similar pattern.

For anyone who is new to this type of mathematical engagement we hope you will take the time to play and explore. If you are a teacher, we encourage you to try the task yourself and to seek creative pathways and multiple answers. We very deliberately do not provide an answer key for this task. The 'right' answer is up to students and the defense they provide for their solution. Many adventures can be had in answering the question: What would the nautilus say?

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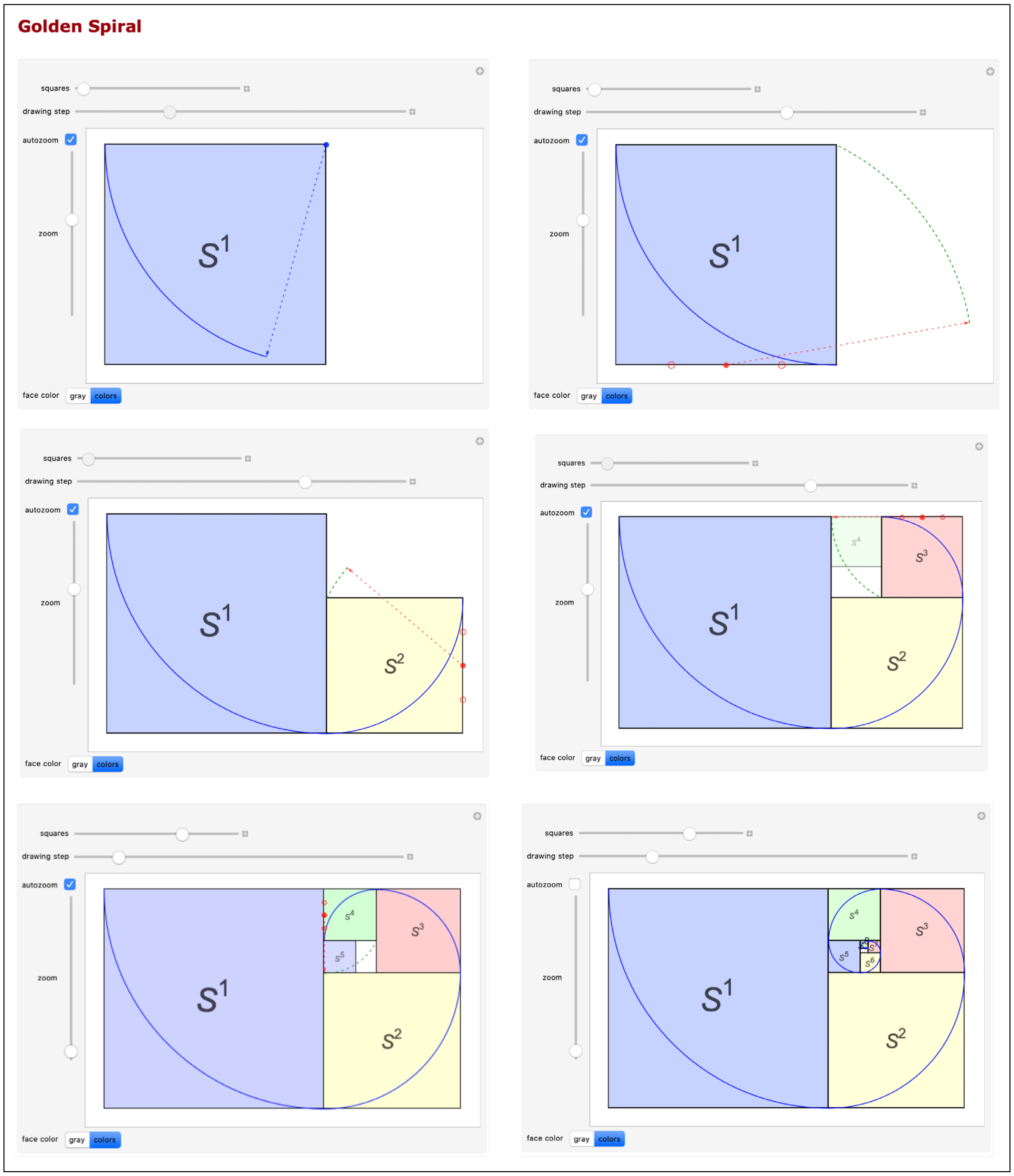
A. Constructing a Fibonacci Spiral



B. The Nautilus Task Description and Handout

What does the Nautilus say?

Logarithmic, golden or something else?

To the teacher: While the nautilus shell is often represented in popular culture as an example of a golden spiral, according to many mathematicians it is not. They acknowledge that it is a classic example of a logarithmic spiral, but claim it does not have the growth factor ϕ (1.618...) required to make it the special case of the logarithmic spiral traditionally deemed the golden spiral. We ask, could an alternative frame or change in axioms give the nautilus shell a golden hue? What other patterns might be hiding in it's shell?

Directions: Using the images below of three different nautilus shells, a calculator/graphing program (desmos, geogebra,..), ruler, protractor, or any other tools you might find useful to make your argument. Is the nautilus shell an example of a golden spiral or just fools gold?

Supply List:

- Images of nautilus shell
- Ruler
- Protractor
- Other optional tools: compass, digital calipers, digital angle finder (protractor), tape measure, string, graph paper
- Any other tools you think might be useful

Directions for the instructor.

What does the Nautilus say?

Logarithmic, golden or something else?

The nautilus shell is often represented in popular culture as an example of a golden spiral, but what does this actually mean. There is currently a debate about if or how the golden ratio appears in the nautilus shell. Using the attached images of real nautilus shells, take measurements and look for patterns. Do you see the golden ratio, ϕ, $\frac{1+\sqrt{5}}{2}$?

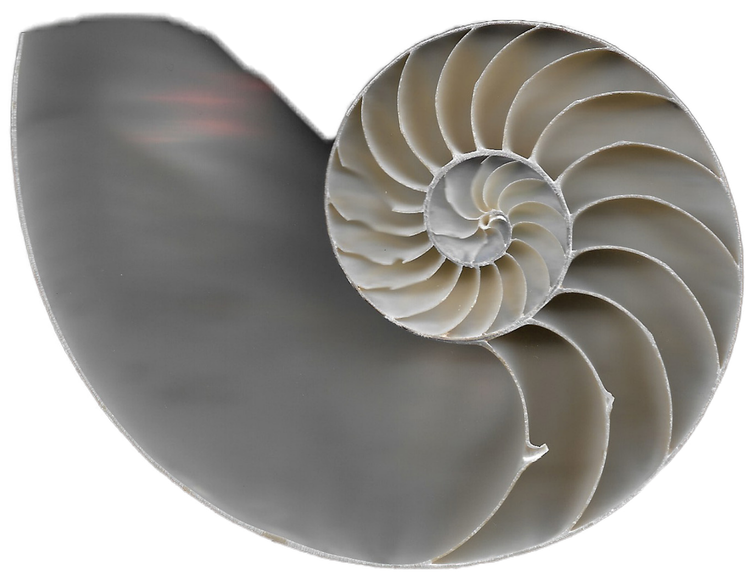
Do you see other patterns? You decide what the nautilus says. Be prepared to justify your conjecture(s).

Student handout with directions.

The first nautilus.

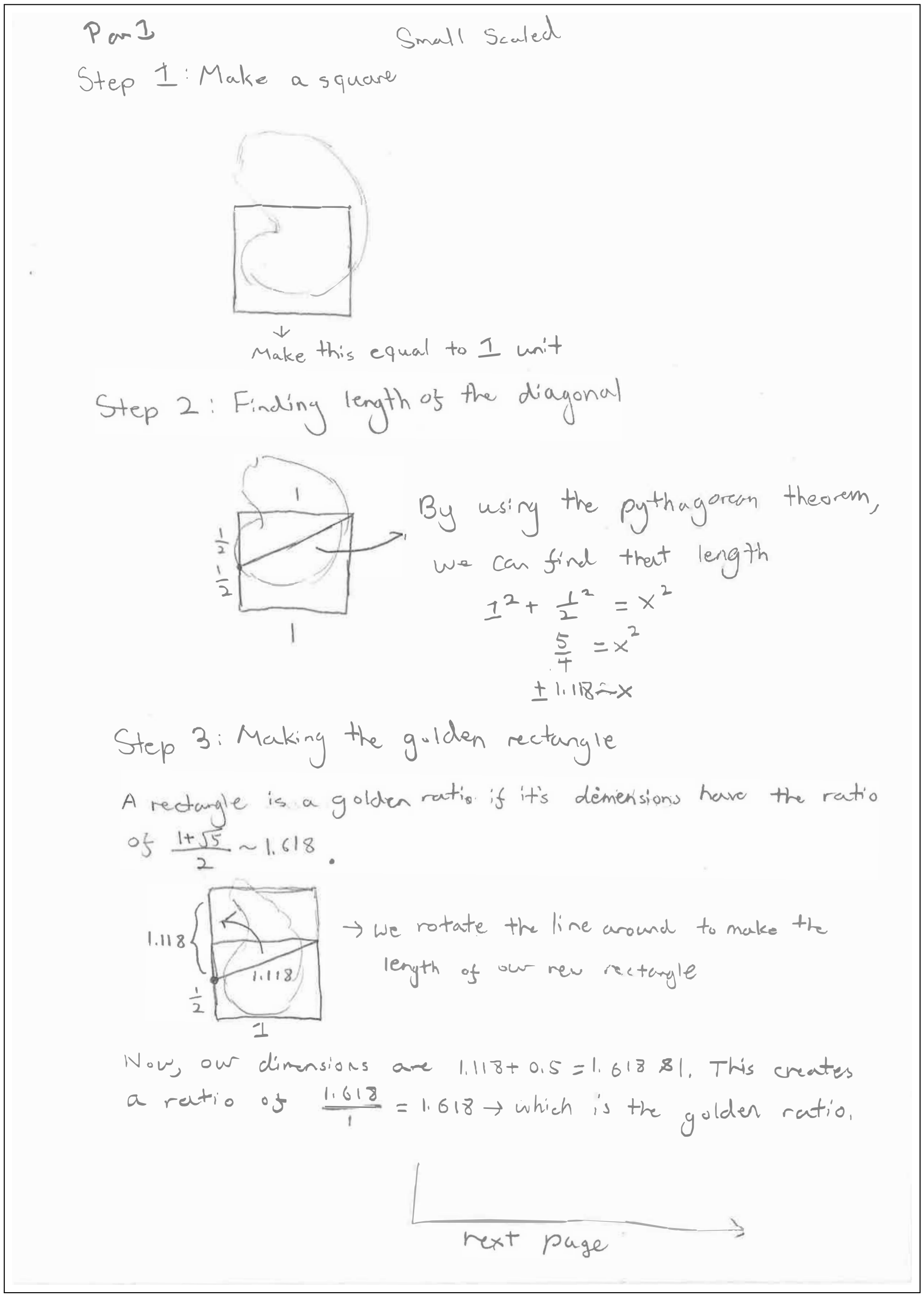
The second nautilus.

Nautilus 3



The third nautilus.

C. A Middle School Student's Work on the Nautilus Task



Part 1 Small Scaled

Step 1: Make a square

Make this equal to 1 unit

Step 2: Finding length of the diagonal

By using the pythagorean theorem, we can find that length

$1^2 + \frac{1}{2}^2 = x^2$

$\frac{5}{4} = x^2$

$\pm 1.118 \approx x$

Step 3: Making the golden rectangle

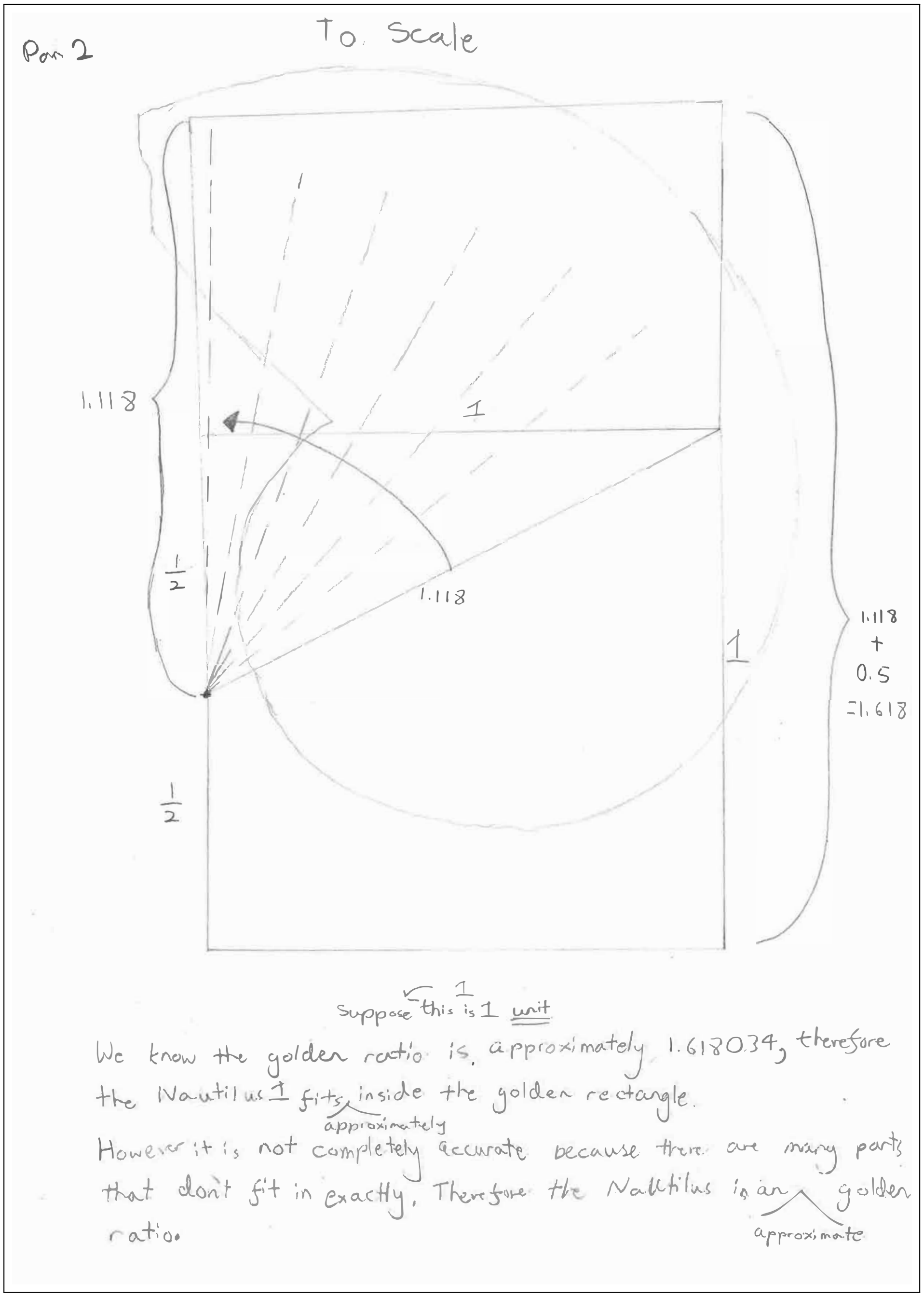
A rectangle is a golden ratio if it's dimensions have the ratio of $\frac{1+\sqrt{5}}{2} \sim 1.618$.

→ We rotate the line around to make the length of our new rectangle

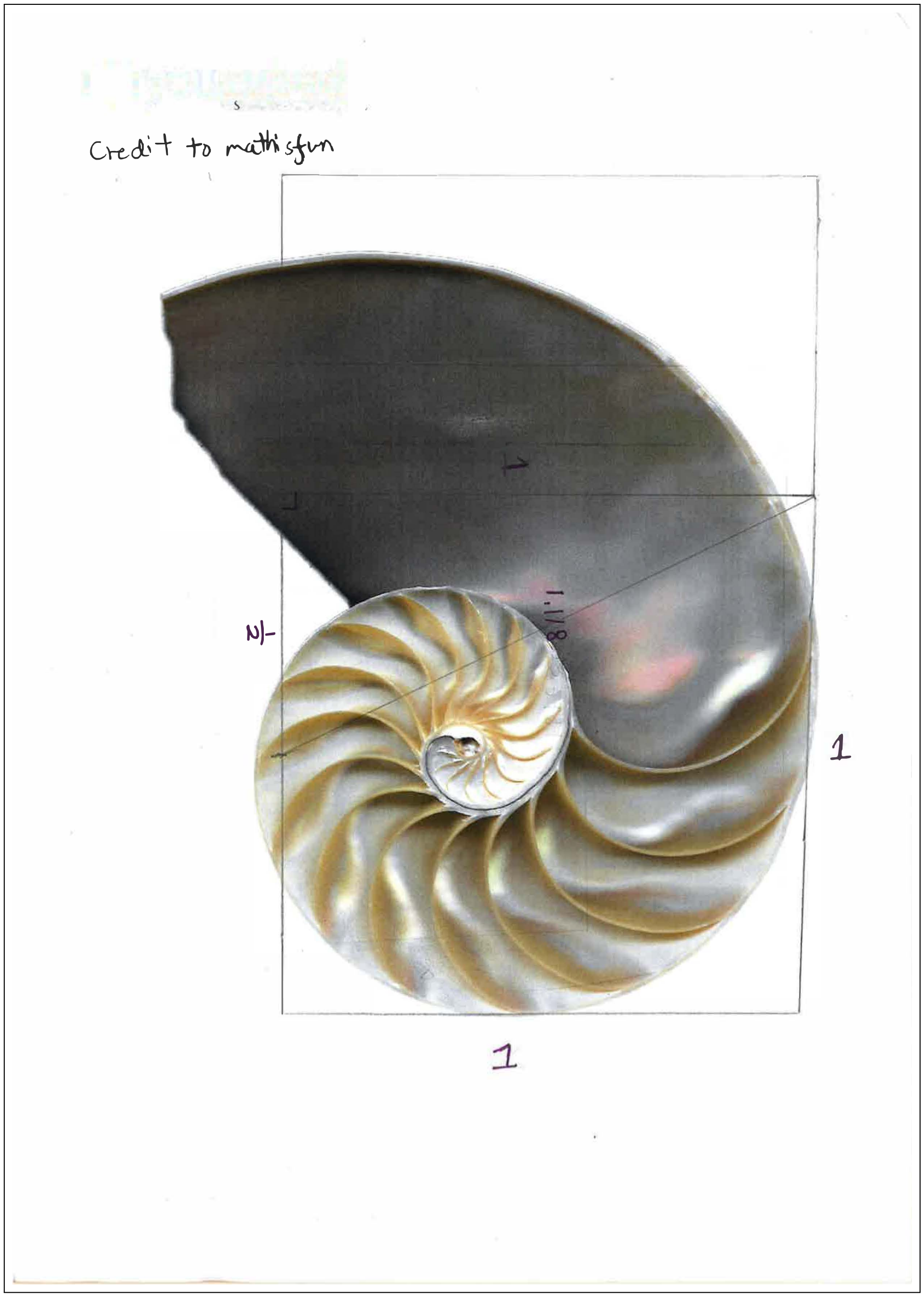
Now, our dimensions are $1.118 + 0.5 = 1.618$ 81. This creates a ratio of $\frac{1.618}{1} = 1.618$ → which is the golden ratio.

next page

Page 1.


Pan 2
To Scale
1.118
1
1/2
1.118
1
1.118
+
0.5
=1.618
1/2
suppose this is 1 unit
We know the golden ratio is approximately 1.618034, therefore the Nautilus 1 fits approximately inside the golden rectangle.
However it is not completely accurate because there are many parts that don't fit in exactly. Therefore the Nautilus is an approximate golden ratio.

Page 2.



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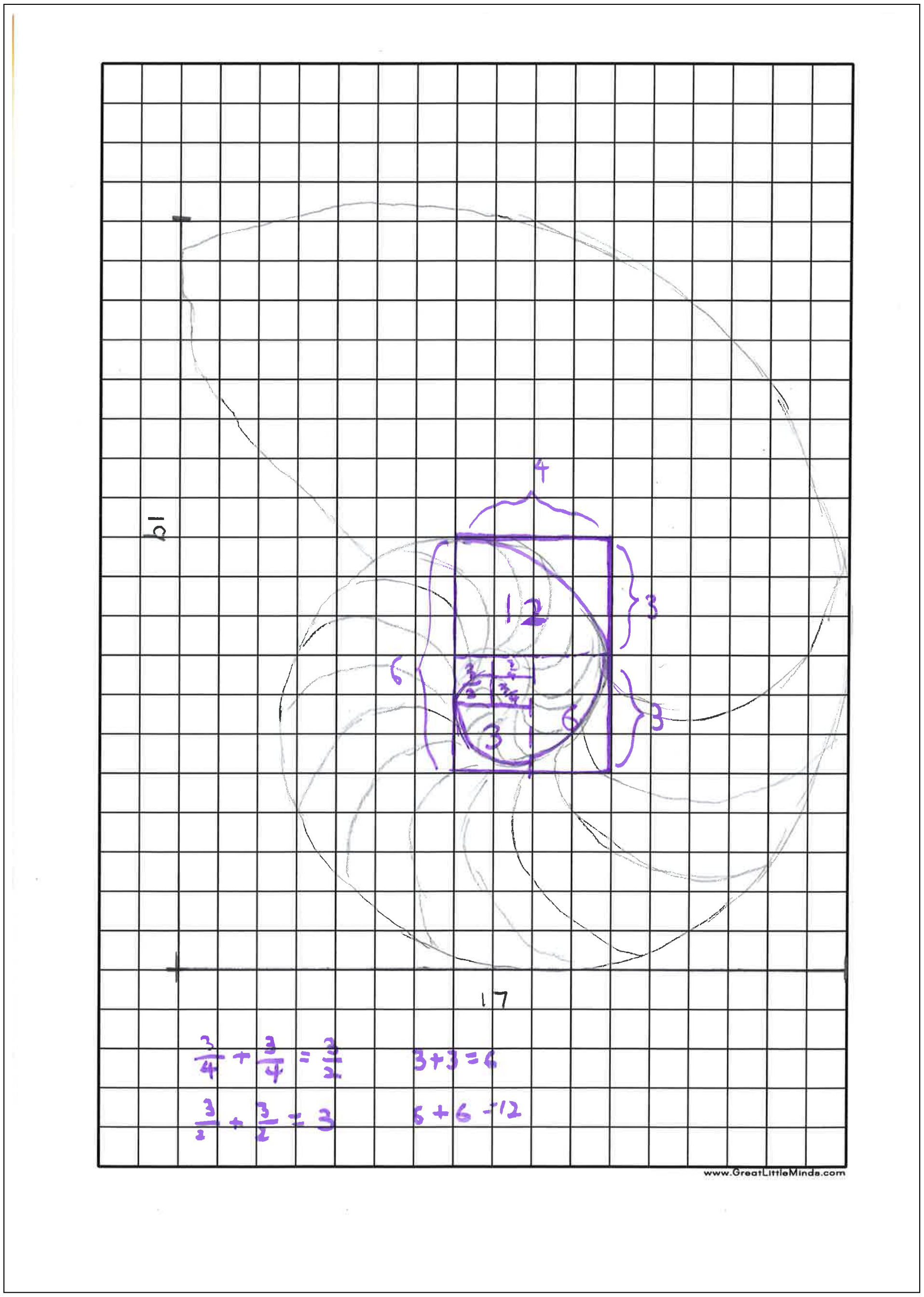
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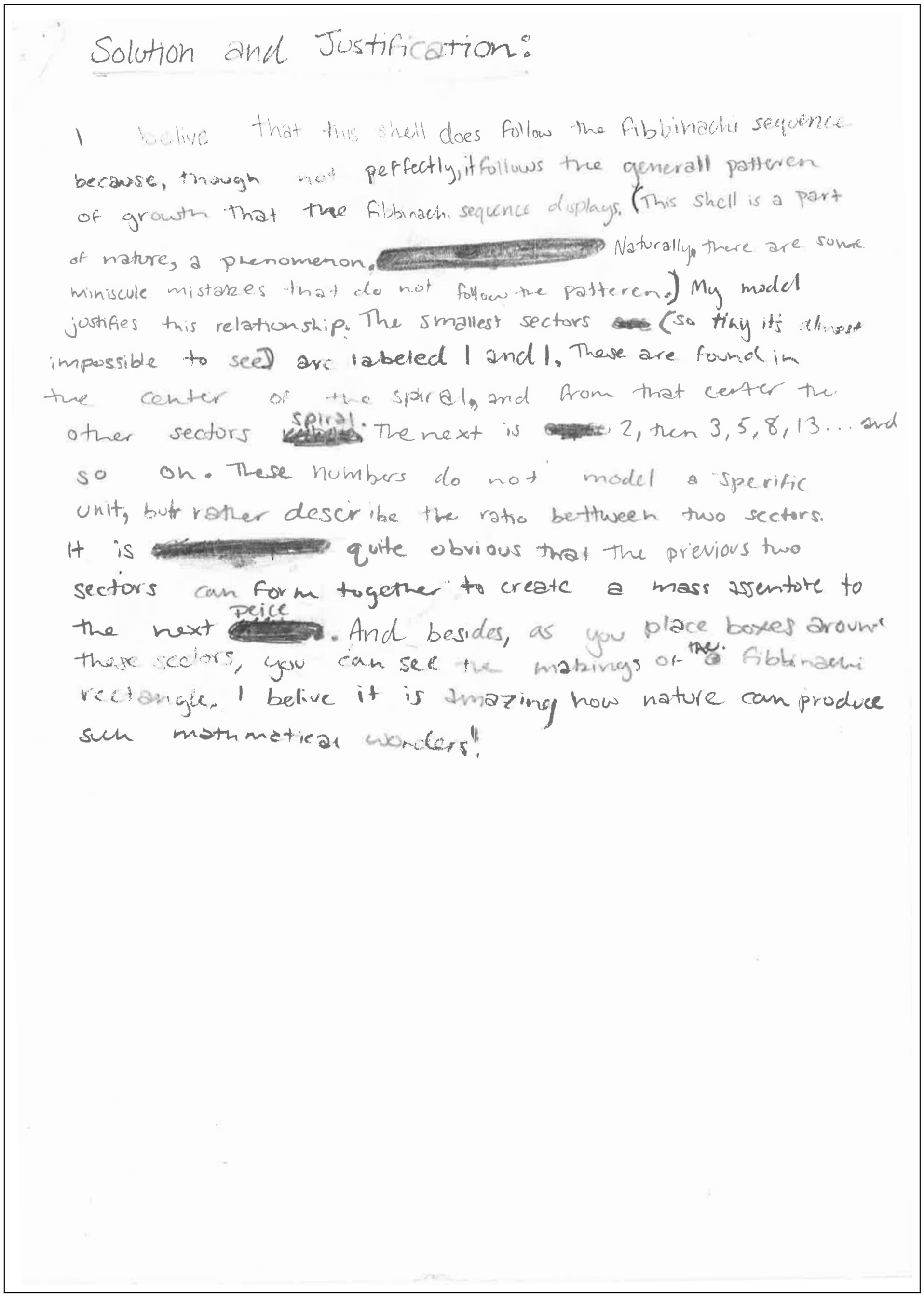
www.GreatLittleMinds.com

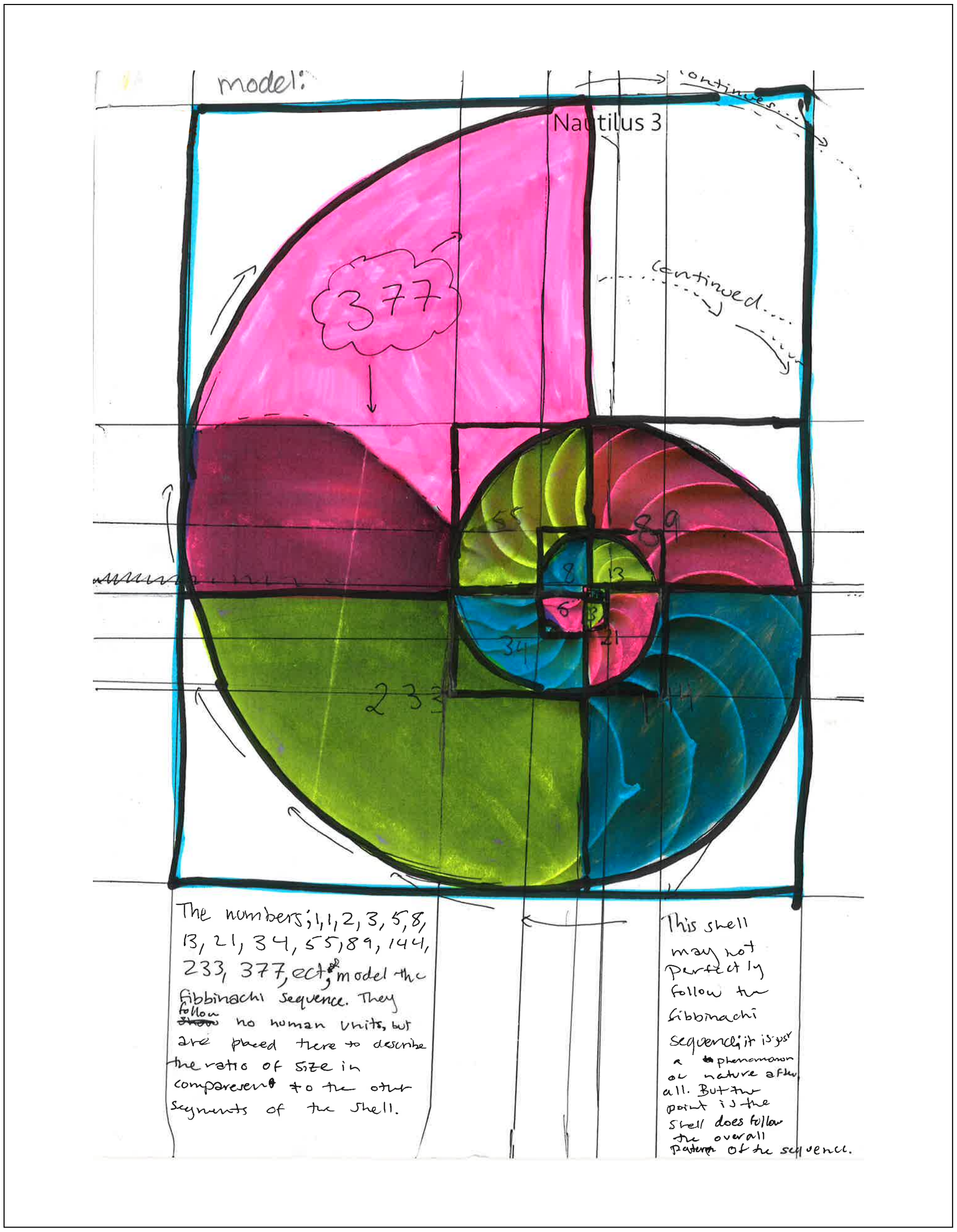
Page 4.

D. Another Middle School Student's Work on the Nautilus Task


Solution and Justification:

I belive that this shell does follow the fibbinachi sequence because, though not perfectly, it follows the generall patteren of growth that the fibbinachi sequence displays. (This shell is a part of nature, a phenomenon. Naturally, there are some miniscule mistakes that do not follow the patteren.) My model justifies this relationship. The smallest sectors (so tiny it's almost impossible to see) are labeled 1 and 1. These are found in the center of the spiral, and from that center the other sectors spiral. The next is 2, then 3, 5, 8, 13... and so on. These numbers do not model a sperific unit, but rather describe the ratio betttween two sectors. It is quite obvious that the previous two sectors can form together to create a mass assentote to the next peice. And besides, as you place boxes around these sectors, you can see the makings of the fibbinachi rectangle. I belive it is amazing how nature can produce such mathmetical wonders!

Page 1.


model:
Nautilus 3
continued....
377
55
89
8
13
21
34
233
144
The numbers; 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ect; model the fibbinachi sequence. They follow no human units, but are placed there to describe the ratio of size in comparesent to the other segments of the shell.
This shell may not perfectly follow the fibbinachi sequence; it is just a phenomonon of nature after all. But the point is the shell does follow the overall pattern of the sequence.

Page 2.