The Emergence of Creativity: Insights from Carnatic Raaga Improvisation and Mathematical Proof Generation

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Abstract

Creativity is a broad phenomenon that scholars have interpreted in a multitude of ways. We notice that a majority of the views describe creativity as something innate. This paper aims to verge from this perspective and explore creativity in terms of the constant mutual interaction of a person and their environment. Using the theoretical framework, enactivism, and the notion of emergence, we investigate the creative processes involved in musical improvisations of south Indian classical or Carnatic music and mathematical proof generation. Interview excerpts from professional Carnatic musicians and research mathematicians on their respective creative processes during musical improvisation and proof generation are analyzed. This study gives a perspective to think about creativity, with an emphasis on emergence. This paper has been partly informed by self-reflections on musical improvisations and mathematical proof generation by the first author, who is a performing Carnatic vocalist and a mathematician.

1. Introduction

Creativity plays an important role in the development of new knowledge in any human endeavor. The definition of creativity, what it is or how to measure it, however, is elusive. One distinction is that there seems to be an over-reliance on creativity associated with the human mind [1]. That is, creativity is often seen as a personal trait or an attribute that one is born with. We usually find studies about the *person* behind a creative act, acknowledged as being *talented* or *gifted* [2] [3]. Less frequently, we find creativity framed in terms of a person's social milieu, like being raised in a certain family environment. For example, socio-culturalists argue that creativity ("imagination" in Vygotsky's terms) develops through the interiorization of cultural tools available to an individual [4]. There are many other challenges in describing creativity or coming to terms with what it could mean in different situations. For example, creativity is often times judged and measured using a finished creative product rather than the creative process underlying it [5]. Is there a way to conceptualize creativity without having to think about the creative product and, if so, what might that look like?

In order to approach these difficult questions and understand creativity to a greater extent, we suggest that emergence might be an important concept to explore. Emergence, according to philosopher George Henry Lewes [6], is an effect that is unpredictable and arises out of the combined agencies of its interactants, but in a form that does not display these interactants [7]. In other words, an emergent effect is neither predictable nor decomposable into its components. Situating creativity in the idea of emergence helps in decentering the human agency involved, thereby providing equal emphasis on other influential factors involved in the creative process. Also, thinking about creativity in terms of the creative process, instead of the finished creative product sheds light on the development trajectory of the product, which is otherwise often hidden if only looking at the product. Put differently, the improvisational aspect of creativity, where the creative process *is* the product, brings forth a perspective that is otherwise concealed when conceptualizing creativity from the product perspective alone [5].

We suggest that the notion of emergence has great potential in informing us about what creativity entails. In this paper, we look at the improvisations in South Indian Classical or Carnatic music and mathematical proof generation. The questions we ask include: What do we notice as emerging in the creative

processes in musical improvisations of Carnatic musicians and proof generation of mathematicians? And, what factors does this emergence depend on, in the creative processes underlying these seemingly disparate streams of knowledge and what might this comparison inform about creativity, in general?

Here, we analyze personal interviews, newspaper interviews and information from autobiographies from both professional Carnatic musicians and mathematicians, to find instances of their respective creative processes. We use enactivism, a modern philosophical framework on interaction as a way to look at creativity as emergence. We begin with a brief exposition of enactivism and its main tenets, to give the reader a theoretical lens to engage with the paper.

2. Enactivism & Emergence

Moving on from the constructivist approaches where the primary emphasis is on knowledge rather than knowing, enactivism was developed to bring forth the idea of the inseparability of the individual and the world [8], [9]. The term *enactive* was coined in the 1992 seminal book *Tree of Knowledge* written by Fransisco Varela and Humberto Maturana [10], to bring forth the view of knowledge not as existing "in any place or form, but enacted in particular situations" [11]. Enactivism is a theory of cognition that comes from a biological viewpoint and posits that *knowing* neither represents grasping knowledge from the outside world nor constructing knowledge inside our minds, but is a result of interactions of an individual with the environment. It further explains that since each individual's personal history and experiences are different, the outcomes of their interactions with the environment will be quite different. In other words, the outcomes of the mutual interaction an organism and its environment are not predetermined; rather, they emerge real-time as a product of the interaction, directly and continually influenced by both [12]. This sits perfectly well with the description of emergence, as proposed earlier, as an effect that is not additive in nature, but a phenomenon that surfaces due to the combined agencies of its interactants. The essence of enactivism is rooted in the unfolding of unpredictable outcomes, the cause of which cannot be traced back to the properties of either the organism or the environment. Enactivism is grounded in Darwin's theory of biological evolution and associated concepts like *structural coupling* and *structural determinism* are discussed below.

2.1. Structural Coupling

The concept of structural coupling is rooted in Darwin's conception of how organisms and environment co-evolve to adapt and become compatible to each other [10]. Enactivism adopts the idea of structural coupling to explain how an organism and its environment co-adapt and mutually influence each other during the process of interacting. This then means that, both the organism and its environment go through constant transformations in their structures, as a result of their interaction with each other. In order to understand what a transformation of structure means, let us take a mathematical problem-solving scenario. For the person, transformations of their structure may occur starting from their way of interpreting the problem, such as making certain assumptions to gain a specific perspective or encountering an emotion or emotions along the way. In terms of the problem, the transformations will be nothing but its continuous morphing into a different mathematical statement at each step of the solution pathway. From an enactivist lens, structural coupling can be seen by watching the development of the solution. The process of solving evolves as the solver and the problem undergo transformations at every step, constantly reshaping each other due to their mutual interaction, to give rise to the final product, which we then call a *solution.* From this notion of structural coupling, we can say that the outcomes of the interaction are highly dependent on both the environment and the organism. This leads us to another important concept of enactivism, namely structural determinism.

2.2. Structural Determinism

The notion of structural determinism addresses the importance of the structural makeup of an organism that guides its transformations and changes [10]. Here, structural makeup not only refers to the physical genetic disposition of an organism, but a collective of the physical, psychological, neurological and sociological make-up of the being. Thus, two people trying to solve a problem, which can be thought of as a *disturbance* from the environment, would engage with it differently based on their structural makeup and their interaction with the problem. This constraint and limitation possessed by each organism is precisely called structural determinism, in enactivist terms. Thus, the interaction between each organism and its environment, is guided very much by the organism's structured tendencies and selections of making sense and understanding, which Varela [13] referred to as "problem posing".

As an emergent effect arises from the combined agencies of its interacting components, structural coupling and structural determinism help us in making sense of what those agencies of the components are. Specifically, though emergence is not predictable from the knowledge of its components and their agencies, we can at least understand what each component brings with itself and how each of these components is transforming, as they all interact. In the following sections, we situate this idea of emergence due to mutual interaction in Carnatic musical improvisation and mathematical proof generation. To give the readers a context, some of the central ideas related to improvisations in Carnatic music are explained briefly, which are then analogously correlated to themes in mathematical proof generation.

3. Methods

Before venturing into the creative processes in Carnatic music and mathematical proof generation, we furnish the methodology adopted to gather and analyze data in this study. A variety of qualitative approaches were employed to gather data for this study. These included conducting personal/email interviews with Carnatic musicians and research mathematicians, looking for data in newspaper interview excerpts of expert musicians and mathematicians and analyzing an autobiographical account for each category. Two professional Carnatic musicians, Swami (vocalist) and Jai (violinist) based out of Chennai, India were interviewed over the phone and two professional research mathematicians, Joe (algebraic topologist) and Kim (mathematics historian and topologist) working at two reputed universities in Canada were interviewed in person/email. All the four participants interviewed have had rigorous training with the fundamentals and are a part of their respective fields for more than 15 years. Apart from this resource, few relevant interview excerpts of two expert musicians and two mathematicians for newspaper articles, magazines and some snippets from an autobiographical account for both categories were collected, to add credibility. In order to maintain the anonymity and privacy of the personally interviewed participants, their names mentioned here are pseudonyms. Table 1 summarizes the data resources used for this study.

The four participants, who were interviewed, were asked a similar set of questions (see Appendix A) regarding their creative processes involved in musical improvisation or proof generation respectively. The questions focused mainly

Name and Specialization	Data collected from
Swami, Carnatic vocalist	In-person/Phone/Email interviews
Jai, Carnatic violinist	
Joe, Algebraic topologist	
Kim, Math historian and topologist	
Kunnakudi M. Balamuralikrishna, Carnatic Vocalist	Newspaper/Newsletter/Magazine interviews
K V Narayanaswamy (KVN), Carnatic Vocalist (late)	
Andrew Wiles, Number theorist	
Maryam Mirzakhani, Topologist (late)	
T M Krishna, Carnatic Vocalist	Autobiographical account
Henri Poincare, Mathematician and theoretical physicist	

Table 1: Data resources for the study.

on the process and little emphasis was given to products. However, the participants shared their view of the product as well as a way to background the process. The gathered data was then analyzed to find traces of the notion of emergence in creativity and then discussed using enactivism. We proceed from here by giving a brief overview of Carnatic music for our readers.

4. Raagas of Carnatic Music

Carnatic music is one of the two major classical music genres of India, specifically belonging to the Southern Peninsula. Its origin dates back to the 13th Century C.E. and Purandaradasa, the saint-singer of the period, is considered the father of Carnatic music [14]. Carnatic music has an oral tradition, usually performed by a vocalist, accompanied by a Mridangam (percussion) artist and a violinist for melodic support, although its development has a close association with *Veena*, the celebrated string instrument of south India. Until early 17th century, Veena was commonly used as the accompanying melodic instrument for vocal concerts. But, as soon as violin was adapted from the West into the Carnatic musical discourse, it took over Veena mainly because of the flexibility it offered to play this type of music and also because of its easy portability to concerts. While Carnatic music came to be identified with south India, the intermingling of the Hindu and Persian cultures in the north resulted in the other main classical genre called *Hindustani music*. Though both these classical music forms of India stem from similar

fundamental concepts and are highly improvisational in nature, there is considerable difference in their aesthetics and improvisational techniques. One of the main factors that differentiates them is the compositional aspect of Carnatic music, which we take up for further elaboration later.

In music, an octave is defined as the interval between one musical pitch and another that is double its frequency. In Carnatic music, an octave contains a series of seven musical notes, called *Swaras*. But by themselves, these seven tonal steps do not reveal the finer nuances of intonation. And so, in order to obtain subtle nuances and semitones, certain musical notes are modified to obtain minute variations. The basic tonic (S) and the perfect fifth (P) do not allow any variations and the remaining five notes allow for two variations each. Thus, in the present-day Carnatic system, twelve *swarasthanas* (semitones) constitute an octave (although it is commonly held that an octave allowed for 22 semitones during ancient times). Also, the basic tonic, S, of Carnatic music, is not tied to any particular frequency and so for any frequency there opens up an octave, with its twelve divisions. Table 2 shows the twelve swarasthanas/semitone division of Carnatic swaras in an octave with their mathematical proportions.

Carnatic Swarasthanas	Mathematical ratio
S	1
R1	16/15
R2	9/8
G1	6/5
G2	5/4
M1	4/3
M2	7/5
P	3/2
D1	8/5
D2	5/3
N1	9/5
N2	15/8
S	2

Table 2: Carnatic swarasthanas in an octave with their respective mathematical ratios.

One of the most distinguishing and outstanding aspects of Carnatic music is the *Raaga* system. A raaga can be conceived as a musical space generated by a collection of swaras, with certain constraints. Every raaga must contain the basic tonic S and at least three other notes of the twelve available. However, raagas entail much more than a collection of notes and are characterized as well by certain note oscillations, sophisticated pathways and distinct emotive quality. The most important quality of a raaga is to please and they are almost like musical organisms which have their own birth, growth and in some cases, death as well [15]. For instance, raagas that existed a few hundred years before are clearly forgotten today and are extinct from the musical discourse. This may be indicative of the fact that the aesthetic value of raagas keeps changing with time, unless, for example, they are reinvented by musicians and composers. The specific characteristics and the key musical phrases of a raaga play an important role in bringing out their emotive essence, much like the properties that give meaning to a mathematical object defined in a certain mathematical space. For example, a musical note, say R, might evoke different emotions when played in the context of different raagas, much like how the meaning of an open set changes with the kind of mathematical space one is working with. Since Carnatic music relies more on melodic improvisations than harmonic progressions, the oscillations called *gamakas* play an important role in giving a specific flavor to a raaga. That is, even if two raagas have the same set of musical notes, the intensity with which the oscillations are performed gives them a completely different color and aural texture. Therefore, in order to get a deep understanding of a raaga with its characteristics and movement pathways, one needs to engage with it, play with its notes and get a feel for its essence. Many raagas have evolved over a long time and composing musical forms in different raagas by great composers has been one of the most important strategies to develop key phrases for their further evolution. For example, almost all of the Carnatic composers have composed songs in the raaga Todi, which is considered to be one of the most popular raaga of all times. Thyagaraja, the 17th-century composer, alone has composed about 32 compositions in this raaga with each composition starting at every single note of the three octaves.

A very interesting aspect of Carnatic music, which differentiates it from fellow improvisational genres like Hindustani and jazz, is the fact that it is as compositional as it is improvisational. That is, raagas are not just introduced as scales and sounds; rather, they are introduced through composi-

tions, which contain their characteristic phrases. Chitravina N. Ravikiran, a popular Carnatic musician, in his short paper on improvisations in Carnatic music [16], mentioned that compositions play an important role in Carnatic music teaching, learning and performance. With each composition, one understands something new about a raaga, regardless of however minor it is and gets better at understanding and accumulating by repertoire of its key phrases.

Each raaga comes with well-defined boundaries and specific grammatical notions, much like a mathematical space, with well-defined objects, rules and limitations. Students are initially taught compositions in different raagas so they can acquaint themselves to the mood, aesthetics and boundaries of each raaga. As one gets more grounding on the key ideas of a raaga accumulated from various compositions, the raw materials such as characteristic/rare musical phrases, general mood of the raaga required for the ability to improvise in the space of a raaga develops. Carnatic music thus has a very interesting balance between learning the motifs or key ideas of a raaga on one hand and improvising on them on the other. It is in this tension, that a learner acquires the freedom to improvise, within the constraints of raagas. A 1986 interview excerpt of the yesteryear celebrated vocalist, KVN, in Sruti Magazine [17] exemplifies the importance of learning compositions in his approach to improvise.

> Mani Iyer would ask me to sing the same song many times. The beauty of this was, when you practiced the song over and over again, you got new ideas every time you sang; you developed fresh sangatis (improvised phrases), enhanced the rendition with each attempt.
>
> KVN [17]

There are a few different kinds of improvisation in Carnatic music and understanding of the basic structure of a raaga is imperative to all of these. Some of them are purely sonic, while some of them are lyrical and rhythmic and some include improvisation on swaras. Of these, the pure sonic and melodic improvisation aspect of Carnatic music, unrestricted by rhythm and lyrics, is called *raaga alapana.* Improvising on the abstract sound, by expressing successive musical phrases that bring out a raaga's essence and aesthetics, marks the complexity of the creative process involved in an alapana. A deep understanding to touch a raaga's soul in the abstract sound, beyond its key

phrases and concrete musical notes, is central to any kind of improvisation. Thus, raaga alapana could be considered as one of the central areas, primary to all kinds of improvisation and could be used as a measure to judge one's understanding of a raaga. Kunnakudi M. Balamuralikrishna, a contemporary musician, in an interview for Indian Express [18], shared his love for raaga singing and touched upon how raaga singing might be the most challenging aspect of Carnatic music.

> I am more into raaga singing. I would like to explore them in detail. In a regular two-hour concert, my main raaga should be at least 15-20 minutes. If somebody could sing raagas in a sampradayik (traditional) style without sounding similar, also not by trying anything "eccentric" why can't we do it now? How did the masters do it? That's how I started exploring raagas. It's about understanding how much this music can offer you without ever going beyond certain boundaries.
>
> Kunnakudi M. Balamuralikrishna [18]

In this paper, we discuss more about this improvisational aspect called raaga alapana and compare it with the inner workings of a mathematical proof generation. With this background information about Carnatic music and its improvisational aspects, let us now see what a mathematical proof entails.

5. Mathematical Proof and Proving

The words *proof* and *proving* are arguably two of the most important words associated with the discipline of mathematics. Though we might think that proofs are the building blocks of mathematics, a peek into the history of proofs reveals that prior to Thales (600 BCE), mathematics was done without proofs [19]. Moreover, the act of proving was executed via different methodologies by different people across the globe.

> Giving an elaborated proof of something that everybody knew, determining precisely the circumstances under which it holds, this was not necessarily a part of the job of a mathematician.
>
> Kim, practicing mathematics historian & topologist

For example, Euclid primarily expressed his proofs through words, whereas in the Chinese and Indian tradition there was extensive usage of visual diagrams

[20], [19]. The formalist view of proof was developed much later, in the early twentieth century, where proofs meant absolute rigor and certainty. Formalization of proofs in actuality means building truths on theorems, axioms and definitions established beforehand, which are considered infallible in principle [21]. However, researchers like Imre Lakatos [22] criticized the reduction of mathematics to formal logic. In his book, *Proofs and Refutations*, he pointed out that the processes involved in the generation of proofs in actual mathematical practice, is almost entirely different from the structure of Euclidean formal proofs that is commonly seen in textbooks and journals. In fact, he argued that mathematicians begin with conjectures and not with axioms and definitions. This view is supported by an interview excerpt gathered from Joe, an algebraic topologist at a reputed university in Canada.

> A lot of creative activity in mathematics is concerned with finding the correct conjectures or statements that one needs to prove in the first place.

> Joe, practicing algebraic topologist

It is possible that the formalization of mathematical proofs may have considerable impact on the way we perceive the nature of mathematical discovery. The evolution of the finished creative product named a proof is not just a mechanical activity based on axiomatic deductive reasoning starting with an infallible truth; rather, it begins with the emergence of a conjecture (as explained in a later section), which is then continuously transformed through the process of structural coupling by its interaction with the mathematician, and through the emergence of supporting lemmas, axioms and counter examples.

In the following section, we discuss the emergence in raaga alapana and proof generation by identifying certain themes and compare them alongside each other. For the purpose of analyzing the analogy between raaga alapana and proof generation, we would like to refer to a specific kind of proofs called *exploratory proofs* [23]. Exploratory proofs highlight the path of exploration whereby the beginning conjecture is tenuous at best and the development of the justification of that conjecture is an emergent process that often ends in surprising new results.

6. Factors Influencing Raaga Alapana & Proof Generation—A Comparative Analysis

In this section, we present six themes that were identified in the data: fundamentals, aesthetics, personal preference, context/setting, collaborations and disciplinary rules. These themes emerged from the creative process of both musicians and mathematicians. Within each theme, we discuss the respective musical and the mathematical aspects alongside each other. These themes are not exhaustive by any means. Since these themes were influential in the creative outcomes of expert musicians/mathematicians who were interviewed or analyzed for this study, we claim that creativity is not a static construct present solely in an individual, rather, it is an emergent phenomenon that arises out of multiple interacting factors.

6.1. Fundamentals

Any form of creativity in a certain field of knowledge, greatly depends on an organized framework of fundamentals, like basic rules, definitions, concepts and schemas [16]. Learning the basic characteristics of a raaga is like learning the fundamentals of a language. Learning musical compositions could be equated to learning grammatical rules and words of a language. However, as learning mere rules, words and sentences of a language does not guarantee the creation of a great piece of prose or poetry, learning key phrases and motifs of a raaga do not guarantee the ability to creatively explore a raaga.

A newspaper interview excerpt [18] with Kunnakudi M. Balamuralikrishna, an expert improviser, sheds light on this balance between learning compositions and using them as baseline to sing raagas:

> You have to get your fundamentals right to know how to approach a raaga. You learn a new raaga by listening to compositions without which you are never going to sing. There are phrases within the song that you take up for raaga singing. For instance, in his Harikamboji composition, Thyagaraja clearly shows – through a certain phrase – how to delineate it differently from raaga Khamas. That phrase gives you the lead as to how to approach raaga Harikamboji. Such knowledge gets accumulated over a period of time. So, when you have a song in a new raaga, analyse the song first, you get the raaga. I may like a song, it

will be running in my mind for some time, and at some stage I start thinking that I would sing it as a raaga.

<div align="right">Kunnakudi M. Balamuralikrishna [18]</div>

Similarly, mathematicians who are crafting proofs will have a necessary set of skills in mathematics. They need to have a sense of what mathematics can accomplish before they set out to develop ideas. For example, a mathematician working in a specific sub-field, say topology, would have to be proficient with the basic definitions, already existing theorems, ways of proving and rules pertaining to that topic, (on top of basic mathematical skills) in order to engage with any proof generating activity in that area of knowledge. Joe expressed this idea in his interview as follows:

> So yeah we have read lots of papers, lots of people who have succeeded in using an idea like this to get here, an idea like that to get there and I think a lot of times we are trying to use such ideas.

<div align="right">Joe, practicing algebraic topologist</div>

From the above excerpts, it seems quite natural that the learning and understanding of the fundamentals in any field of knowledge is imperative to engage creatively with it.

6.2. Aesthetics

The aesthetic appeal of music performance may be an important factor that plays a major role in the creative endeavors of people engaging with it. As discussed earlier, each raaga has a certain aesthetics and emotionality attached to it, and appreciating this innate beauty has a huge influence on how well that raaga is being improvised. TM Krishna, one of the leading Carnatic musicians, shared about his opinion on Alapana in his 2013 book [24], *A Southern Music: The Karnatik Story* as follows:

> Musical interest arises purely through intimate connection with the aesthetic flow of the raaga—spinning phrases out like strokes on a canvas.

<div align="right">TM Krishna [24, p. 113].</div>

In this quote, Krishna described the unfolding of a raaga, based on the aesthetics of the phrases and as something that is exploratory without a predetermined plan. Schachter [25] explained Krishna's conception of raaga alapana as something that is more instinctive, phrase-based, and not depending on a formal structure.

Similarly, notions of aesthetics like *beauty* and *elegance* are very prevalent in the mathematical discourse and mathematicians often use elegance as an aesthetic measure to judge a proof. For example, in *A Mathematician's Apology* [26], mathematician G. H. Hardy famously claimed that, "there is no permanent place in this world for ugly mathematics" (page 85). So, it may be posited that the aesthetic appeal of mathematics is a driving force for mathematicians to engage with it creatively. Sinclair [27], described how mathematicians go through a phase of *getting a feel for* or *playing around*, trying to fit things together and generate patterns, when they are aesthetically drawn to a certain mathematical terrain. She also mentioned Johan Huizinga's [28] theory of play, which discusses play as the free, orderly (within certain pre-defined constraints) and aesthetic exploration of a situation. An interview excerpt with Kim, reflected this idea.

> Before you are trying to prove something, you are looking around and seeing that it in fact holds. The proof comes in stages, but starts with a conviction or a question, that it seems to me this is probably so. Is it really so? And the way you look and see if it's really so, there is the improvisation.

<div align="center">Kim, practicing math historian & topologist</div>

Thus, we might say that improvisations in a raaga or finding a correct conjecture do not happen accidentally; rather, they emerge from the deliberate musical or mathematical play when one engages within a certain terrain. As posited by Huizinga [28], play is neither random nor possesses an ultimate goal. What matters is its unpredictability, which is motivated by aesthetic explorations. This is why many musicians and mathematicians consider stumbling upon a raaga or a potentially productive conjecture to work with as one of the important creative acts.

6.3. Personal Preferences

Though any trained musician can technically improvise on any raaga, to engage with certain raagas, personal preferences arise due to bodily limitations such as vocal range for vocalists, or skin texture for instrumentalists. The personality and attraction towards the moods of certain raagas will also be a preference. These personal preferences can be mapped to the concept of structural determinism mentioned in a previous section, whereby the structural limitations of the individual exert agency on the possibility of their engagement with their surroundings. Thus, we might say that an artist's personal preference or choice of raaga also plays an important role in giving them opportunities to engage with their music creatively.

In an interview with the first author, Jai said that he liked to explore raagas that he likes or is more comfortable with, while sticking to the structure in unfamiliar ragaas.

> It depends on the raaga. I like some raagas and on those raagas I tend to explore more. But in some raagas where I am not so comfortable, I try to stick to the norm and not do too much creativity. So, it's a mix of both.
>
> Jai, practicing Carnatic violinist

Similarly, Sir Andrew Wiles, the famous mathematician who proved Fermat's last theorem, expressed this idea of having a natural flair for certain mathematical area of knowledge in his detailed interview for European Mathematical Society Newsletter [29].

> I started off really in the dark. I had no prior insights how the Modularity Conjecture might work or how you might approach it. To start with, there are three ways of formulating the problem, one is geometric, one is arithmetic and one is analytic. I think I was a little lucky because my natural instinct was with the arithmetic approach and I went straight for the arithmetic route, but I could have been wrong. Partly out of necessity, I suppose, and partly because that's what I knew, I went straight for an arithmetic approach.
>
> Andrew Wiles, *EMS Newsletter* [29]

Thus, in both the above examples, we see how the personal preferences and limitations of the individual has a great impact on what is being explored. This is consistent with Varela's notion of *structural determinism*, where the organism's structural make-up, with its physical/psychological features, allow for a certain kind of action. Thus, the knowledge base of a person in a certain domain combined with their personal preference and taste is directly influential in helping them to explore their desired terrain much better. Structural determinism, however, may give the wrong sense of what is happening. Enactivists posit that although people have certain *structures* they are constantly changing. That is, as the individual gains a deeper understanding of their subject area over time they structurally change alongside their subject.

6.4. Context/Setting

The other major theme that was found to be influential in a raaga alapana was the context or setting; where it was being performed. For example, performing on-stage versus performing off-stage will influence the nature of improvisation very much, as performing on-stage will require the musician to adhere to certain rules and stage etiquettes that may impact their mood considerably. Carnatic musician TM Krishna [24] criticized the standardization of raaga improvisation approach that has come to be the norm of improvising during on-stage performances.

> The primary problem is that adopting a linear ascending approach to alapana exploration affects the aesthetic content of the raaga, because the phrases have not evolved with the scale of the raaga in mind. This also leads to a certain distortion in the raaga identity. But a level of standardization has, over time, come to be adopted in the alapana format, whereby most musicians take recourse to the step-by-step approach to raaga alapana for almost all raagas ... leading to the loss of many important phrases. They use unnecessary stresses on regions and swaras that are of no relevance to the raaga, leading to a loss of aesthetic identity.
>
> TM Krishna [24, p. 114]

On a positive note, performing on-stage may also invoke feelings of anxiety or enthusiasm in the performer, that can further influence their work in unexpected ways. For instance, during a performance of Carnatic music

that the first author was involved as the main singer a few years back, two contemporary expert Carnatic musicians arrived at her concert venue quite unexpectedly when she was improvising a raaga. In this unanticipated situation, based on her first-hand experience, the first author describes that the enthusiasm she experienced with their arrival, the inspiration and approval she received from their expressions for her music, along with the slight nervousness to sing well and prove herself, significantly boosted her energy levels to improvise with much more passion and her entire team's efficiency to collaborate was also undoubtedly much higher than what it was before their arrival.

Along the same line, we also find that the way that we can think of a proof, a formal proof as opposed to an informal proof significantly depends upon the context or the stage at which it is progressed. For example, *writing* formal proofs for a paper might be more predictable or mechanical than the zig-zag exploration of generating the essence of a proof. Joe, a practicing algebraic topologist, mentioned this point quite clearly in his email conversation:

> I would like to distinguish between coming up with a proof, and writing up of a proof. Usually, coming up with a proof involves a lot of trial and error, one tries all the approaches one can think of, using all the techniques one is familiar with. Writing up details for the paper is lot more predictable, often routine and boring, but necessary for a successful paper, i.e., one which isn't only quoted but also read by others.
>
>   Joe, practicing algebraic topologist

Thus, the context in which a raaga alapana or proof generation takes place has a considerable influence on the nature of the creative output that comes out. More precisely, the setting in which a creative act occurs has the power to alter the nature of the work itself.

6.5. Collaborations

Another important theme that impacted the work of these musicians and mathematicians was found to be their collaborations with their peers and their respective fields, in general. An on-stage performance is a teamwork, and all the musicians accompanying the main singer contribute equally to the raaga alapana that unfolds on stage. Jai said,

> Most of the times, I see it off others. When I am accompanying someone, I try to compliment the phrases that the vocalist sings and it's quite unexpected. The unpredictability is definitely there when you are improvising alone, but it's more frequent when working with others.

> Jai, practicing Carnatic violinist

Similarly, Sir Andrew Wiles [29] described how the work of other people and his collaborations with his peers were influential in leading him to the path that he was seeking for proving Fermat's last theorem.

> With John Coates I had applied Iwasawa theory to elliptic curves. When I went to Harvard I learned about Barry Mazur's work, where he had been studying the geometry of modular curves using a lot of the modern machinery. There were certain ideas and techniques I could draw on from that. I realized after a while I could actually use that to make a beginning—to find some kind of entry into the problem.

> Andrew Wiles, *EMS Newsletter* [29]

Thus, we might say that collaborators and peers contribute significantly to one's creative endeavor in any field. This is true, even if a person is building on the work of another person, who is not physically present. Although one is technically not collaborating in the sense of a stage performance, the exchange of ideas from fellow musicians/mathematicians even during a casual conversation might play a significant role in shaping one's thoughts unexpectedly. Moreover, according to vocalist Swami, the collaboration could be either for the good or bad, as he said, "If an accompanist is not very knowledgeable and receptive, their misrepresentations of a raaga can hinder your creativity and may distract your improvisation. His opinion portrayed the unpredictability involved in an improvisational setting and touched upon the idea that no matter how much one is prepared for a performance, there is always going to be an element of the unknown, both to the person engaged in the process and also to whatever/whoever is in the audience.

6.6. Disciplinary Rules

In a non-collaborative setting, where an individual is generating a proof or improvising a raaga in solitude, there is still a collaboration between them and their respective discipline, which exerts an agency in the emergent creative process. In Andrew Pickering's [30] terms, this is called disciplinary agency. He claimed that a discipline's concepts exert a force, or a way or working, that is necessary for consistency. This aligns with Swami's idea of thinking about a raaga as a person and an alapana as a negotiation. He said, "You have to think of a raaga as a person. Sometimes while negotiating, you have to push their boundaries, but you also have to learn to compromise in the mutual negotiation to keep the overall aesthetics intact.

Likewise, Field's medalist Maryam Mirzakhani [31] relied on the agency of the problems and says,"I would prefer to follow the problems I start with wherever they lead me."

In both these instances, we see how these people gave a significant amount of emphasis on their co-evolving creative entities, the music or mathematics that they work with, which constitutes the environment that they are in. This co-evolution of the individual and their environment is what we identify as *structural coupling*. We also possibly sense the individual's momentary decentering of themselves from the creative act, as they give away their agency to the problem/raaga or the collaborative setting that they are in, to lead them into some unknown territory as Mirzhakhani does when she said she would follow her problems wherever they lead her to. For example, the serendipitous discovery of quaternions by Sir William Rowan Hamilton and the mathematical agency that led him to such a historical discovery speaks volumes about how the discipline of mathematics itself can be influential in shaping the outcomes of a mathematician's creative work.

7. Discussion - The Emergence of Creativity: An interaction of multiple factors

From the previous section, we understand that the outcome involved in a creative act depends on multiple factors that come together at any certain point in space-time. During musical improvisation, the raaga phrases that emerge depends on every factor, like the logic/emotion the player at that time or the setting of the performance or the rules of the raaga itself.

Often times, emotional progression is much more aesthetically pleasing than mere technical and mathematical patterning, as the latter has a possibility to be somewhat predictable if overused. In either case, the sequence in which the improvisation unfolds is partially unpredictable to the musicians themselves and most of the times, informal improvisation, where there are no strict start/end points and time factors are much more exploratory than a formal on-stage performance.

Comparably, the way a mathematician explores a conjecture, at any given context; either while working alone or with peers, plays a role in the way the act of proving proceeds. In enactivist terms, play can be seen as a type of problem posing. Varela proposes that conjecture or problems do not exist "out there"; rather, they are brought forth, posed and looked at through specific ways, because of both the limitations and possibilities of both the individual and the context they are placed in [13]. For example, let us consider some different ways of approaching the proof of uncountability of the real number set $\mathbb{R}$. One of the most common proofs that we find is proof by contradiction. That is, by assuming $[0, 1]$ to be countable and then proceeding from this assumption using the diagonalization argument to show that $[0, 1]$ is uncountable and hence $\mathbb{R}$ is uncountable. However, there might be few other formal or intuitive ways to proceed with this proof. For example, one could try to generate a direct proof, by proving $[0, 1]$ to be uncountable and thus $\mathbb{R}$ to be uncountable. To do this topologically, one may begin to think about what kind of a space is $[0, 1]$ and may proceed towards proving it as a non-empty, compact, Hausdorff space with no isolated points, and thus uncountable. However, in order for the proof to evolve this way, one needs to know what each of the terms- non-empty, compact and Hausdorff space mean in a topological sense. This is where the individual's knowledge base comes into play. This approach may thus lead to a different pathway to the proof as a whole, and might involve unpredictable statements emerging at each stage, due to various interacting factors acting in that context. Another intuitive or informal way of proving this would be proceeding with the idea that rational numbers are countable. The set of all real numbers is the disjoint union of the sets of all rational and irrational numbers. If one proceeds with this assumption, they might then have to prove that the set of all irrational numbers is uncountable, which would finish the proof. Now, this approach has led to a completely different path to envision this proof.

Thus, we posit that that the unpredictability that underlies the unfolding of a creative process is one of the most exciting and stimulating factors for the person engaging in a creative act. This is because if the sequences in a raaga alapana or a proof are predictable and mechanical, it will most likely not be interesting for the musician/mathematician anymore. This is analogous to how Schoenfeld [32] defined a 'problem', as something to which there is no clear-cut path to solve. This makes sense, because if there was a clear path with predetermined procedures to reach to the solution of the problem, it will not remain a problem anymore, but will just be an exercise. Hence, we claim that unpredictability in a creative process plays a vital role in keeping the individual engaged in their activity.

If this is true, then the individual, in some way, recognizes that they are not central for the creation to happen and the unpredictable outcomes are not deliberate choices made by them. However, this dissolution of self during creation does not happen without a preparatory stage of developing knowledge, like gaining expertise through proper training, expanding knowledge base by reading/collaborating with people etc. In an autobiographical letter, Henri Poincare [33], the famous mathematician, expressed how unpredictable outcomes can emerge from repeating fundamental ideas over and over. He said, "It seems to me then, in repeating a reasoning learned, that I could have invented it. This is often an illusion; but even then, even if I am not so gifted as to create it by myself, I myself re-invent it in so far as I repeat it. (page 5). A similar idea on repeated training on fundamentals and key ideas, is emphasized by the Carnatic vocalist, KVN, in [17] as,"Mani Iyer would ask me to sing the same song many times. The beauty of this was, when you practiced the song over and over again, you got new ideas every time you sang." There is then, clearly a tension between the importance of the self and its dissolution during the process of creation. Sawyer [5] seemed to agree in his exposition of the aesthetic theories of Dewey and Collingwood. When writing about the role of ready-mades/cliches in improvisation, he talks about Collingwood's notions of *false art* and *art proper* that rests on the presence and absence of ready-mades, respectively.

> Art is false when the creator uses a ready-made *language* which consists of a repertoire of cliches to produce states of mind in the persons upon whom these cliches are used.
>
> Collingwood, 1938 [5]

However, Sawyer also noted that Collingwood later acknowledges the fact that an artist has to speak the language of his domain. This means the artists depend on a shared body of knowledge, conventions and techniques. This is and probably will be an unresolved mystery in Collingwood's theory, as Sawyer rightly said that the tension between false art and art proper is where creation happens. Vocalist Swami aligned with Sawyer's idea of creation beautifully, as he said that the rules and key phrases of a raaga rest on one hand, and its very expression or the musicality rests on the other hand and the creation happens when a proper balance is achieved.

> There is always that separation between exploring different facets
> or different combinations and figuring out how to express, which
> is on one side and then on the other side, it is the expression
> itself that comes out. So, the expression comes out because of
> the other side being coherent and properly looked through.

Swami, practicing Carnatic Vocalist

In a sense, these ideas of art proper and false art clearly expound the current state of mathematical belief and understanding in classrooms, where mathematics is perceived as a way of doing, but not as a way of thinking [34]. A good balance between art proper and false art is imperative for any kind of improvisation to happen and unfortunately, this balance is rarely attained in mathematics classrooms, as it is often taught as a rule-based, linear, and a strict subject, where only the correct answer is valued. To extend Collingwood's idea, we think of mathematics as predominantly imposed as a false art on the learners. Students are often caught up in the world of rules and algorithms, and never get to taste the soul of mathematics. We suggest that understanding creativity as emergence has significant impact on the pedagogical and philosophical aspects of mathematical creativity, as this perspective brings forth the importance of providing right contexts for creativity to emerge and thrive, as opposed to conceiving it as a static entity that someone is born with.

Thus, we could possibly say that the emergence of creativity is a product of multiple interacting factors, for which optimal conditions might be created, but can seldom be reduced or dismantled to the properties of the component parts. This is quite consistent with the notion of emergence mentioned in a previous section as something that is not additive in nature. In other words, though emergence of creativity depends on the properties of the in-

teracting entities, it might not be possible to describe it solely in terms of these properties. This might then be one of the caveats of using enactivism as a theoretical lens to view this idea, as it rests on the basic assumption of existence of 'objects' prior to the interaction. Zooming in further into the internal mechanisms of the interacting objects calls this theoretical framework into question, as enactivism's basic ontological unit of analysis is objects and not the processes that give rise to the objects.

8. Conclusion

Once a product is formed, we tend to overlook the dynamic emergent processes that were involved in the creation of the product and look at it as a static entity. However, creative processes are seldom linear, because the product thus formed is due to the result of constant emergent deformations and re-formations caused by the interaction between the individual and their environment. In this paper, we analyzed the similarities in the processes involved in the musical improvisations of Carnatic music and the generation of mathematical proofs, and advocate for the idea that creativity, in general, is an emergent phenomenon that unfolds due to the constant mutual interactions between a person and their environment. Two professional Carnatic musicians and two research mathematicians were interviewed, and the participants were asked to reflect on their own thoughts about their creative processes while engaging in raaga improvisation and proof generation, respectively. Additional data from newspaper interviews, magazines and autobiographical accounts of expert Carnatic musicians and mathematicians were also analyzed. Several statements from both the musicians and the mathematicians emphasized the role of their collaborators, mood, personality and their disciplines in the emergence of their creative process.

By comparing the inner workings of musical improvisation and mathematical proof generation in this study, we saw a great resemblance between artistic and mathematical creativity. The descriptions from the expert musical and mathematical creators, starting from having an aesthetic attraction towards a certain musical space or a mathematical area, to the unpredictability and suspense experienced during a raaga improvisation or proof generation, and finally, losing themselves in the act of creation sheds light on the closeness between their personal experiences during their respective creative process. On the surface, mathematical proofs, appear exceptionally precise and logical; however, the inner core of their improvisational aspect is as imprecise,

unambiguous and uncertain, as a free-flowing musical improvisation. And, on the other hand, though musical improvisations appear imprecise and unorganized at the outset, their inner core is highly guided by logic, rules and regulations. Mathematical and artistic creativity could be considered as two sides of the same coin: a kind of unity with a diametrically opposite front and back.

Lastly, all the participant statements in the study were suggestive of embracing the balance between mastering the fundamentals of their field, increasing their knowledge base, and letting it go during the actual process of creation. This proved to be a very powerful insight in emphasizing our aim to describe creativity as emergence. The involvement in deliberately equipping oneself with the raw materials required for the creation, by personal choice, and finally dissolving the Self during creation, further suggests that the creative outcome is emergent, in that it cannot be traced back solely to the properties of the interacting entities. Thus, we could say that the unpredictability of the outcomes is imperative for an individual to feel the liveliness in their creative process, for which the dissolution of the Self has to happen.

We wrap up this paper with a compelling quote by Jiddu Krishnamurti [35], one of the most original and influential thinkers of India, who wrote extensively about the self and the society. The following lines beautifully convey the idea that a creative outcome is not caused by a human creator, but emerges as the Self dissolves itself to the moment in space and time.

> To be creative is not merely to produce poems, or statues, or children; it is to be in that state in which truth can come into being. Truth comes into being when there is a complete cessation of thought; and thought ceases only when the self is absent, when the mind has ceased to create, that is, when it is no longer caught in its own pursuits. When the mind is utterly still without being forced or trained into quiescence, when it is silent because the self is inactive, then there is creation.
>
> Jiddu Krishnamurti [35]

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A. Appendix

1. What, according to you, is a raaga alapana/proof and how do you see yourself as creative while engaging in this aspect of Carnatic music/mathematics?

2. What do you focus on while singing each phrase in an alapana/proof? Do you think about the next phrase or the relationship between the current and the next one? Do you anticipate anything at all?

3. What guides you primarily while singing an alapana or generating a proof? Logic, emotions or aesthetics?

4. Do you think the context that you are situated in and your circumstances, mood, accompanists etc., affect the quality of your alapana/proof and your ability to improvise/generate a proof?

5. Have you noticed differences between the creative processes involved in expanding a raga on-stage vs expanding it during your practice sessions/generating a proof and writing a proof? More precisely, do you feel restricted by the need to follow some sort of a structure for raga alapana on stage/writing proof for paper?

6. Do you start a raga alapana/proof with a conviction to engage with it a certain way or do you let your imagination run wild?

7. When do you feel your raga alapana/proof is novel? Is it when you encounter something unexpected as you are singing/generating or is it when you look back after the alapana/proof has been completed?

8. How do you judge an alapana/proof as beautiful or, what according to you is beauty in a raga alapana/proof?