Journal of Humanistic Mathematics

Volume 10 | Issue 2 July 2020

Fostering Student Discovery and Conjecture in Multivariable Calculus

Aaron Wangberg
Winona State University

Recommended Citation
Aaron Wangberg, "Fostering Student Discovery and Conjecture in Multivariable Calculus," *Journal of Humanistic Mathematics*, Volume 10 Issue 2 (July 2020), pages 315-347. DOI: 10.5642/jhummath.202002.15. Available at: https://scholarship.claremont.edu/jhm/vol10/iss2/15

JHM is an open access bi-annual journal sponsored by the Claremont Center for the Mathematical Sciences and published by the Claremont Colleges Library | ISSN 2159-8118 | http://scholarship.claremont.edu/jhm/

Fostering Student Discovery and Conjecture in Multivariable Calculus

Cover Page Footnote

We are grateful for the meaningful discussions with Elizabeth Gire, Tisha Hooks, Tevian Dray, Corinne Manogue, and Robyn Wangberg about designing small group activities, and to Jason Samuels and Brian Fisher, collaborators on the Raising Calculus project. Partial support for this work was provided by the NSF TUES program (DUE-1246094).

Fostering Student Discovery and Conjecture in Multivariable Calculus

Aaron Wangberg

Department of Mathematics & Statistics, Winona State University, Minnesota, USA
awangberg@winona.edu

Synopsis

Who owns the mathematical ideas in the undergraduate classroom??Certain types of mathematics classroom and curriculum impose several barriers that prevent students from discovering and engaging with mathematical concepts. Definitions, notations, and theorems require mastery before students can work meaningfully with the underlying mathematical concepts. Raising Calculus to the Surface utilizes a different approach by providing students multiple entry points to engage meaningfully with mathematics ideas in a multivariable calculus course. It allows students to promote meaningful ideas and conjectures into the classroom discourse to formalize their explorations. In this paper, I describe several characteristics built into the Raising Calculus project materials, including a rubric designed to encourage student discussion in small groups and for the whole class. I illustrate with three vignettes how these features allow students to explore mathematical ideas using their own creations leading to conjectures which are promoted and shared with the whole class, thereby making students involved in creating the course's mathematics content.

Keywords: classroom discourse, creativity, Raising Calculus to the Surface,

1. Introduction

The act of forming conjectures, of finding patterns, suggesting relationships between mathematical objects, and proposing proof statements can incorporate student creativity into mathematical practices even before students are able to prove theorems. Indeed, even when students are unfamiliar with the technicalities of proof techniques, they can utilize creativity to engage deeply with the material. While this engagement can happen spontaneously, it can also be fostered in a classroom and by an instructor to align with the course's content enabling students to discover and take ownership of some of the content. This requires a classroom setting supportive of such student engagement - including peer and instructor support for discussing both correct and incorrect ideas. It also requires tasks which make the mathematical concepts and ideas more accessible to all students.

Raising Calculus to the Surface is a project which utilizes small group, active engagement activities designed to help students discuss and explore multivariable calculus concepts prior to the introduction of formal definitions by the instructor. The project consists of activities designed to support discussion in both small groups and as part of the whole class. The project utilizes several manipulatives, including dry-erasable surfaces (henceforth referred to as Surfaces, see Figure 1), contour maps, coordinate grids, and measurement tools which provide means for students to engage with concrete representations of multivariable calculus concepts.

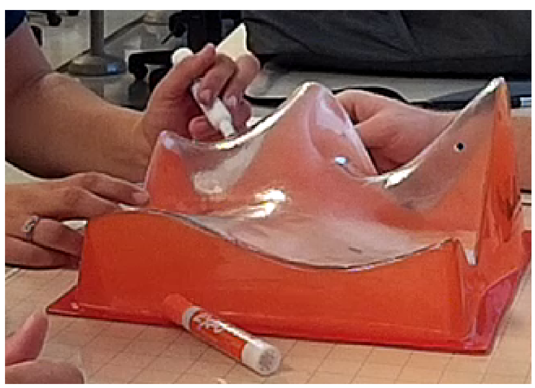


Figure 1: A Surface represents the graph of a multivariable function of two variables. It has a dry-erase finish and is large enough to be a common work area for a small group of students.

In this paper, I set out the features built into the activities and project materials that are designed to help students explore, discuss, and generate mathematics (e.g., definitions, patterns, and conjectures). After describing the rubric used to evaluate the activities, I then illustrate the creativity exhibited by students as they discuss possible conjectures resulting from those tasks. I conclude with a discussion of the possible factors that help students engage with their peers and contribute content to the course.

2. Raising Calculus to the Surface

Raising Calculus to the Surface [7] is a project funded by the National Science Foundation (NSF DUE #1246094) which utilizes physical manipulatives and open-ended group activities designed to help students discover new multivariable calculus (MVC) concepts prior to lecture. Large three-dimensional plastic dry-erasable surface manipulatives represent the graphs of abstract multivariable functions and are paired with dry-erasable contour maps. Students use tools such as an inclinometer, ruler, and rectangular and polar coordinate grids to measure quantities like instantaneous rate of change, distance or height on the surfaces and contour maps. With these tools, students study not the surfaces themselves but rather the relationships between the mathematical objects represented on the surfaces.

The activities and materials developed for the project represent multiple external representations (MERs) and are intended to help students engage meaningfully with mathematical content. *DeFT*, a conceptual framework proposed by Ainsworth [2], delineates learning with MERs with respect to Design, Functions, and Tasks. *DeFT* considers the Design parameters of the external representation tools that are unique to promoting learning with these particular MERs, the Functions that MERs serve in supporting learning, and the cognitive Tasks that must be undertaken by a learner interacting with MERs. The distribution of information across representations plays a key role in student learning with MERs. Ainsworth [1] notes combining MERs allows a second representation to (a) support complementary processes or information contained within the first representation, (b) constrain the interpretation of the first representation, or (c) support the construction of deeper understanding when learners achieve insight using a second representation.

Certain features built into the Surface materials help distribute information across the MERs. The surface manipulatives are coordinate-system free, meaning students can utilize them with rectangular, polar, or without a coordinate system. There are two distinct contour maps (Figure 2). A *fine*

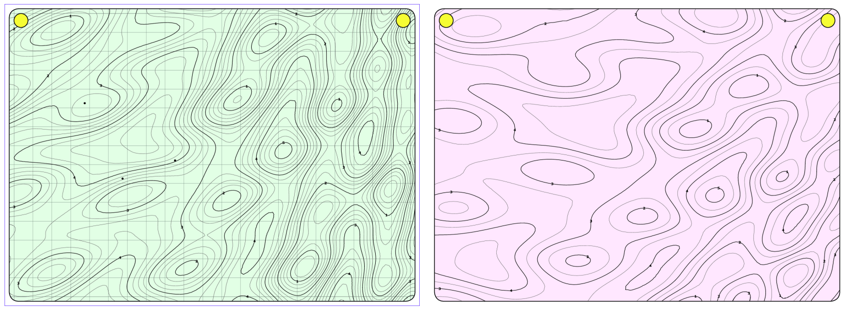


Figure 2: The *fine* contour map (left) includes a rentangular coordinate grid and twice as many contours as the *course* map (right). The *coarse* contour map, which is rotated, has no such restriction.

contour map incorporates a rectangular grid and three dots which align with three dots on the surface, while a *coarse* contour map is free of both dots and the rectangular grid. The formulas for the functions behind the surface manipulatives are never revealed in the activities. Then, when student incorporate two representations using derivative, for example, they must draw upon other conceptions of derivative than computational rules. This distribution of information is designed to encourage student exploration of mathematical concepts before utilizing algebraic and symbolic manipulation during lecture.

I include two important notes regarding the materials: First, the activity sheets incorporate context which typically changes from one activity to the next. Second, the surface manipulatives were designed to be free of symmetry, so that students finding patterns or relationships between mathematical concepts are doing so for the abstract situation. These choices help make the materials be flexible for several activities while still allowing specific information to be included (when necessary) on the activity sheet.

3. Activity Rubric

The activities developed for Raising Calculus to the Surface use a rubric developed to foster exploration of new mathematical ideas and promote discussion in both small groups of students and with the whole class. This rubric (Figure 3) is adapted from guidelines presented on [3] to convert standard homework problems into open-ended contextualized small group activities to foster group discussions. After eliminating the physics-specific criteria from [3] list of 21 characteristics, the remaining criteria were collapsed into five categories and assigned half-integer scores from 0 (not present, can be solved without attention to this feature) to 4 (significant – must be incorporated to make sense of the problem or solution) for each activity. Typical scores for Raising Calculus activities range from 13 to 16 out of a possible 20 points using the rubric.

The five categories are Open-ended prompts, Meaningful context, Fluency with representations, Measurement and mathematization of observable quantities, and Geometric relationships. Each category helps promote discussion amongst students in the following ways:

- **Open-ended prompts:** Mathematics questions are often carefully worded – but an important act of mathematics is precisely defining the question. Mathematically vague or imprecise statements provide students opportunities to discuss meaning and understanding of quantities, concepts, and approaches to the problem. This is a feature, rather than a bug: It can promote discussion and help students see the implications of making changes to their assumptions.

- **Meaningful context:** Incorporating context into a problem often provides additional opportunities for students to engage with the content. This can happen when they are uncomfortable with the technical content: They can ask "Where is the temperature increasing most quickly" rather than asking "Where does the gradient vector point?" Context can also help students formulate questions which are meaningful and/or which help organize different mathematical quantities (e.g. The location is measured in meters, but the density is measured in grams per square meter.)

  Context should incorporate meaningful situation for students. The most familiar context, height of a hill, is likely familiar to all students. If the height and location are measured with similar units (e.g., meters), then the context requires care when referring to different quantities (e.g., was 5 meters the height or change in a coordinate direction?) Other contexts are less familiar to students (e.g., concentration of lead in the soil, amount of gold in the ground, energy absorbed on a solar panel) which have additional meaning for students (e.g., increased levels of lead are bad for children; increased amounts of gold in the ground are generally valuable). The additional information context brings to a problem can be used to generate discussion within the group (e.g., should a winter down-hill sledding park be built where lead levels are high?) which might conflict with their intuition about the situation (e.g., a high point on the surface represents elevation).

- **Fluency with representation:** Some questions are more easily answered with one representation than another. For instance, finding the steepest direction or comparing surface area might be easier done on the surface manipulative than on the contour plot. Activities which utilize multiple representations provide students different ways to solve a problem which then provide additional opportunities for students to contribute ideas into the classroom.

- **Measurement and mathematization of observable quantities:** Quantities such as rate of change or parallel / perpendicular relationships play an important role in multivariable calculus. The inclinometer tool enables students to measure rate of change on the surface manipulative, and the dry-erasable materials let students draw curves and vectors to find relationships existing on the surface or on the contour maps. Mathematization of these quantities produces important mathematical concepts (e.g. a dot product between vector quantities being 0) which are fundamental to the physical system, rather than the physical situation being used to illustrate an abstract mathematical relationship.

- **Geometric relationships:** Although appearing 'square' or 'rectangular', the surface materials do not incorporate an explicit coordinate system. If an activity requires students to solve a problem with an explicit set-up (e.g. locate the highest point at the origin) or must include directions to set up a coordinate system (e.g. set up a coordinate system and draw a line from *(x=3, y = 4) to (x = 7, y = 9)),* then the problem likely focuses on the coordinate-system specific formalization of the problem. It is fine for students to decide they need a coordinate system, and to set it up – but then subsequent discussion can address whether their answer would depend upon the location of the coordinate system chosen, the coordinate system's origin, or the group's choice for selecting coordinate directions.

  From a broader perspective, the coordinate system is used to describe physical systems, not define them. Relationships between quantities, say between a gradient vector and the level curves of a function, occur independent of the coordinate system used to describe that relationship. Thus, when students find a relationship which holds independent of their choice of coordinate axes, origin, or coordinate system, they have likely arrived at an important mathematical result.

The complete task design rubric for activities which promote student investigation of mathematical ideas as part of small group and whole class discussion is as follows.

Feature	Rating, description, and example
Open-ended prompt	0: explicitly directed (*Use the method of Lagrange multipliers to find the optimum solution to* $T = x^2 + y^2$ *subject to* $y = 3 - x$.) 1: Limited interpretation (*Draw a path from the red star to the blue circle on the surface.*) 2: Decide about procedure/concept or representation (*Draw a path from the red star to the blue circle.*) 3: Decide about procedure/concept *and* representations (*Which path contains the most berries?*) *Note: A vague or ill-defined statement increases the rating one unit. Such a statement might not specify the target variable, require interpretation about the question, or use common language instead of mathematically precise language. (*Which path should you choose to pick berries for making a pie?)* The assumptions for this question are discussed in Section 5.

Feature	Rating, description, and example
Meaningful Context	0: no context (*Find* $\delta G/\delta y$)* 1: Superficial, non-meaningful use (*Find* $\delta G/\delta y$. *Include units.*) 2: Domain / range organizational tool (*Estimate the gold density at the blue dot.*) 3: Rate or direction organizational tool (*Estimate how the gold density changes at the blue dot toward the green triangle.)* 4: Coordinating quantities with meaning (*Estimate the gold density 1.2 miles north and 2 miles west of the blue dot.)* * Note: The context requires distinguishing between domain and range, measurements and units for the derivative ($\delta G/\delta y$, $\delta G/\delta x$) and displacement ($(\Delta x, \Delta y)$) quantities, and how to combine those quantities together to answer the question.
Fluency with Representations	0: Symbolic representation only (*Compute* $\delta T/\delta x$ *if* $T = x^2 + y^2$*)* 1: Utilizes contour map or surface area, not both. (*Measure* $\delta T/\delta x$ *on the surface.*) 2: Connections between representations (*Measure* $\delta T/\delta x$ *on the surface ... Measure* $\delta T/\delta x$ *on the contour map.*) 3: Unspecified representation— students must make choice about representation (*Is* $\delta T/\delta x$ or $\delta T/\delta y$ *larger* at the blue dot?* [students choose the representation for this ranking task.] 4: Misdirected representation: Problem more easily solved with a different representation than the one specified. (*Which region on the contour map has the most surface area?*)

Feature	Rating, description, and example
Measurement and mathematization of observable quantities	0: Not present. (*Compute* $\delta T/\delta x$ *if* $T = x^2 + y^2$)) 1: Sign of quantity (*Find a location where* $\delta T/\delta x > 0$ *and* $\delta T/\delta y < 0$?) 2: Comparison (*Rank the three points based on* $\delta T/\delta x$.) 3: Combines quantities for measurement (*Using the inclinometer, measure* $\delta T/\delta x$ at the blue dot on the surface.*) 4: Combine multiple quantities for comparison or for mathematization. (*Estimate the gold density 1.2 miles north and 2 miles west of the blue dot.)* [This initially requires measurement of partial derivatives using the inclinometer or using the contour map, combining those measurements into a new quantity, and combining those new quantities with other information to arrive at an estimate. This result can also directly lead to an abstract formulation of linear approximation.]
Geometric Reasoning	0: Specific axes and origin for a coordinate system are defined in the activity. (Place *the surface so the tall corner is located at (0,0) and the surface is aligned in the first quadrant of the Cartesian coordinate system.)* 1: Freedom of origin, but the axis directions are specified in the activity. (*Align the surface so* $\delta T/\delta x$ *and* $\delta T/\mathrm{dy}$ *are both positive at the blue dot.*) 2: Freedom of origin and axes. (*Place the surface on the rectangular coordinate grid.)* 3: Freedom of coordinate system (*Describe how the temperature changes moving one unit in each coordinate direction.)* 4: The activity does not require a coordinate system to be defined but investigates relationships between mathematical quantities independent of coordinate description. (*Draw a loop around three dots on the contour map and surface.)*

4. Classroom Setting

The Raising Calculus curriculum spans the multivariable calculus content; each semester, seven activities were used within the first seven weeks of the semester. The activities were used to help students investigate mathematical concepts prior to those concepts being formally introduced by the instructor.

The instructor's role in each activity was to listen to arguments being discussed within groups of three students and organize such arguments across groups for the whole-class discussion. At times, the instructor also listened for student ideas or confusion which needed to be further developed or explored by the group or the whole class. The intent of this listening by the instructor was to bring forth discussion of important and troublesome ideas into the broader classroom.

To explore the ways in which the Raising Calculus materials develop students' conceptual understanding of multivariable calculus, we conducted research studies. In these studies, data were collected from in-class implementation of the Raising Calculus materials in the form of video-recordings of small and whole-class discussions at a medium-sized midwestern regional university. In this paper, I focus on three activities and share excerpts of student interactions to demonstrate student creativity and conjecturing that was promoted by these three activities.

5. Student Creativity and Conjectures within the Multivariable Calculus Classroom

Three activities from the Raising Calculus to the Surface project (The Park, The Roller Coaster, and The Boysenberry Patch) are described in this section. The activities are compared to standard multivariable calculus activity, and both types are compared to the rubric shared in Section 3. Excerpts of student interactions are used to demonstrate the creative work generated by students with a focus on the conjectures proposed by students as part of the task.

5.1. The Park*: Conjectures involving contour plots and graphs of multivariable functions*

Figure 3 provides two versions of an activity in which students explore the relationship between a multivariable function, its graph, and the concept of level curves. Both activities can be used prior to the formal introduction of multivariable functions and can be used to introduce students to the concept of level curves. I first discuss the differences between the two activities according to the rubric shared in Section 3, then share the conjectures proposed by students as part of the classroom discussion with the activity incorporated as part of Raising Calculus to the Surface.

The standard activity would score relatively poor using the rubric. Most obviously, the questions are fairly direct and lack context. Although the contour line concept is present, it primarily uses one representation (the surface) in addition to the symbolic representation. By asking students to focus on the surface's maximum or minimum values, the final question focuses student attention to one of several relationships between level curves and surfaces. The activity provides limited opportunity for students to share their findings with their peers from other groups: Each group will arrive at the same conclusions for question 3. Indeed, because of the directed nature of the tasks, students would likely be able to complete the entire activity in isolation.

Consider, instead, the rewritten activity for Raising Calculus to the Surface. The activity includes several prompts, two of which are fairly open-ended. First, the prompt in "On your Mark" asks students to mark all points on the surface with the same lead concentration as the park. This question is vague because it draws student attention to points, rather than curves. This language introduces opportunities for confusion within a group ('Why are you drawing a curve – it says mark points!'). The subsequent negotiation within the group often leads to discussion among students of the relationship between points and curves (which are made up of points). The second main prompt, included in **Get Set**, asks students to locate the surface on the contour map without providing any process for doing so. It provides opportunities for students to develop their own conjectures as to how to match the two representations.

The context in the problem, concentration of lead in the soil, is unfamiliar to students. In addition to helping distinguish between the domain (location)

Standard activity: Level curves

1. Draw a curve on the surface T connecting all the points at the same height h_0 as the blue dot.
2. The curve $T(x, y) = h_0$ is a level curve. Draw three more level curves on the surface.
3. What do the level curves do near a maximum or minimum point on the surface?

Raising Calculus to the Surface activity: *The Park*

On your mark: You work for Granite Falls, a town which needs to move a playground due to harmful levels of lead (Pb) in the ground. The surface's height represents the concentration of lead (in $\frac{g}{m^2}$) in the topsoil at every location in Granite Falls. Lead levels range from roughly 0.5 $\frac{g}{m^2}$ to 6 $\frac{g}{m^2}$.

1) The park is currently located at the red star. Mark all points on the surface with the same concentration of lead as the park.
2) Mark all points with lead concentration 1 $\frac{g}{m^2}$ higher and 1 $\frac{g}{m^2}$ lower than at the park.
3) Could these curves intersect? Why or why not?

<< *Class discussion about intersecting curves*>>

Get Set: Granite falls is a 10 km × 10 km town in the 15.5 km × 21.5 km Rock County. Find where the town is located in the county and place it there. Explain how you know you found the right location.

<< *Class discussion about aligning the surface with the contour map.*>>

[Third and fourth parts removed for brevity.]

<< *Class discussion: On the surface, point to the best location for a down-hill sledding park.* >>

Figure 3: Activity defining level curves and the surface/contour map relationship.

and function values (level of lead) for the function, the use of units (1 $\frac{g}{m^2}$ higher and 1 $\frac{g}{m^2}$ lower) indicates students work with different output level curves rather than input trace curves. Context has a very important role in the final discussion question, as students disagree whether the downhill sledding park should be located at the highest or lowest point on the surface.

Measurement, in the form of comparisons, are fairly important aspects of the task. Students have to interpret questions about scale, increased or decreased amounts of lead concentration, and what it means regarding lead concentration for different curves to intersect. These comparisons appear in several of the activity's tasks.

Multiple representations, on the other hand, are highly important and prominent to this activity. The main question

> *Find where the town is located in the county and place it there. Explain how you know you found the right location.*

in **Get Set** asks students to connect the surface and the contour map. The surface and contour maps contain different information about the underlying function. As further explored in Wangberg [6], students utilized primarily five methods (illustrated in Figure 5) to locate the surface on the contour map:

1) Match Values: Students match contour line values to high and low points on the surface, knowing the surface extends between contours 0.5 and 6.

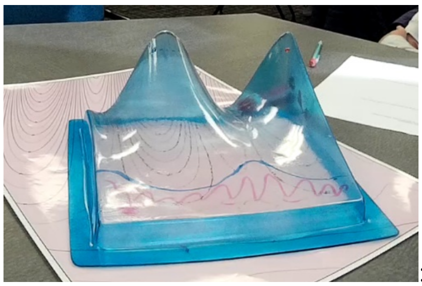
2) Repeating pattern: Students match a repeating pattern in the contour map to a repeating pattern on the surface.

3) Prominent feature: Students identify a prominent feature to match between the surface and contour map. This might involve an oddly shaped contour line containing several prominent bumps and indentations.

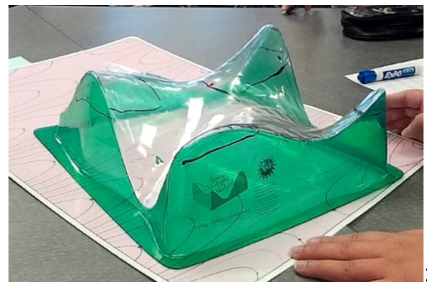
4) Rings around local extrema: Contours form closed loops around local extrema if they occur within the interior of the surface.

5) Spacing of contours: The surface changes very little when the contours are spaced apart.

While not seeming present in the written activity, the geometric relationship

;

(a) Alignment using *contour values*.

;

(b) Alignment using *repeating pattern*.

;

(c) Alignment using *prominent feature*.

;

(d) Alignment using *rings around local extrema*.

;

(e) Alignment using *spacing of contours*.

Figure 4: Aligning surface manipulatives to the contour maps.

between the contour map and the surface manipulative is very prominent in the classroom discussion of matching the surface manipulative with the contour map.

Features of the surfaces make some of the conjectures more visible to students than others. Repeating patterns (which are not symmetric) are built the green, blue, and purple surfaces. Local extrema occur on the interior of the red, yellow and purple surfaces, while the orange, green, and blue have high and low points primarily isolated on the edge of the surface. Almost every surface contains two or more of these features – the orange surface typically can only be placed on the contour mat by considering the fifth method (spacing of contours).

This distribution of information across the materials provides opportunities for students to propose, test, and verify or refute conjectures in the classroom. These conjectures, formulated by the students and formalized by the instructor in the following lecture, then become important qualities of the relationship that exists between graphs of functions and their contour map.

5.2. The Roller Coaster*: Creativity and Conjecture with Lagrange Multipliers*

Figure 5 shows two activities involving constrained optimization, a problem which is typically solved using the method of Lagrange multipliers. The practice problem in the standard activity would be utilized after students had typically been shown the method of Lagrange multipliers. In contrast, *The Roller Coaster* activity is intended to help students discover and propose geometric relationships underlying solutions to constrained optimization problems.

The rubric in Section 3 scores these two problems very differently: The standard problem is not open-ended; It directs students to use a procedure which must previously have been defined in class. The procedure requires students to view the constraint, $xy = 2\sqrt{2}$*, to be one specific curve of a more general multivariable function* $z = xy$*, and to recognize that the solutions to the problem occur when the gradient vectors of* f *and* z are proportional, hence generating $\nabla f = \lambda \nabla z$ *using the proportionality constant, or Lagrange multiplier,* λ*. This conceptual knowledge, however, is not required to perform symbolic manipulations needed to solve the problem.* This activity does not make use of measurement or multiple representations, and the geometry is hidden from the student. Because it requires prior knowledge of the method of Lagrange multipliers, this activity provides little opportunity for students to develop ownership of the solution method.

Standard Activity: Lagrange Multipliers

1. Suppose $f(x, y) = 700 - 20x^2 - 40y^2$. Use the method of Lagrange multipliers to find the maximum and minimum values of f subject to $xy = 2\sqrt{2}$.

Raising Calculus to the Surface activity: *The Roller Coaster*

On your mark: Draw a *big, smooth, (interesting)* ride around the dots on your contour map.

Get Set: Mark spots where your curve is perpendicular to level curves with an X, and places where your curve is parallel to level curves with an O.

Go: Transfer your path (carefully!) to the surface. What happens at each of the marked points? Write down any relationships between the function and path.

<<Classroom discussion about relationships.>>

Challenge: Find the highest point for the ride given by the path $xy = 2\sqrt{2}$ where the ride's height is given by $h(x, y) = 700 - a\ x^2 - b\ y^2$ with $a = 20\ \frac{\text{ft}}{m^2}$ and b= $40\ \frac{\text{ft}}{m^2}$ and x and y are measured in meters.

Figure 5: Problem investigating constrained optimization.

In contrast, *The Roller Coaster* activity is designed to help students propose the geometric relationship occurring behind constrained optimization; The mathematical understanding developed in *Go* can be used by students to solve the constrained optimization problem presented in the challenge. The activity scores much higher according to the rubric from Section 3. The main question is very open ended (*What happens at each of the marked points [at the X's and O's]?*); The set-up questions of **On your mark** and **Get Set** are directed to focus students to important relevant relationships without needlessly telling students which curve must be drawn. Students have to interpret the context of this curve within their groups as well as the notion of a curve parallel to, or perpendicular to, a level curve. The activity also explicitly incorporates multiple representations—not just between the surface and contour map, but also between different symbolic representations of mathematical objects. The notion of measurement occurs as a comparison between function values along the roller coaster curve. The geometric relationship is more obvious (when are curves parallel or perpendicular?), although the instructor's help is needed to formulate these relationships into mathematical notation.

Figure 6 illustrates several of the paths generated by students for this activity. Students are free to draw any roller coaster they like, provided it is not a simple circle. One group, which proposed the self-intersecting path in the third image in Figure 6, discussed whether this was possible (and safe!) for an actual roller coaster. Extreme cases like self-intersecting roller coasters or the *Superman Roller Coaster*, whose track doubles back on itself, provide interesting scenarios to analyze mathematically and compare to real life. Nearly any smooth path can be analyzed in the activity.

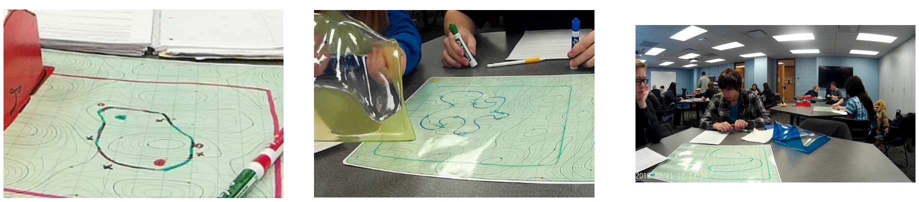


Figure 6: Several roller coaster rides generated by students.

In the excerpt below, a group has proposed the path shown in the middle image in Figure 6. Students Evan and Hayden are responding to the prompt, What happens at each of the marked points? Write down any relationships between the function and path." In this excerpt, students have drawn a path on both the surface and the contour map, and they have placed the surface directly on top of the contour map. The students are primarily looking at the surface or down through the surface to the contour map.

EVAN: [*reading the prompt*] What happens at each marked point? Write down the characteristics. So...

HAYDEN: [*grabs marker, and adjusts path on the surface to match the path drawn on the contour map below.*]

EVAN: [*looking down at the paths on the surface and contour map*] Kind of looks like where its parallel, it kind of flattens out there just for a minute. Like here [*points to a low point on the path*] it just sits in the valley ... [*points to a high point on the path*] here it just kinda sits here. [*points to another low point on the path*] and here it just sits there.

HAYDEN: uh-ha ...

EVAN: Like it just kind of steadies out.

In the excerpt above, Evan has used the shared information (the path) drawn on both the contour map and the surface manipulative to investigate what happens at the 'O' locations on the path. Evan proposes that the path levels off at these locations, which is the correct: The surface may not be flat at those locations, but it is flat (e.g. not changing) when restricted to the path.

Evan and Hayden continue their investigation by looking at the points marked with 'X', which occur where the path is aligned with the gradient vector for the surface. Gradient vectors point in the direction of greatest increase, so at these locations the path should increase or decrease along the surface at the greatest possible rate.

EVAN: Everywhere else it is perpendicular stuff is changing. Like here [*Evan points to a place on the surface along a curve just past a low point*] it's decreasing and like here [*Evan points to a place on the surface along a curve just past a high point*] it's, uh, increasing.

HAYDEN: Where ever it's perpendicular, it's ... oh, so, it's ... where ever it's perpendicular the, like ... [pauses] the gradient is perpendicular to the level curve. It's pointing in the steepest direction.

EVAN: Oh, yeah!

HAYDEN: Isn't that the definition of the ...

EVAN: yeah. [*Evan looks at the surface at two separate places marked with an 'X' and uses her finger to point in the gradient direction, which also points along the path*] Are all of those pointing in this direction - in the steepest direction?

HAYDEN: ... at that point.

EVAN: yeah. [*Evan points to a third place marked with an 'X' on the path*] This is pointing up that way [*Evan motions with hand in gradient direction on the surface.*]

In the dialogue above, Evan immediately remarks on a significant difference to the behavior of the path at the points marked with 'X' as compared to what was found at the 'O' markings, noting that "it is perpendicular" and "stuff is changing". Evan checks several other points, gathering evidence for the claim that categorizes the behavior at the 'X' and 'O' points. At this stage, Hayden then connects that evidence to their prior understanding of the gradient vector for the surface, focusing on the relationship that gradient vectors are perpendicular to level curves. In this way, Hayden introduces mathematical definitions as they formalize the conjectures proposed by Evan. Evan then checks several places to verify Hayden's claim.

At this stage, Evan and Hayden have recognized the rate of change for the surface along the path has different behavior at points marked with an 'X' and with an 'O'. Although they do not verbalize these claims so succinctly, they have roughly recognized that the path is flat at the points marked with an 'O', and that the path is pointing in the gradient direction at the points marked with an 'X'. Next, they work to identify the specific relationships between the surface, path, contour lines and gradient vectors. In what follows, the term *it* refers to the value of the surface along the path:

HAYDEN: I kind of like your point with the parallel, it's kind of like ... at that point [*motions horizontally with his hand*] it's levelling off.

EVAN: it's staying steady for just a second.

HAYDEN: kind of like at a low or a high ...

EVAN: even if it's, just, like if it's just like ...

HAYDEN: right at the point [*inaudible*]

EVAN: yeah, right there. It still becomes steady for just a second ...

EVAN: So parallel points ... [*Evan begins writing down the relationship on the activity worksheet.*]

HAYDEN: [*Hayden investigates the claim at several points on the path drawn on the surface.*] 'Cause even right here [*points to a high*

> *point on a path*] you see it [*the path*] comes through the highest point right here [*the path doesn't go through the highest point on the surface*] and stops and goes down.

EVAN: yeah. yeah.

Hayden and Evan use their work with the manipulatives to investigate and form two conjectures (Figure 7). Evan describes features which are common across the places labeled with X's, saying, "It's decreasing and like here… it's uh, increasing". After Hayden recalls that the gradient is perpendicular to level curves, Evan then confirms this relationship for the path by checking several places on the surface, and they make a point in their discussion to emphasize that this relationship happens for an instance at a point. The ability to recognize seemingly unrelated knowledge (the concept of the gradient) into this activity, connect it to the X's located on the path, and verify the claim with (previously) generated examples indicates a creative level of thinking on the part of students [4].

Go: Transfer your path (carefully!) to the surface. What happens at each of the marked points? Write down equations characterizing these relationships.

O Parallel points: slope levels off for just a moment
X perpendicular: Path goes in steepest direction on Surface
perpendicular = $\vec{\nabla} f$
Parallel = $df(x,y)=0$

Go: Transfer your path (carefully!) to the surface. What happens at each of the marked points? Write down equations characterizing these relationships.

- parrallel points the graph seems to level off, maxs and mins of the path
- at perpendicular points the graph seems to go in the steepest direction

$\perp = \vec{\nabla} f$
$\parallel = f'(x,y) = 0$

Figure 7: Hayden and Evan's conjectures for the relationship between a path and a multivariable function's gradient vector and level curves.

The emphasis of this activity to explain what happens at the important locations (the X's and O's) provides students the opportunity to propose conjectures and to potentially disagree with each other. In the following excerpt, the students in the group have labeled the X's and O's on the contour map. One student, Logan, makes a proposed conjecture which is refuted by a second student, Morgan, while a third student, Parker, works to transfer the path onto the surface. It is important to note that Logan, who makes the conjecture, is not paying attention to the surface or contour map as he remembers and connects prior concepts covered in the course:

LOGAN: [*Logan is not looking at the surface or contour map*] Perpendicular places, that'll be—slope of, er, it won't be changing, then. Parallel places, it'll be changing a lot.

MORGAN: I think it's the other way around.

LOGAN: No, cause the ... remember when we did the [*puts one finger pointing upward and another finger pointing perpendicular to it.*] ... when it's going this way, it's not changing.

MORGAN: But the ... going perpendicular to the level curve was the quickest—was the gradient.

PARKER: [*Comments while drawing ...*] Perpendicular to the curve is the gradient—that's the steepest direction.

Logan proposes a conjecture while Parker draws the curve and marks the points on the surface. Logan's proposed conjecture is incorrect—he has recalled the relationship between a gradient vector and a level curve for a surface explored in a previous activity. In the current activity, Logan has confused the role of the level curve and the path, which is not a level curve for the surface. Morgan challenges Logan's conjecture that there should be no change in the perpendicular direction, noting that the gradient direction should be the quickest rate of change. Parker provides a memorized fact about gradient vectors to support Morgan. The group is temporarily at a stand-still trying to recall information regarding Logan's incorrect claim.

When Parker finishes drawing the path and comes back to Logan's conjecture, the group makes progress:

PARKER: [*reads off the activity sheet*] What happens at the marked points?

MORGAN: Well the change in Z should be 0. Logan: For the circles or for the X's? Morgan: For the circles. [*Parker and Logan check the points marked 'O' on the surface*]

LOGAN: Yep, the slopes are like 0 pretty much at the O's.

In this interaction, Parker redirects the group to the activity. Morgan proposes that there should be no change in the path at the O's, a claim which opposes Logan's original claim. Logan and Parker check various points on the surface to verify this claim, finding both that it was correct and that Logan's original conjecture was incorrect. It is noteworthy that Logan was not convinced by Morgan and Parker's correct facts about gradient vectors, but by the evidence generated within the group using the manipulatives.

The vignettes above show that students can form conjectures between the surface, contour map, path, and rate of change. However, it is challenging for students to express these conjectures symbolically. Using ideas shared from students, the instructor can ask leading questions which help students restate the property about curves being parallel into statements about the orthogonality of tangent vectors to the path and gradient vectors for the surface.

The Raising Calculus activity includes several places where students can utilize mathematical creativity: drawing their roller coaster, interpreting the concepts of parallel and perpendicular between level curves and the path (especially in regions where the path lies between nearby level curves), investigating cases (e.g. the reversal point for the Superman roller coaster). Students make conjectures about the locations of the X's and O's, and can suggest patterns for these locations (e.g. do the X's and O's alternate?). These conjectures can be used to solve the activity's final constrained optimization problem, providing an opportunity for students to apply their discoveries to problems independent of the manipulative materials.

5.3. The Boysenberry Patch: Conjecture with contour lines and path integration

Both the standard activity and the Raising Calculus activity shown in Figure 8 focus on the concept of scalar path integrals without explicitly writing down an integral sign. The standard activity focuses on both the set-up of the integral and computing the answer using appropriate integration techniques, while the Raising Calculus activity places emphasis upon the various quantities (path, length, density of berries, total amount of berries) incorporated into, and are a result of, path integrals.

The standard problem for integration scores highest of all the standard problems using the rubric in Section 3. It incorporates a context (mass of a wire) and utilizes a shape which can be described with either rectangular or polar coordinates, but it likely requires, a priori, knowledge of the relevant integral form for mass and center of mass. Further, the use of context is relatively weak: No meaningful units are given in the problem, removing a means for students to check the reasonableness of their integral set-up and final answer. The activity could receive relatively high scores if students solved the problem not only by drawing the wire curve but also the underlying linear density function—but the problem does not motivate students to incorporate such representations as a requirement.

Standard Activity: Path integration

A wire takes the shape of a semicircle $x^2 + y^2 = 1$, $y \geq 0$ and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$. [5]

Raising Calculus to the Surface activity: The Boysenberry Patch

On your mark: You plan to go on a hike and pick Boysenberries along the way. The contour map shows the density (in) $\frac{L}{100\ m^2}$ of Boysenberries along the trail. The path is divided into four segments, and you plan to pick berries along each. Rank the four segments according to the total amount of berries you'd pick.

Amount of Picked Berries: Least __ __ __ __ Most

Why did you rank it this way?

<< *Classroom discussion about ranking strategies*>>

Get Set: Path A is 1 km long. Estimate the total amount of berries on paths A and C.

<<*Classroom discussion of approach, generalizing to integral formulation of the problem*>>

Go: Exactly how many 3 liter buckets are necessary to pick berries along Path C? Explain how you arrived at your answer.

Figure 8: Problem investigating path integrals.

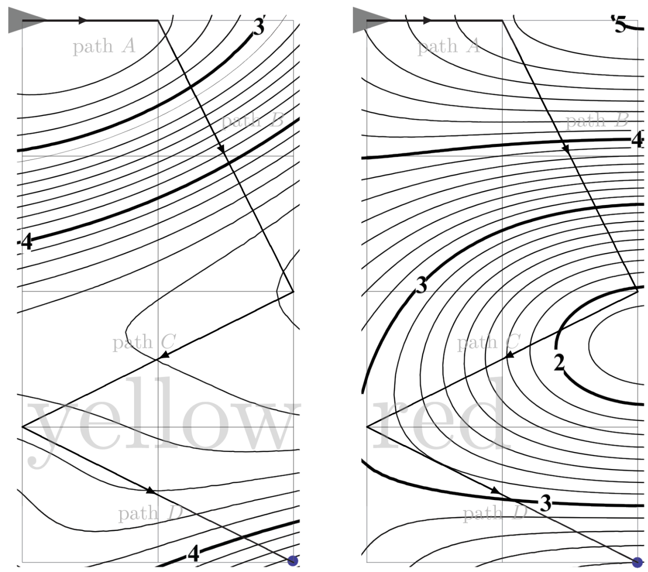


Figure 9: Four paths drawn on level curves for the yellow and red surfaces. Path B crosses the most contour lines for both surfaces.

In contrast, the problem for the Raising Calculus project incorporates several features from the rubric. The main question focuses on determining which path has the most amount of berries. While this is a fairly direct question, students are not told how to solve such a problem. Thus, they have to develop their own procedure and utilize relevant information from the contour map or surface, both of which are provided. As will be seen in student work below, the problem provides opportunity for students to propose connections between the amount of berries and one of two values related to the contour map (e.g. the value of the contour, or the concentration of the contours). The problem also utilizes measurement and context subtly: Paths B, C, and D are more than twice as long as Path A, a fact which can be seen from the contour map and which is helpful in making comparisons for the ranking task. The units chosen for the density of berries, while reasonable for describing berries growing over an area, presents an apparent conflict when students analyze their answers to **Get Set** and **Go.** Students must also account for how far off the trail they reach to pick berries—they pick berries across an area with one long dimension (measured in km) and one much smaller (measured as a meter or less). As students chop up the paths into smaller pieces in order to answer **Get Set** and **Go**, the abstraction process needed to estimate the amount of berries (chop the path into fine segments described by $d\overrightarrow{r}$, find the mass on each segment, and add to find the total amount of mass) naturally leads to formulations of a line integral.

The first question, however, provides opportunities for students to propose conjectures which can be shared with the classroom. Consider how two different groups of students determine the amount of berries on the paths for the red surface, whose contour map is shown on the right in Figure 9. In the first vignette, student Tony joins Evan and Hayden, whom we first saw in The Roller Coaster activity. The three are deciding which path contains the most amount of berries are focusing only on the contour map:

HAYDEN: So, we're just going by the more contour lines you cross?

TONY: yeah. Which ...

EVAN: So the steeper you go the more Boysenberries you get, right?

TONY: yeah.

EVAN: Ok, so cause the more lines you cross, the more berries.

HAYDEN: B would

EVAN: B would probably be the most. yeah.

This argument is one that is encountered by multiple students in the classroom. The conclusion that Path B has the most berries is correct, but for the wrong reason.

At this stage, the students have made the correct conclusion that Path B has the most berries for the wrong reason. The level curves indicate the density of berries. As indicated by Evan's proposal that "the more lines you cross, the more berries", Hayden and Evan have confused the density of the berries with the density of the contour lines. The density of the boysenberries on the path that matters for this problem, and Tony raises this issue with the group:

TONY: er, actually, no. It would be the higher, like the higher ... um, contours. [*points to dark contours and values on the contour map*] This level is 4, and this level would be 3. So B would be the most.

HAYDEN: yeah.

TONY: No wait, Path A is going to be the most [*inaudible*], like on the top right corner it's five.

HAYDEN: Oh, I suppose. Yeah. It's super dense [*along Path A*].

In the exchange above, Tony makes three significant actions. First, Tony recognizes that the value of the contours, not the density of contours, influences the amount of berries on a path. Second, Tony re-evaluates the group's previous answers with this new information. Third, Tony notices that the Boysenberries are most dense along Path A. At this stage, the group should now discuss whether there are more berries along Path B or along Path A, which is much shorter. However, Evan holds onto the notion that the density of contour lines matters:

EVAN: [*Referring to Path A*] But you're only crossing—It's dense, but you're only crossing a very few. If you're crossing a lot and it's a less, what's going to be more?

TONY: Yeah, that's true.

EVAN: I feel like, I understand that it's less but there's a lot more that it just adds up to be more.

TONY: Yeah.

EVAN: I think Path A would definitely be second most.

TONY: Yeah. And then B and then C?

EVAN: Yeah.

TONY: You look at the level of the contour lines and not how many contour lines you're crossing.

In the exchange above, Evan and Tony agree with their order for paths but do not reconcile their underlying reasoning. Evan makes a point to reference the number of contour lines crossed while Tony holds to the idea that it is the level (value) of the contour lines that determines the amount of berries on the path.

The confusion about whether the amount of berries is proportional to the value, or the density, of level curves is fairly common and occurs within

Figure 10: A student uses the surface to point out that the berry density is high along Path B (which runs along the ridge) for the yellow surface, even though the path crosses very few contour lines.

several groups of students. Evan, Hayden, and Tony eventually resolved this issue by using the problem's context and units when trying to calculate the actual amount of berries along Path B. Another technique, shown in Figure 10, is to incorporate the surface. Here, one student traces out each path on the surface and shows that Path B, which runs along the ridge, has the highest density of berries even though it does not cross many contour lines on the contour map. By utilizing another representation (the surface), the student was able to discuss the issue and make their argument with their classmate in a convincing way—even though nothing in the question told them to use the surface.

The instructor can promote a classroom discussion surrounding the competing conjectures by carefully organizing the classroom discussion which occurs immediately after this ranking task. Some of the surface manipulatives, like the red surface discussed by Evan, Hayden and Tony, have one long path (Path B) where there is both the highest values of contours and the greatest density of contour lines. Other surfaces, like the yellow surface (Figure 10), show clear differences between the path which has the most contours (Path B) and the path which has the highest value of contours (Path C, the left contour map shown in Figure 9). By arranging the group presentation order, the instructor can promote classroom discussion by have student groups introduce both the incorrect and correct conjectures (say for the Red surface). Since both conjectures suggest Path B for the red surface, additional arguments from students investigating the issue on other surfaces are needed. Students from a different group can then contribute their findings to support one conjecture and contradict the other. In the example above, the Yellow group's argument of considering the actual path on the surface helps convince classmates about the importance of the contour's value to the situation.

6. Discussion of student creativity in these tasks

The tasks above illustrate several examples where activities written according to the rubric guidelines specified in Section 3 enable students to explore mathematical concepts and incorporate creativity as part of the mathematical solution process. In some cases, this might involve students creating some of the content which will be studied in the activity, such as the roller coaster curve in the second activity. While not observed in the excerpts shared above, the activities sometimes provided opportunities for students to develop and name new concepts, such as the concept of level curves or contours as explored in The Park when they drew points at every location containing the same lead concentration as at the park.

More importantly, however, the examples of students' interactions above illustrate how students were allowed to create mathematical conjectures from each activity. In The Park, student groups proposed five different strategies for aligning the surface with the contour map. In The Roller Coaster, the activity drew student attention to locations where the function could have a local minimum or maximum when constrained to their path, but left it to the students to recognize and propose this pattern. And in The Boysenberry Patch, the activity and surface models provide the opportunity for students to propose competing conjectures which can be discussed, in an authentic way, at the classroom level. Because the conjectures are aligned with the content of the course, they not only further the development of the course using student ideas, but they also help promote the notion that mathematical content is a creative endeavor which is accessible and can be investigated by students.

7. Conclusion

The rubric given in this paper highlights features which can be used to promote meaningful student discussion within small groups as well as whole class discussion as students investigate new mathematical concepts. The focus upon multiple representations, geometric reasoning, measurement, context, and the use of open-ended questions provided activities in which students could formulate and propose conjectures based upon their insights and scenarios created within their small groups. Productive use of these student conjectures requires careful attention by the instructor, but can lead to a classroom which is supportive of student creativity—namely supporting mathematical conjecture—in a mathematics course which is usually focused upon computation and devoid of mathematical proof.

8. Acknowledgement

Partial support for this work was provided by the NSF TUES program (DUE-1246094).

References

[1] S. Ainsworth. The functions of multiple representations. *Computers and Education*, 33(2-3):131–152, 1999.

[2] S. Ainsworth. Deft: A conceptual framework for considering learning with multiple representations. *Learning and Instruction*, 16(3):183–198, 2006.

[3] Heller and Heller. Rating the difficulty of context-rich problems. URL http://groups.physics.umn.edu/physed//Research/CRP/crjudge.html.

[4] G. Karakok, M. Savic, G. Tang, H. El Turkey, D. Plaxico, and E. Naccarato. A rubric for creativity in writing proofs. *Mathematical Association of America Focus Magazine*, 36(1):42–43, 2016.

[5] J. Stewart. *Calculus: Early transcendentals.* Cengage Learning, Boston, MA, USA, 8th edition edition, 2016.

[6] A. Wangberg. Student reasoning with graphs, contour maps, and rate of change for multivariable functions. In TBD, editor, *To appear in Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education*, pages XX–XX, Boston, Massachusetts, 2020.

[7] A. Wangberg and B. Johnson. Discovering calculus on the surface. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 23(7):627–639, 2013.

A. Activity sheets

Standard activity: Level curves	Raising Calculus to the Surface activity: *The Park*
1. Draw a curve on the surface T connecting all the points at the same height h_0 as the blue dot. 2. The curve $T(x, y) = h_0$ is a level curve. Draw three more level curves on the surface. 3. What do the level curves do near a maximum or minimum point on the surface?	**On your mark:** You work for Granite Falls, a town which needs to move a playground due to harmful levels of lead (Pb) in the ground. The surface's height represents the concentration of lead (in g/m^2) in the topsoil at every location in Granite Falls. Lead levels range from roughly 0.5g/m^2 to 6 g/m^2 . 1) The park is currently located at the red star. Mark all points on the surface with the same concentration of lead as the park. 2) Mark all points with lead concentration 1 $\frac{g}{m^2}$ higher and 1 $\frac{g}{m^2}$ lower than at the park. 3) Could these curves intersect? Why or why not? << *Class discussion about intersecting curves*>> **Get Set:** Granite falls is a 10km x 10km town in the 15.5km x 21.5km Rock county. Find where the town is located in the county and place it there. Explain how you know you found the right location. << *Class discussion about aligning the surface with the contour map.*>> [Third and fourth parts removed for brevity.] << *Class discussion: On the surface, point to the best location for a down-hill sledding park.* >>

Standard Activity: Lagrange Multipliers	Raising Calculus to the Surface activity: *The Roller Coaster*
1. Suppose $f(x, y) = 700 - 20x^2 - 40y^2$. Use the method of Lagrange multipliers to find the maximum and minimum values of f subject to $xy = 2\sqrt{2}$.	**On your mark:** Draw a *big, smooth, (interesting)* ride around the dots on your contour map. **Get Set:** Mark spots where your curve is perpendicular to level curves with an X, and places where your curve is parallel to level curves with an O. **Go:** Transfer your path (carefully!) to the surface. What happens at each of the marked points? Write down any relationships between the function and path. *<<Classroom discussion about relationships.>>* **Challenge:** Find the highest point for the ride given by the path $xy = 2\sqrt{2}$ where the ride's height is given by $$h\,(x, y) = 700 - a\ x^2 - b\ y^2$$ with $a = 20\frac{\text{ft}}{m^2}$ and $b = 40\frac{\text{ft}}{m^2}$ and x and y are measured in meters.

Standard Activity: Path integration	Raising Calculus to the Surface activity: *The Boysenberry Patch*
A wire takes the shape of a semicircle $x^2 + y^2 = 1$, $y \geq 0$ and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$. [5]	**On your mark:** You plan to go on a hike and pick Boysenberries along the way. The contour map shows the density (in $\frac{L}{100\ \mathrm{m}^2}$) of Boysenberries along the trail. The path is divided into four segments, and you plan to pick berries along each. Rank the four segments according to the total amount of berries you'd pick. Amount of Picked Berries: Least __ __ __ __ Most Why did you rank it this way? << *Classroom discussion about ranking strategies*>> **Get Set:** Path A is 1 km long. Estimate the total amount of berries on paths A and C. <<*Classroom discussion of approach, generalizing to integral formulation of the problem*>> **Go:** Exactly how many 3 liter buckets are necessary to pick berries along Path C? Explain how you arrived at your answer.