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Everyday Examples in Linear Algebra: Individual and Collective Creativity

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Abstract

This paper investigates creativity in students' constructions of everyday examples about basis in Linear Algebra. We analyze semi-structured interview data with 18 students from the United States and Germany with diverse academic and social backgrounds. Our analysis of creativity in students' everyday examples is organized into two parts. First, we analyze the range of students' creative products by investigating the mathematical variability in the more commonly mentioned examples. Second, we unpack some of the collective processes in the construction of students' examples. We examine how creativity was distributed through the interactions among the student, the interviewers, and other artifacts and ideas. Thus, in addition to contributing to the process vs. product discussion of creativity, our work also adds to the few existing studies that focus on collective mathematical creativity. The paper closes with connections to anti-deficit perspectives in mathematics education and some recommendations for individual and collective creativity in the classroom.

Keywords: basis, everyday examples, distributed creativity, collective processes

1. Introduction

Mathematics as a discipline is inherently creative. We continue to investigate ways that the learning that happens in the classroom can promote and embody this idea. Scholars have noted that opportunities for going beyond acquiring a set of predetermined set of ideas in the classroom are still relatively rare [1, 2]. In the creativity literature, such tension is reflected in the contrast between an absolutist definition of creativity and a relativistic definition of creativity [3]. Whereas mathematicians prioritize grand discoveries of mathematical ideas (an absolutist definition), mathematics education scholars focus on *students'* novel ideas and approaches within the constraint of school mathematics (a relativistic definition).

In the context of undergraduate mathematics, mathematics education researchers have focused on students' creativity in the context of writing proofs and solving formal mathematical tasks [4]. This is similar to works at the K-12 level that focuses on divergent thinking in problem solving (e.g., [5]). In this paper we contribute to existing research about mathematical creativity by exploring students' engagement with a task that extends a formal mathematical idea into novel contexts that might not be inherently mathematical. We discuss creativity in students' constructions of everyday examples for the concept of basis in linear algebra.

Our analysis for the current paper builds on Adiredja and Zandieh [6] and Zandieh et al. [7]. The former is our work with female undergraduate students of color in the United States and their construction of everyday examples for basis. Our participants were predominantly mathematics majors whose racial and ethnic backgrounds include African Americans, Asian, Asian Americans, and Latinx. The latter is our work with male undergraduate and graduate students in Germany and their construction of similar examples. We have previously noted the creativity in our students' everyday examples in these previous works. In this paper we dig deeper into these creative products and unpack some of the collective processes [8] that might have supported their constructions. In the next section a literature review offers a brief overview of existing research about the teaching and learning of basis, and particular findings from our previous studies that we build upon in the current paper. It is followed by the theoretical frameworks that guide our investigation. We present our research questions after these discussions.

2. Literature Review

In this section we discuss literature on student understanding of basis and to a lesser extent related concepts such as span and linear independence. Even though we did not find any articles that explicitly applied the creativity literature to learning basis, in this literature review we highlight aspects of existing studies that illustrate student creativity.

2.1. Basis Literature 2008-2017

A recent survey of the research literature on the teaching and learning of linear algebra [9] found 15 articles published between 2008-2017 that focused on span, linear independence or basis. Stewart et al. [9] noted that very few of these explicitly mentioned basis and called it out as "an area that requires more attention" from researchers (page 1022). The two papers from that timeframe that focused most explicitly on basis were Stewart and Thomas [10] and Bagley and Rabin [11]. Neither of these papers explicitly addressed creativity, but each of the papers highlighted some student ideas related to basis that might be considered creative.

Stewart and Thomas [10] hypothesized ways that students might develop a richer understanding of basis by focusing on relationships between mathematical entities. One unique task in the paper was to have students create concept maps to express the connections they saw between different aspects of linear algebra, including basis. In this way students had the opportunity to reflect on, and illustrate in a creative way, their own conceptions and the connections between them.

Bagley and Rabin [11] considered when and to what extent symbolic calculations about matrices might show a rich understanding of basis. We note that in doing so Bagley and Rabin [11] implicitly discussed creativity in something that might be presumed to be not creative: computational procedures. Students were asked to generate a set of basis vectors in $\mathbb{R}^4$ when given two specific vectors that must be included in the basis. Students shared novel strategies, which included sophisticated versions of guess and check as well as methods relying on leveraging row reduction and linear combinations of vectors.

In re-examining a larger set of papers on span and linear independence both Stewart et al. [9] and Adiredja and Zandieh [6] have noted that there are a range of problem types used in these research studies. For example, Stewart et al. [9] described a range of tasks in their surveyed articles including "checking whether a property holds, creating examples, making connections between ideas, and generalizing to create conjectures" (page 1022). The task in Bagley and Rabin [11] fits what Adiredja and Zandieh [6] call an example generation task.

2.2. More Recent Literature about Basis

In the remainder of this section we discuss two papers, Adiredja and Zandieh [6] and Zandieh et al. [7], that used an example generation task related to basis but in a less formal context. Adiredja and Zandieh [6] and Zandieh et al. [7] share the aim of understanding how students make sense of the concept of basis in linear algebra. The two studies explored students' explanations about the concept using contexts from their everyday lives with two different populations of students. Adiredja and Zandieh [6] analyzed individual interviews with eight female undergraduate students of color in the United States, who were mathematics majors and minors. Zandieh et al. [7] analyzed individual interviews with eight male advanced undergraduate and graduate students in Germany who were in a Computational Sciences and Engineering program. Although the two studies did not set out to investigate students' creativity, they documented students' constructions of creative examples.

Following the anti-deficit framework for studies of sense making from Adiredja [12], the intentional selection of subjects in Adiredja and Zandieh [6] supported the study's goal of challenging existing deficit narratives about the contribution and participation of women of color in mathematics. Adiredja and Zandieh [6] found a high-level of engagement with mathematical ideas from this group of women, which resulted in rich and creative explanations for basis. The eight women in the study constructed explanations and explored boundaries of the concepts of span, linear independence, and basis using more than 20 distinct everyday contexts, including family chores, friendships, religion, and the world market (for an overview see Table 1 in Appendix A). Many of the students also used artifacts that were present during the interview like pens and the room of the interview to construct their explanations.

Adiredja and Zandieh [6] also developed analytical codes to capture some of the intuitive ideas in the women's explanations. These codes described necessary *characteristics* of basis vectors (*minimal, non-redundant, different, essential*), and different *roles* basis vectors played in relation to the vector space (*generating, covering, structuring, traveling*, and *supporting*). These codes uncovered the richness of the concepts of basis, and linear independence and span, the two defining concepts for basis. Adiredja and Zandieh [6] posited that these codes might also represent aspects of basis that can help students extend the concept to novel contexts [1].

Zandieh et al. [7] applied and further refined the methods from Adiredja and Zandieh [6] with the students in Germany. We analyzed the data from Adiredja and Zandieh [6], first, contrary to the date of the publication. The goal of the study was to apply the previous methods and analytical codes to gain insights about students' understanding of basis with a different group of students, but not to compare the two populations of students. The eight students in Zandieh et al. [7] used eight different contexts, including building a house, bits in binary numbers, starfish, and the value of money (for an overview see Table 2 in Appendix A). The study found that students commonly described basis in the context of *generating* a physical structure, like a house or a tower.

Adapting the analytical codes from Adiredja and Zandieh [6], Zandieh et al. [7] uncovered other themes in the data. Students emphasized the notion of *difference* as an important characteristic of the building blocks for the structure. Other students were able to assign different *roles* to basis vectors in their examples. For example, one student, Andreas discussed basic elements in chemistry as a way to *generate* molecules by adding elements together, as well as a way to *structure* molecules by identifying what elements made up a particular molecule.

Zandieh et al. [7] also found that in a number of instances students constructed a new example due to a mathematical issue with their first example. For example, one student, Fritz initially came up with rules to move around in the room as his first example, but noticed that such an example was restricted to a three dimensional space. He then came up with a new example discussing factors that contribute to the real value of money, which allowed for considerations of higher dimensions.

Adiredja and Zandieh [6] also noted how some of the women addressed a mathematical issue in their original example by constructing a new example. For example, a student, Jocelyn introduced cooking as a new context to describe basis to address an issue with capturing all linear combinations in her fashion example. Jocelyn explained that with clothes, one would always need a top, a bottom, and shoes, which did not allow for the use of zero for any of her basis vectors in their linear combinations. Ingredients to make a dish addressed this concern because one had the option of not using some ingredients.

2.3. Building on Two Themes from Recent Literature

In the current paper, we focus on analyzing creativity by building on two emerging themes from Adiredja and Zandieh [6] and Zandieh et al. [7]. The first theme stems from a finding from [7]. They found that there were some contexts that were commonly used by the eight students in the study. For the current study we became interested in exploring in more detail the variety within these common examples and how these played out across more students.

The second theme is related to instances when additional examples were being generated during the interview. In Adiredja and Zandieh [6] and Zandieh et al. [7]) the focus was on the generation of a new example as being motivated by a mathematical issue that a student noted in their previous example. For the current study, we conceptualize the generation of a new example as a moment of creativity. We are interested in exploring other factors that might have motivated the generation of a new example. Before we share details about our analytical methods, we elaborate on some of the theoretical frameworks that influence the analysis.

3. Theoretical Frameworks

Originality is one aspect of creativity that has been the focus of prior investigations into creative products [13, 3]. For example, Sternberg and Lubart [14] have defined creativity as "the ability to produce work that is both novel (original or unexpected) and appropriate (i.e., useful or meets task constraints)" (page 677). In our investigation of creativity we also emphasize the reciprocal relationships between insights about students' mathematics and those about their creativity [3]. In this way, we focus on the potential for students' creative products to reveal insights about their mathematical understanding, and also the way that mathematical analysis of these products might reveal insights into students' creativity. As we noted in the introduction, we focus our investigation on students' construction of everyday examples as creative products.

Our view of students' everyday examples is influenced by the work on conceptual metaphor [15], [16], which focuses on the use of a metaphor to unpack the structure of a concept and ways that entities in the concept interact. In the undergraduate mathematics education literature conceptual metaphor has been used to examine student understanding of a number of concepts including function (e.g., [17]), limit (e.g., [18]), and derivative (e.g., [19]). We consider students' everyday examples as potential conceptual metaphors that allow insight into each student's creativity as they discuss their understanding of basis.

Part of our analysis focuses on the notion of *collective creativity* [20] by focusing on the interactions between the interviewers, the student, and other artifacts (see also "distributed creativity," [21], page 82). We did not organize our data collection to study how different students equally contributed to, or collaborated in a group to produce a creative product or solution (e.g., [22], [21]). Having said that, our data still has the potential to offer insights into "moments when the creative insight emerges /. . . / across the interactions of multiple participants in the process" ([8], page 484). As we alluded earlier, our analysis focuses on the transitions between examples as moments of creativity. We argue that these transitions were not an independent individual cognitive act by the student but instead "insights that emerge in the interactions between the individuals." ([8], page 484).

To analyze the collective creative *process*, Sawyer and DeZutter [21] have recommended a focus on the interactions between members of the group, as such interactions are a "substantial source of creativity" (page 83). They highlighted "moment-to-moment contingency" (page 82) "the dependence of one person's action on the action that came just before" as an aspect of a collective creative process. Hargadon and Bechky [8] have noted a similar type of contingency in interactions. They highlighted how a particular suggestion by an individual at a given moment could be easily be dismissed as crazy, and ultimately forgotten. Or, the same crazy suggestion is considered and built on by others, becoming more realistic and, ultimately, leading to a creative solution. Here the original comment takes on new meanings-becoming creative-through the mindful interactions of participants in the problem-solving process ([8], page 487).

We observed a similar contingency structure in our interactions with students during the interview. Some of the ideas students presented could as easily be dismissed as nonsensical or non-mathematical. Students themselves often wanted to dismiss their ideas due to some mathematical limitations in their examples. In most cases we pushed students to suspend their disbelief and further explore their ideas (see also "tolerance for ambiguity," [2] page 435). We also observed cases where students built on others' ideas. Such situations motivated our analysis to focus on the interactions that occurred during the interview.

The literature review and the theoretical frameworks both frame and contextualize the following research questions for the paper:

1. What is the range of creative products that we can observe from students' common everyday examples?

2. What insights into the collective creative process can we glean from analyzing moments of creativity during the interactions between the student, interviewers and other artifacts?

4. Methods

4.1. Data collection

For the current paper, we consider data from individual semi-structured interviews with 18 students with diverse academic and social backgrounds. Names of students in this paper are all pseudonyms. We add to the data from the eight American female students of color in Adiredja and Zandieh [6] and from the eight German male students in Zandieh et al [7], two additional interviews that Zandieh conducted with two female students in Germany that we have not previously reported. All of the interviews were completed around the same time. The interviews in Adiredja and Zandieh [6] were led by Adiredja but Zandieh also asked follow-up questions. The interviews in Zandieh et al. [7], as well as the interviews with the two female students were led by Zandieh with the help of her collaborator in Germany. In other words, all of the interviews involved interactions between multiple individuals, and not a one-on-one interaction.

The two studies used the same interview protocol (see Appendix B), though follow-up comments or questions might differ with the different lead interviewer. For the purposes of this paper, we focus our analysis on students' responses to the following questions:

Q2. (a) Can you think of an example from your everyday life that describes the idea of a basis? (b) How does your example reflect your meaning of basis? What does it capture and what does it not?

Q4. Can you see a basis as a way to describe something? If so, what is the something? How?

Q5. Can you see basis as a way to generate something? If so, what is the something? How? Prior to answering these questions, students completed some exercise problems about basis, and shared with us what basis meant to them (Q1a). We gave students access to a written definition of a basis after they answered question Q1a.

4.2. Data analysis

With the new aims of the current paper, the first step of our analysis was aligning the data from Adiredja and Zandieh [6], Zandieh et al. [7] and the additional interviews with the two female students from Germany. We identified everyday examples from the two students using the same method used for the other 16 students as described in our previous papers. A complete list of the examples from the two new students is included in Table 2 in Appendix A. Each example required one aspect of the context to serve as the basis vector(s), another aspect to serve as the vector space spanned by those vectors, and some description of the relationship between the basis vectors and the vector space. We used the revised roles codes from Zandieh et al. [7] to analyze examples from the two new students and reanalyze the data from the eight students in Adiredja and Zandieh [6]. For the full code descriptions and examples, please see Table 3 in Appendix C.

Having aligned the data, we began our analysis for our two research questions. To answer the first research question we focused our analysis on commonly mentioned examples across the 18 students. We first noted what students most commonly used as the basis vectors, the vector space, and the relationship between them. We determined the most commonly used relationship utilizing our revised roles codes. Afterwards, we examined the variability of the mathematics discussed within each of the common examples to reveal the range of creative products.

To answer the second research question, we focused on analyzing moments of creativity wherein students generated additional examples during the interview and how the new example came about. Zandieh et al. [7] reported cases when the construction of a new example was specifically motivated by the student's dissatisfaction with the ability of their current example to address some aspect of basis. Here we examined this kind of transition as well as other types of transitions between students' examples.

We distinguished transitions that occurred independently from the interviewer, and those that were motivated by an interviewer's question or comment. The number of transitions is smaller than the total number of examples listed in the Appendix. Sometimes students came up with multiple examples at once so there were no transitions between those examples. We also counted multiple examples that emerged from Q4 and Q5 as one transition because sometimes these questions were asked together during the interview. In sum, we focused on the transitions that occurred as a result of interactions between the student, the interviewers, and the mathematics.

5. Results

5.1. Observing Creativity in More Common Examples

We found three common characteristics that were shared in students' examples. The first set of examples shared an everyday object as a basis vector. A number of students used pens as basis vectors. The second set of examples used a room or often the room in which the interview was conducted as the vector space. The last set of examples similarly described the relationship between the basis vectors and the vector space using the roles code generating. We found that students commonly used the context of cooking/recipe and building a structure with this code. In Adiredja and Zandieh [6], we discussed ways that the recipe example can play out differently with two of the American students. Here we focus on the variability with the building a structure.

5.1.1. Pens and pencils as basis vectors

Four of the 18 students created an example involving pens or pencils. The general shape of a pen or pencil may have an association with the shape of a drawn vector. For example, when asked by the interviewer for "something in the real world that's sort of like a basis," one of the German male students, Colin suggested,

> Maybe you can explain it with pencils? If the pencils are in this direction [*arranges two pencils next to each with tips in the same direction]* they are linearly dependent because they are in the same direction. But these two [*arranging the pencils in a V shape*] would be independent because they are different directions.

Colin used this example to explain basis, specifically highlighting the notion of linear independence. For Colin the goal was to use the pencils to describe all possible directions that he could point to. Tobias was another German student who used pens as directions. However, for Tobias it was to describe different ways to move in the room.

Eliana, one of the American students also used the pens in her first example. However, instead of focusing on constructing different directions with the pens, Eliana focused on discussing the notion of span. She conceptualized the span of the vector as the space needed to support the vector. Earlier she had made a distinction between the span of two pens vs. one pen. Here, she summarized her conceptualization with the interviewer.

ELIANA: You can carry the two pens. Like in a two-dimensional plane. With one [pen], you can carry it in a line. And so.

ADIREDJA: Right. Whereas if you have a line, you can just like carry that with just a single, like, if you have a string, you can attach that and carry that, with just that thing.

ELIANA: Yeah.

ADIREDJA: Whereas with two pens, you need the whole plane or the whole paper to in order to accommodate or to support it. Right.

ELIANA: That's kinda what I was trying to say.

Morgan, another American student went an entirely different direction with her pen example. Morgan conceptualized linearly independent vectors as vectors that could "stand on their own." By this she meant that they could not be obtained from the other. She kept using the phrase so we asked her if she could provide an example to explain this notion of standing on their own. She offered her pen example.

MORGAN: Yeah just standing on their own. Um so you have? Let's see. It's hard because I have to compare like a set, maybe. Um I'm trying to think. So maybe like a group. Pens. That's what that came to me. So I think, I don't know, you have a group of two pens, one pen. Like each group doesn't stand on its own because, I don't know, you can have, you have two groups of one pen and get the same group of two pens. I feel like that's really simplistic.

ADIREDJA: Say a little more about that.

MORGAN: You have a group of, you have one pen on its own. And that's like one group. And then you have a group of two pens. Um those two groups wouldn't stand on their own because you could have like two groups of, like, group 1. That would be the same as group 2. When we asked if she could apply basis to the pen example, she said,

MORGAN: Um, let's see. See, when I think of a basis, I think of a span now. So, it's kind of hard to, um, maybe I guess if the span is just like as many pens as you want. Like from zero to infinity then it's just like, what? $\mathbb{R}^1$? Um then you only really need group 1. Morgan, instead of using the pens for directions, she used it to construct groups of pens. She essentially argued that a group of one pen and a group of two pens were not independent. But then from there she constructed a new span, which was a group of as many pens as she wanted. She equated this group to the one-dimensional space, $\mathbb{R}$.

All four students used pens as basis vectors in their everyday examples, but there was variability in both the mathematics they emphasized in their examples and how the pens represented basis vectors. Colin and Tobias used pens to mark directions. Morgan used pens as objects in a set, whereas for Eliana, the pens were objects to carry. With respect to the mathematics, Colin and Morgan both prioritized discussing linear independence with their pen example. However, Colin focused on the pens pointing in different directions to illustrate independence, whereas Morgan used different groups of pens to illustrate a dependent set of vectors. Unlike Colin who focused on describing all possible directions as the span, Eliana conceptualized the span as the space needed to carry the pens. Altogether the variability in these examples reveal a range of the creative products from students.

5.1.2. The room as the vector space

Seven students used a room in their everyday example. Our coding using the roles codes revealed two ways that students utilized the room context. Some students focused on ways that different objects could serve to structure the room (e.g., students' arms and legs, the edges of the floor and walls as potential "coordinates" for points in the room). Other students focused on directions to travel around the room (e.g., Tobias' earlier example to use pens as directions to move around the room). Now, we focus on ways the room example helped students navigate different mathematical ideas related to the concept of basis. Again, focusing on the mathematics showcase students' creativity through its variability in their room examples.

Eliana, one of the American women started with the idea of her arms and legs structuring the room. She then came up with another example where she talked about the dimensions of a storage room as its basis. She explained,

> You can fill [a storage room] based on how much you have. How much room you have. Sort of.. I don't think that really applies because a basis is the least amount in order to cover an entire space. That doesn't really work because the space would be limited. And with a basis it's just a representation, like a representation of an unlimited area.

Here we observe Eliana recognizing one limitation of the room example: the room represented a finite space, which a vector space was not. This seems to lead her to dismiss the example. Unlike Eliana who found an issue when analyzing the room example, Morgan resolved a mathematical issue using the room example.

For most of the interview, Morgan implicitly assumed that a basis had to span the entire vector space, like $\mathbb{R}^2$ or $\mathbb{R}^3$. She took several turns to make sense of subspaces having a basis. She came up with an appropriate example of a plane in $\mathbb{R}^3$ that she knew had a basis, but it did not immediately convince her. It was not until she mentioned column space, null space, and eigenspaces that she felt that the issue was resolved. She then immediately offered a new everyday example with the room where we held the interview: "I just imagine a physical 3D space, like this room /. . . / I'm thinking that point (pointing to the corner of the room) is the origin. And it would just have positive coordinates."

Morgan used points in the room as an example of a "subspace" of $\mathbb{R}^3$ that had a basis ("three coordinate vectors"). She provided a constraint that the points had to have positive coordinates. Resonating with Morgan's example, Alina, one of the German female students came up with a unique example that could be interpreted as a subspace within the room context. She described a plane within the room using eye contact between people as a basis.

> I wanted to, something like, someone sitting there, I'm sitting here, and you're sitting there. And I draw lines between us and look at each other and we all do in the same plane. Height. /. . . / We're all the same height. And this plane's um, something like, um yeah what we want to get, and eye contact is the basis.

Alina effectively created a subspace within the room using eye contact.

Stacy, another American student, also used a room in her everyday example. Her idea resonated with Morgan's idea in that she also discussed the notion of positive and negative coordinates. Stacy initially discussed the room example in terms of the directions she moved in her room in the morning as she was getting ready. The use of her personal room instead of the room of the interview was unique.

> So I have my bedroom and I start from my bed. I go one way and that there is a vector for me, which then I see I don't need

> to go that way, which then I kind of go another way, which side is a basis in that plane of my room.

We then asked her how she might describe basis to her mother, and she said,

> Um, well first I would tell her to think of the floor as a plane. The floor is a shape of a square, parallelogram, and that, say the vectors start off in the corner of the two corners of the wall. The two vectors create a basis. And then I could use those two basis [vectors] to help go to the back kind of thing, the negative side, which I would be, like, it's the other side of the room. It'd be the other room next door. It'd be the same shape, just the negative way.

In this example, she incorporated the notion of "negative side," which most students did not consider in their examples.

This set of examples from the seven students used a room as the vector space. We observed a range of variability in the creative products through the different roles codes represented. Some students focused on the objects in the room (e.g., students' own arms and legs) playing the role of *structuring* the room, while others focused on *traveling* within the room. Analyzing the mathematics further revealed the creativity in students' examples. Eliana's storage room example solidified for her the need for a vector space to be infinite. Morgan solidified her newly developed understanding of the basis of a subspace using her example. Alina might have also played with subspaces by effectively creating one within the room. Beyond acknowledging the constraint of having positive coordinates in the room as Morgan did, Stacy accounted for negative constants by going to the room next door. The range of creativity revealed in these examples also highlights the reciprocal relationship between insights into students' understanding of the mathematics and their creativity.

5.1.3. Building a structure

Seven students used building as a context in at least one of their examples. These examples also share *generating* as a code describing the relationship between the basis and the vector space, with the specific verb to build. The structures that the students described as being built and the materials that the students build with varied. The structures students used included a house, a tower, a shelf, an essay, or a dinosaur Lego, and so the materials varied from bricks and other building materials to Lego pieces. As with the pens and the room, here we focus on the mathematical ideas students explored while discussing their building example.

We begin with a description from one an American female student, Morgan and German male student, Andreas to give a sense as to how students used building to explain basis. Morgan shared how she thought of basis,

MORGAN: I think of them as like building block vectors. /.../ So, there'd be a basis for something /.../ and the basis is like the rudimentary building blocks for that space. /.../ Like the 3 by 2 Lego and you have like a 2 by 2 Lego. You can just like, build on to that to create that space that you have (emphasis added).

Morgan immediately connected to the building idea and mentioned Lego blocks to discuss basis. Andreas similarly used the idea of generating for basis and said, "Yes, generally it's like you have a lot of bricks. And you can *build* something with it. And each brick is unique. And then you can build something bigger. And this [brick/basis] is kind of a fundamental thing?" (emphasis added).

Different students navigated the notion of linear independence, span, and vector space differently. Stefanie, a German female student discussed building a wooden shelf.

STEFANIE: Some wooden boards and screws to like, to try to build, I don't know, a shelf? And still say, ok, like, anything I could build with it are linear combinations of these. Cause maybe I need like 4 wood boards and some screws or ... I mean, so the basis are the screws and the boards. Cause these are like the elementary things I need. I might need multiples of, I need the kinds of boards and screws. You can add them different [sic]. Having specified the basis, she then explained what the vector space would be, and examined if her example could account for the linear combination idea.

STEFANIE: I'd say in our vector space, anything in our vector space is a shelf. We build it from wooden boards and screws. So, the first [the notion of span] is definitely true.

ZANDIEH: The linear combination feels pretty true. It's working. Yeah.

STEFANIE: Yeah. Having just one possibility, it kind of- We probably have to say, like, so if I make, like, some shelves of just like two screws for the same board instead of one. It's still the same shelf.

ZANDIEH: Oh. Right. What you count as the same shelf.

STEFANIE: Yeah, cause that would make the second property [linear independence]. ...So, you're saying, ok, because like what was once was not the same shelf anymore...it also has the uniqueness thing then if I say it looks the same.

Stefanie felt that the notion of span (or linear combination) was captured by her example. She included stipulations to determine if two boards are the same (i.e., same board but different number of screws), to account for the notion of linear independence. She explained that her vector space would be a space of all shelves.

Simon, a German male student grappled with what amounted to be the vector space in his example. Initially, Simon used the house as the vector space built from wood, stones, and glass as the set of basis vectors. He highlighted the way that his example captured the notion of linear independence, but "maybe can't get the vector space of the house. It's maybe a bit difficult to see that." Perhaps consistent with the requirement for vector spaces to be infinite, as discussed in Eliana's storage room example, Simon shifted the vector space. He said, "I guess it would be everything you can build up from stone, glass, and wood. It could be a house; it could be something else." Another German student, Colin uniquely introduced people as an additional basis vector in the building context, which has the potential to start a discussion about nuances with the notion of independence. For example, one cannot get people from bricks, yet are the amount of materials independent from the number of people? How does this relate to basis?

The seven students used building a structure as a context to describe the concept of basis. We observe variability in the details within students' building examples as related to the structure itself (e.g., shelf vs. house) and the building materials (e.g., bricks vs. people). At the same time, when we focus on the mathematics in their examples, we observe the differences in the way that students constructed the object that represents the vector space (e.g., Stefanie's shelf vs. Simon's and Colin's house). While Colin did not explicitly discuss linear independence as Stefanie did, his addition of people as one of the basis vectors would likely generate a different discussion about its role in his example.

While the idea of *building blocks* is an intuitive way to think of basis, our students often seemed dissatisfied with some aspect of their building example. In particular, a number of them raised concerns related to the action of combining the building material (e.g., how to combine Lego bricks that lie on the same plane). Stefanie, one of the German women, preferred her Pancake recipe example because it allowed her to "really combine the ingredients," as well as to use more ingredients, thus working with higher dimensional vector spaces. This example adds to the cases from Zandieh et al. [7] where a mathematical issue motivated a generation of a new example. We turn our analysis to this process of generating a new example.

5.2. Creativity in and through Interactions

The transcripts from the first section of the analysis already showed a few examples of the interactions between the interviewer and the students during the interview. We found that the majority (21 out of 27) of instances when students constructed a new example after having shared one occurred in response to a particular interaction. This shows that the creative products were not constructed by students on their own, but that they were influenced by the interactions during the interview.

Analyzing the development of new examples, we found that students constructed a new example as a result of: 1) the request of the interviewer for another example without any specific guidance (3 instances); 2) the request of the interviewer for an example for a specific context or audience (6 instances); 3) the suggestion of a different structure or idea to think about basis by the interviewer or the protocol (8 instances); and 4) an interviewer's mention of an everyday example by a student from another interview (4 instances). These four cases can be interpreted as different types of moves in an interaction that illuminate some of the collective creative processes.

The first type of move offers minimal guidance for students. We left space during the interview for students to share as many examples they could. Sometimes we would just remind students that we would be happy to hear another example if they had one. Fritz' weather example, and Stephanie's and Jocelyn's respective cooking examples emerged in this way. In what follows we focus on the other three types of interactional moves, where additional information enters the interaction whether through the interviewers or some of the present artifacts.

5.2.1. Requesting for a change in some aspects of the context

One specific context we asked students to shift was the audience for the everyday example. We asked students to consider a situation where they needed to explain the concept of basis to someone outside of the classroom or STEM community, such as their grandparents, non-science or mathematics majors, or a much younger student. Doing so supported some students in the study to generate a context that appeared "less" mathematical. Linus' shift from moment of inertia example to moving around the room was one example. Eliana's shift from the "least amount of myself to cover the room" to the "dimensions of a storage room" was a result of being asked to construct an example more accessible to someone who might not know as much mathematics.

Sometimes we asked students to move away from a specific context or particular way of thinking about basis in their first example. For example, Eliana's shift above was also in response to our request for an example that was less vector-like than her original example. Matthias's first example, like many others, used a "spatial" or "movement" context (e.g., the room or driving). He shifted his map example to the building/tower example after we asked him to consider an example that's less about movement. The next set of instances of new example generation involved more interaction of ideas, which came from either the interviewers or the protocol.

5.2.2. An introduction of a specific idea to think about basis

In eight interactions, students worked with an additional piece of information in the form of a specific structure to think about basis. The information came from the following sources: 1) two particular questions from the protocol (5 instances); 2) the written definition of a basis used in the interview (1 case);

and 3) students' general conceptualizations mentioned during the interview (2 instances).

We found that question four and five in our protocol led a number of students to construct a new everyday example. These questions asked if students could see basis as a way to "generate" something or "describe" something. In retrospect, it made sense that these two questions offered two different ways to look at the concept. A number of students constructed an example that represented each idea after we asked them this question. As we have discussed in Zandieh at al. [7], Andreas bricks example and molecules examples were constructed to explain the notion of generating the vector space and describing the vector space, respectively. We use Nadia's example to illustrate how a student used new information from the other sources aside from the questions from the protocol. Nadia's construction of a new example was influenced by the written definition we provided her during the interview and the way she made sense of this definition. In this definition, a basis, B is defined for a subspace H, within a vector space V. Nadia read this definition after she explained what basis meant to her. And reflecting on the written definition and how it related to her understanding, she said:

NADIA: I think of [V] as this big blob thing. And like H is in there. And it has like, these vectors. And if they're linearly independent, and you can linearly combine them to make all of the H. then it's a basis of H.

During the interview this got referred to as the "blob within a blob" idea, which she ended up using in her everyday example.

Nadia's first everyday example focused on the room, specifically the floor of the room we conducted the interview. She recalled this example from her linear algebra class. At some point we asked her if there was a vector space that was not "geometric" (away from her initial conceptualization). She then offered $\mathbb{P}_n$, the vector space of polynomials with degree n or less. From these spatial and polynomial examples, we asked her if she could use her "blob within a blob" idea to explain basis to someone who might not be familiar with mathematics. She responded with another example with planets and stars as a basis for the universe. However, incorporating the definition of basis from the text, she ended up generating yet another example: natural elements as a basis for the earth.

NADIA: Yeah, it's like. Maybe I'm thinking more like, earth as a subspace within the universe. Like, the V is the universe and H is like the earth. So, earth is like, maybe B is like, [laughs], I'm making stuff [up].

ADIREDJA: This is fun. Keep going.

NADIA: So B would be things that describe earth, but you wouldn't make—It's like the list of things that are like, maybe it's like, the list of the elements of the earth or something. And they're, like, unique to the earth. But you only list each one once. You wouldn't say water twice, or hydrogen twice, or whatever. Because that would be extra.

We considered the planets and stars and the elements for earth as two different examples because each uses a different everyday object as the basis and the vector space. Among the students, Nadia was unique in explicitly incorporating this definition into her everyday example.

The development of Nadia's two examples highlights Nadia's interaction with two sources for a new structure to think about basis: the written definition and her own sense making of that definition. It also highlights the reciprocity between the two. The "blob within a blob" arose from the definition, which helped produce the planets and stars example, which was later modified to account for the specifics in the definition! Nadia's case is an example where the interviewer reflected back the student's way of conceptualizing basis, which resulted in a new everyday example that accounted for specific ideas in the definition the student read. The next interactional move also offered a new idea of thinking about basis, except that it is in the form of a complete example from other students.

5.2.3. Introducing other students' examples

With a few students, the interviewers opted to share other students' examples to support students in generating their own everyday example. Sharing examples was not part of the protocol. Johann's recipe example, Colin's building a house example, and Stacy's marching band and house chores examples were all constructed after hearing some version of other students' examples. Here, we share Stacy's examples because of how different the examples were compared to what she heard. Both Johann and Colin stayed in the context that was suggested by the other students' examples, baking and building a house, respectively.

Stacy started with a room example. She explained basis in terms of the route she took in walking in her room every morning. However, after the interview we continued to chat with her and shared some of the examples we had heard from other students. Stacy happened to be a student in Adiredja's linear algebra class. So, at this point, the interaction turned instructional. The first author started by sharing Jocelyn's fashion example of tops, bottoms, and shoes to generate the set of all outfits. From that example Stacy constructed her marching band example:

STACY: Well, I remember in marching band where one, uh, we had to create a circle. And then we kind of, you need the people to create the circle. And you need the field. And then you have to have the music, which then, those are the three and that's what can help to move on to the different parts of the field for marching band.

ADIREDJA: That's an interesting way of thinking about it. Sort of different components that are necessary to create the space in some ways, you can think about it that way. I was thinking about a band. You need the vocalist; you need a bass player. You need a drum player.

STACY: A drummer [*correcting*].

ADIREDJA: Um what else do you need? Probably need a guitarist or something. You can have these different things, right? Because then together they can then create the music. And each one is necessary. You cannot remove one. You cannot remove the drummer because then it'll just be a weird thing. You cannot remove the bass player because then you're missing the bass in your song, or the music that you perform, right?

The first author happened to know that Stacy used to be a drummer for a band at some point, hence the connection. After this, Adiredja also shared Leonie's friendship example, and Lilianne's basic moral teachings of a religion example.

ADIREDJA: Can you think of anything else that works like that?

STACY: Uhm. My mom telling us to do all the chores?

ZANDIEH: [*Laughs*]

STACY: Each of us had a different thing to do in the house.

ADIREDJA: Oh!

STACY: So like mine was taking out all the trash from the house. My brother was vacuuming the living room. And then Richard's [her other brother] was cleaning up the kitchen. speakAdiredja Ok. And all of you together, you have a clean house.

STACY: Yeah.

ADIREDJA: But if you remove one of them, there's gonna be a piece missing, right? And also, all three of you, I mean of course you can help each other out, but the idea is that you're given an independent task, right? That you can do yourself, right? That's sort of the idea of having a basis. Is-Is that helpful?

STACY: Yeah.

From this particular example, we observe that sharing other students' examples can be really productive in generating additional creative products, ones that were quite different from the presented examples. The case of Stacy also showed a co-construction of examples, albeit not a prolonged one. With the band example, the interviewer took a different direction from the student's marching band example, while appealing to the student's interest. With the chores example, the interviewer made sense of the brand-new example while simultaneously adding related ideas into the example (e.g., the necessity of all the basis vectors, and linear independence).

6. Discussion and Implications

6.1. Creativity in Unique and Not-so-unique Examples

In this paper we dug deeper into the creativity in the students' constructed examples. The examples were clearly imaginative, clever, elegant, or surprising, beyond analytical thinking [23]. For the first part of the analysis we found that when students' examples shared common basis vectors, vector space, or even the notion of "building," the details and the mathematics of their examples further revealed students' creativity. We found quite a range of creative products even within those that shared common characteristics. Presenting the "common" examples together revealed the variability in their examples.

Analyzing the mathematics in students' example further showcased students' creativity. We observed instances when students used their creative products in making sense of the mathematics. Making an everyday example to effectively represent different aspects of basis is also another creative (and challenging) act that many of the students engaged in beyond generating novel examples [3]. This highlights the reciprocity between the process of gaining insights into students' mathematics and creativity.

From the transcripts, we also got a sense of the way that these students constructed the examples on the fly. They did not always have all the components of their examples figured out. In fact, a few of them dismissed their examples altogether after finding a mathematical issue. In other words, the "survival" of the creative ideas was at times contingent upon the interviewers' responses to them, which highlights the critical role of the interactions that occurred.

6.2. Collective creative process

The second part of analysis focused on the creative process by examining the nature of the interactions that occurred during moments of creativity. In contrast to the focus on the creative product in the first part of the analysis, we explored aspects of the interactions that contributed to the generation of new examples, or the collective creative process.

It turns out that in our context, simply asking for another example led to a generation of a new creative product with some students. Shifting the audience or asking students to move away from particular ideas in their original example also generated new examples. Asking students to explain mathematics to a less knowledgeable friend is a known and productive interview method to access their understanding. In our context, such a question was also productive to generate new creative products. These moves did not introduce any new ideas into the interaction.

We learned that some of the written artifacts and the questions from the protocol about basis as a way to generate and to describe a space introduced particular ideas into the interaction. Students took up these ideas in interesting ways, generating a range of creative examples. We also observed productive ideas that emerged from students themselves (e.g., Nadia's "blob within a blob" idea, or other students' examples). Asking students to reflect on these ideas during the interactions supported the construction of additional examples. In other words, our analysis observed moments when these creative insights emerged during interaction and illustrated how creativity might have been distributed across individuals and artifacts [8], [21].

6.3. Countering a Deficit Interpretation and Deficit Narratives

Focusing our analysis on the mathematics in the students' examples also challenges possible deficit interpretations of the data. A reader might focus on the mathematical imprecisions or even misconceptions from the students (e.g., finite vector spaces). Our analysis highlights the fact that students themselves identified many of these issues. Students' critiques went beyond connections to the definition of basis, span or linear independence and went to thinking critically about other generalizations (subspaces and other related theorems).

A newly constructed example cannot immediately address all aspects of the concept. In this way some mathematical limitations should be expected. That said, what we have seen in our data is that beyond being expected, mathematical limitations in fact generated more creativity. Sometimes this happened as students resolved the mathematical issue within the context itself. Other times we saw some students generating a new example entirely after recognizing a mathematical issue. Adiredja [12] has alluded to fixation with mathematical consistency as one value within the mathematics community that might inadvertently contribute to deficit interpretation of students' work. The findings of this study suggest that temporary suspension of mathematical precision might in fact be productive for creativity. Fixation with inconsistencies might be the antithesis of creativity.

Creativity has been tied to mathematical ability and giftedness (e.g., [24, 25]). Adiredja [12] has also highlighted the role of research studies that focus on mathematical sense-making to challenge deficit narratives about the mathematical ability of particular groups of students. Intentionally selecting students who were women of color in the United States in Adiredja and Zandieh

[6] supported the effort in challenging deficit social narratives about the contributions of women of color in mathematics and their mathematical ability. Thus, highlighting the creativity of all of the students, including the women of color and the female students from Germany further challenge any deficit narratives about them in the United States [26], and globally [27].

6.4. Classrooms as Potential Sites for Individual and Collective Creativity

Generating everyday examples for basis can be a fun activity for creativity in an undergraduate mathematics classroom beyond proofs and traditional problem solving. Adiredja, Bélanger-Rioux, and Zandieh [28] documented one mathematician's implementation of this example construction task as part of homework in her course. The study also found a diversity of examples that students constructed, not all of which were documented in our previous studies.

Bélanger-Rioux shared her ideas on how she could build on students' examples. Much like how it happened after the end of Stacy's interview, classroom teachers, instructors, and students can be creative collaborators. We note that students in Adiredja, Bélanger-Rioux, and Zandieh [28] had more time to construct their examples as a homework assignment compared to students in the current study. The fact that we found similar patterns in our data with the little time students had during the interview does not take away from the creativity reflected in those classroom examples. Instead it highlights how accessible students' creativity is with this seemingly "simple" task.

Contrasting the mathematics students discussed in our analysis also shows that students' examples can be put in communication with one another. Shared contexts provide a common language for students to discuss relevant mathematical ideas. While one student might feel that their example was restricted in its capacity to capture an aspect of basis, another student might have addressed the issue in theirs (e.g., Morgan's positive coordinates restriction in her room example vs. Stacy's going to the other room to consider the negatives). In other words, if a student had an issue with their example, another student might have a way to resolve it in the same context. In the classroom this can manifest into a collective generation of creative products.

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A. Everyday Examples from Germany and the US

Table 1. *Everyday examples from the American women of color in [6].*

Student	Context (for basis and vector space)
Leonie	• different individuals representing different desired personalities in a friendship • three friends creating a cube of desirable characteristics • different continents contributing to the world market
Morgan	• 2x2 and 2x3 Legos to build a dinosaur driving (on a grid) to get anywhere in the city • sugar and eggs for making a recipe • a single pen to generate groups of pens • a corner (basis vectors) for a physical 3D space (e.g., a room) • 1x1 Lego to generate all Legos
Annissa	• different ideas to solve a problem with multiple solutions
Eliana	• a piece of paper (space) carries two pens (basis vectors) • using length of arms and body to cover the space of the room • dimensions of a storage room or a house • building from a skeleton • writing from an outline of a paper
Nadia	• floor (as basis for the ground) • edges of the floor (as a basis for the floor) • planets and stars (as basis for the universe) • natural elements (e.g., water or hydrogen, as basis for the earth) • syntax in a programming language (as a basis for a program code)

Jocelyn	• pieces of clothing (as a basis for different outfits) • cooking ingredients (as a basis for different recipes you can make) • art sculpture & collage (capturing the Spanning Set Theorem)
Stacy	• directions to walk in a room as a basis for (all the places you can get in the room) • edges of the floor (as a basis for the room and the room next door) • band, music, and the field (basis vectors) as components for a marching band formation (space) • self and siblings covering all of mom's chores assignments
Liliane	• using vectors to go home/house at a non-origin location • most basic teaching of a religion (as a basis for all decisions)

Table 2. *Everyday examples from the German students (including the two women). The table is partially adapted from Zandieh et al. (2019).*

Student	Everyday context
Andreas	The masts of a sail spanning the sail Legs of a starfish as directions to access all of the sea floor Bricks to build something bigger Elements as basis for all molecules
Colin	Pencils pointing in different directions Stones and people to build a house
Fritz	Two rules to get to any point (on the floor of) a room Economic factors that influence how much money is worth The conditions which are needed to describe the weather at a particular point

Johann	Eight binary bits to describe a space of numbers The ingredients needed for a recipe
Linus	Using principal axes to describe moments of inertia Directions to go to find a point in the room
Matthias	North and east directions (or northwest and northeast directions) to make any vector on a map Bricks to build towers
Simon	Stones, glass and wood to build up a house or other item built with these materials
Tobias	Directions to reach every point in the room
Stefanie	Wooden boards and screws to build a shelf Ingredients to cook a dish or ingredients to make pancakes
Alina	Eye contact between multiple people to describe the plane at the eye level Finite elements (numerical) method to solve problems: subdividing a large system into smaller simpler parts called the finite elements

B. Basis Interview Protocol

Q1. (a) What does a basis mean to you?

Follow up:

(i) [*If students felt like their course did not cover basis formally*]: Some students have said that a basis of a vector space is a linear independent set that spans the vector space. What do you think that student means by that? Then go to (iii).

(ii) [*If they only mention one but not the other]*: Some students have said that a basis of a vector space is a linear independent set that spans the vector space. What do you think that student means by that? Then go to (iii).

(iii) [*If they mention both span or linear independence*]: What does each of those things mean to you?

Q1. (b) How would you explain it to a student who is about to take a Linear Algebra course?

Q2. (a) Can you think of an example from your every day life that describes the idea of a basis? (b) How does your example reflect your meaning of basis? What does it capture and what does it not?

Q3. Could basis be relevant for any of the tasks you did? If so, how?

Q4. Can you see a basis as a way to describe something? If so, what is the something? How?

Q5. Can you see basis as a way to generate something? If so, what is the something? How?

Q6. Go through each task, and ask if they CAN possibly see basis in them. Follow up: Can you express #3 in parametric form?

[If time permits] **Q7.** (a) Some students say that a basis is a minimal spanning set, what do you think the student means by that?

(b) Some students say that a basis is a maximal linear independent set, what do you think the student means by that?

C. Codes

Table 3. Codes description, common indicators, and examples Roles

Roles	Information	Characteristics	Information
Generating	Creating the space through combining vectors. E.g., **make** a recipe, **build** with blocks.	Minimal	*The set* being the smallest size or having the least number of vectors to fulfill the role in the space. E.g., **minimum** number of pieces of clothing, **least amount** of myself to cover the space.
Covering	Filling or encompassing the space. E.g., **cover** the room, **fill** in the gaps	Non-redundant	Not wanting extraneous objects *in the set*. E.g., a skeleton would have **extra information you don't need**; you don't want to have the same pair of shoes

Structuring	Organizing the space by using crucial vectors from which the rest of the space can be delineated. E.g., shows you **where everything is**, **describes** everything you need.	Different	*The object* being sufficiently distinct compared to other objects in the set. E.g., each of us had a **different** *chore* to do; each continent is physically **independent** from the others
Traveling	Moving through the space (physical or metaphorical). E.g., to **get anywhere** in the world, **to reach** a decision based on scriptures.	Essential	The critical importance of each *object* in the set. E.g., **crucial** information to define your life, you **need all three of them**.

Supporting	Contributing an important aspect to the space. E.g., **to get something** from each friend group; each continent **contributes** to the world market.	Systematic	The purposeful choice of basis vectors to make the relationship between the vectors and the space to be orderly and efficient. E.g., operations that **works every time**; axis about which you can rotate **easily**

Note: this table is adapted from Zandieh et al. ([7]).