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Inspiring Mathematical Creativity Through Juggling

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Synopsis

The goal of the Creativity in Mathematics and Science project, funded by the National Science Foundation's [NSF's] Improving Undergraduate STEM Education program, is to reconsider how we teach mathematics at the collegiate level. Over the last three years, we have developed interdisciplinary modules that seek to encourage students, including non-STEM majors, to see mathematics in unexpected places, make connections to their own interests and disciplines, and explore creativity in mathematics. Relying on traits of creativity such as the ability to connect ideas, be inquisitive, question norms, and have flexibility [1], we encouraged students to participate and understand mathematics in unconventional ways. The scheduling of a professional juggling company's performance at our on-campus theater inspired us to create a module connecting mathematics and juggling for both a general education mathematics course and a mechanics course. We drew from research on the mathematics of juggling [2, 3] to develop a module that encouraged students to explore the patterns, notations, and mathematical elements of juggling in a variety of ways. Their final projects, representing further explorations, were displayed in our theater's lobby and featured interactive displays and demonstrations. In this paper we describe our experiences developing and implementing this juggling module, students' experiences with the modules, and their development of final projects.

1. Introduction

Although creativity is notoriously difficult to define [4][5][6], one element of creativity that many in the field agree upon is that creativity allows for unexpected connections to be made [7]. Juggling, a physical skill often associated with recreation, performances, circus acts, and sports, might not immediately bring to mind lessons in mathematics, but it does have connections to mathematics and physics [3][8]. In fact, the link between mathematics and juggling is strong, and some famous mathematicians, including Claude Shannon and Ron Graham, have been known to be skilled jugglers. Research on juggling ranges from exploration of simple juggling patterns [8] to using physics and mathematics to test possible juggling patterns [9].

In an effort to encourage students to see mathematics in unexpected places, we developed a module to explore connections between mathematics and juggling. The module was implemented in an introductory mathematics course as well as an early classical mechanics course, and involved collaborations with other university faculty, a world-famous juggling company, and our university's performing arts center. In class, students were asked to think differently about mathematics as they learned to juggle and analyze juggling through mathematics. As part of the module, students also attended a juggling workshop and performance by Gandini Juggling and contributed a final interactive project for an exhibit displayed in the theater lobby during the performance dates. In this paper, we describe our experience creating and implementing a module on the mathematics of juggling, paying particular attention to students' reactions to the module.

2. Motivation

The activities described in this paper are a part of an NSF-funded project called Engaged Learning through Creativity in Mathematics and Science. A major objective of the project is to reconsider undergraduate mathematics education in a way that emphasizes creativity by engaging students in processes and mindsets associated with creativity. Some of the traits associated with creativity include being inquisitive, connecting ideas, questioning norms, and having flexibility [1]. Our motivation for creating a module on the mathematics of juggling was rooted in three events: news of a juggling company scheduled to perform on campus, our collaborations with a campus-wide Creative Thinking Group, and our ongoing attempts to seek mathematics in unexpected places. The Gandini juggling company [10] was scheduled to perform their "Smashed!" production at our university's performing arts center at the end of the semester. Their scheduled performance led us to think about the connections between mathematics and juggling and the possibilities for using juggling to motivate mathematical thinking. We felt that juggling was both an unexpected place and an area with rich mathematics for students to explore, further motivating us to develop and implement the module.

3. Course Descriptions

We designed a juggling module for two undergraduate courses—a general education mathematics course for non-STEM majors and a mechanics course for physics majors.

3.1. Contemporary Mathematics for Everyone

Contemporary Mathematics for Everyone is a terminal mathematics course for non-mathematics majors. Students enrolled in this course come from various majors (e.g. communications, theater studies, criminal justice) and often are reluctant to take the course. To help them see mathematics differently, we developed several modules that encouraged the exploration of mathematics in atypical ways. These modules had students explore non-Euclidean geometry using beach balls, understand and use data to create infographics, and read and conduct book club discussions regarding interesting mathematical discoveries. Each module was created with an interdisciplinary focus on creativity in mathematics and science and was designed to encourage students' creative thinking, especially around mathematical questions.

3.2. Classical Mechanics

Students in a sophomore-level physics course, Classical Mechanics, also participated in this collaboration. The course is a core requirement for physics majors and the first in a sequence of courses that explores intermediate to advanced topics in physics (e.g. differential equations, Newton's Laws, Oscillations, Normal Modes, Lagrangian Mechanics) and is taken by physics, mathematics, and computer science majors. The course requires a final project which was dedicated to the theme of juggling in this semester. A few labs and assignments were designed to focus on this topic to have students become comfortable with connections between juggling and physics. In addition to contemplating issues of physics related to juggling, students were asked to attain at least very basic juggling skills. To encourage students to make connections between physics and other disciplines, we invited a guest lecturer from the Department of Exercise Science and Physical Education to speak about the physics of sports and the biomechanical aspects of juggling.

4. Juggling Module

We spent the summer preparing the juggling module by learning how to juggle and studying the underlying mathematics of juggling. We used online videos [11] [12], juggling kits, and a book on juggling for beginners to learn how to juggle. To further explore the mathematics of juggling, we relied on articles and videos about existing connections [3][2]. We were in contact with the director of the "Smashed!" performance, Sean Gandini, who shared insightful videos and information on juggling notation and various possible patterns [13]. Lastly, before the start of the semester, we held a faculty workshop for our physics and mathematics colleagues where we shared the beginning stages of our juggling module and encouraged them to implement the module in their classes.

Because mathematics and juggling can become complicated quickly, we kept it manageable for students with limited juggling backgrounds and customized the modules to the content of each course. In the weeks preceding the juggling module, students in both courses were asked to purchase a juggling kit and use an online tutorial [11] or the instruction guide that came along with the suggested juggling kit, "Juggling for the Complete Klutz" to become comfortable with 2- and 3-ball juggling. The purpose of the juggling module was for students to understand the importance of definitions (i.e. defining juggling), explore juggling patterns, and represent patterns both physically and through the siteswap notation, a mathematical notation for juggling [3]. Siteswap is a simple numerical notation that essentially answers the question of how many beats later an object gets re-thrown in a given juggling sequence. We drew on our exploration of existing literature on siteswap and other notations to generate learning objectives for this module. These objectives ultimately encouraged students to: (a) make connections to mathematics, (b) make connections to other disciplines, (c) make sense of juggling, (d) make sense of mathematics, (e) be flexible and persevere, and (f) question norms.

4.1. Module implementation in Contemporary Mathematics for Everyone

We began the module by having students practice 2- and 3-ball juggling in groups and make posters of diagrams, notations, and mathematical doodles to think about what was happening as they juggled. To connect themes of creativity to mathematics and juggling, the instructor provided pointed questions for students to consider, including:

1. What information could be collected from seeing someone juggle three balls?
2. How might juggling be represented in a mathematical way?
3. What mathematical questions could be asked about juggling?
4. What do you wonder about as you juggle?

After discussing their posters, the class generated a list of mathematical questions related to juggling. The posing of questions was incorporated into our module to align with other characteristics of creativity that include encouraging multiple approaches [14] and problem posing [15]. Some questions the students generated were:

1. Is there a perfect height to throw the bean bags and does it impact the way someone juggles (i.e. height of throwing balls, each ball following the same cycle)?
2. Is there an ideal timing and height combination for juggling?
3. What direction should the balls be thrown to avoid collision?
4. Does the weight of the ball affect the arc you should throw?
5. Do we measure the pattern of the balls in the air as a circle or triangle?
6. What physical laws apply when juggling?
7. How does time impact juggling?
8. What is the time frame between throwing and catching balls for juggling?
9. Does adding more balls make juggling more challenging?

10. How does distance of the ball in the air and time impact juggling patterns?

For homework and in preparation for the next class meeting, students were tasked with finding a video on juggling and writing a description of something mathematical about it.

Building on students' mathematical observations and connections between juggling and mathematics, the second day of the module began with a segment of a juggling video [12]. As they watched, students were asked to identify patterns of the juggler's hands, movement of the balls, and visually represent these ideas. Next, students were shown a video on siteswap notation and discussed the 3-ball cascade, a specific type of juggling pattern. Once students were familiar with the siteswap notation for 3-ball cascade, they practiced juggling three balls, keeping in mind the beat and location of each ball, and making conjectures about when a cascade pattern might not be possible.

The module was intended to take three days to implement. However, given the time constraint of each session, module implementation actually happened over four sessions. During this time, students were asking questions, exploring existing notations, and trying to develop and understand if patterns were possible given various notations. We created a worksheet (Appendix A) to guide students through their exploration of various siteswap notations.

4.2. Module implementation in Classical Mechanics

In the mechanics course, the first lab session (lasting about 2 hours) was spent on students exploring the juggling of three balls. Students were asked to practice juggling and periodically showcase their skills to the class. This performative aspect of juggling brought an element of play to the course, creating a reprieve from the more abstract and theoretical aspects that dominated the course.

During the lab, students discussed how Newton's laws of motion applied trivially when analyzing the motion of a single ball. However, when more balls were added, the questions soon became more sophisticated: (i) in the case of two or three balls, how quickly should one release a ball in the air so there is no more than one ball in one's hands at any given time? (ii) What might be an optimal height to achieve for the balls? and (iii) What might be the optimal frequency of the ball and the juggler's hands? Because the answers to these questions were difficult to provide theoretically, students conducted experiments using a video recording of a juggling performance (some students chose online videos while others recorded themselves juggling), extracting image frames from the video, and analyzing them using the software ImageJ (Figure 1).

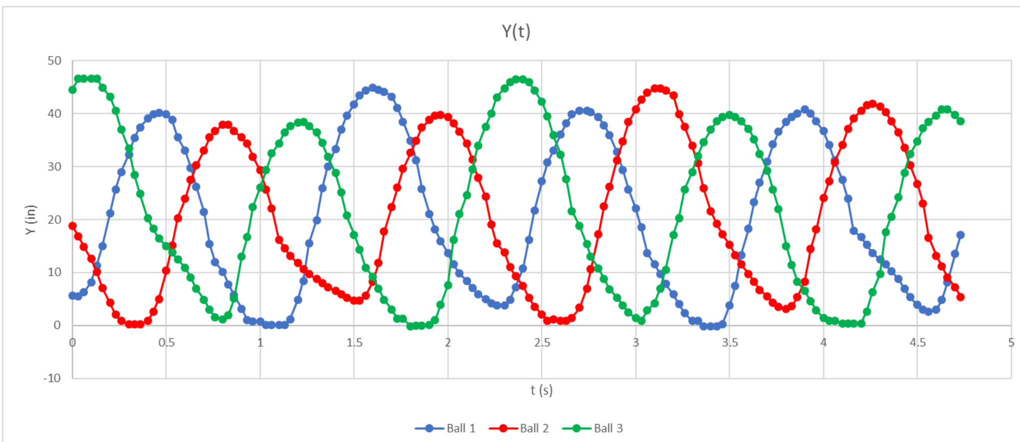

Figure 1: Results from the juggling lab of a Classical Mechanics student. The figure shows the outcome of a video analysis of juggling, showing the changes in vertical position of the three balls as a function of time.

4.3. Final projects and final performance

The final project for the general education mathematics course required students to add a new dimension to the analysis of the mathematics of juggling, or connect mathematics and juggling to another discipline. Students developed a "product" and were informed that they would be entered into a contest to have their work displayed prior to the "Smashed!" performance. We were purposefully vague in describing the "product" because we wanted students to have the freedom to make videos, write poetry, create PowerPoint presentations, or come up with other creative ways to display the connections they researched. Students were expected to explain the mathematics and the connection to juggling both in written form and in an oral presentation. Lastly, we asked students to reflect on the experience of exploring unorthodox connections to mathematics and how they might apply this perspective in other courses.

The final project for Classical Mechanics was inspired by the theme of juggling, but extended more generally to 'The Physics of Toys'. The topics students pursued included the physics of skateboarding, juggling and stability, the biomechanics of juggling, a computer analysis of possible 3-ball juggling patterns, and the idea of juggling as simple harmonic motion. One group decided to design a video game on juggling, while a second group developed a virtual reality setup which captured and analyzed the physics of a live juggling performance (Figure 2). In the true spirit of inquiry-based learning and building engagement with the course material, students were given the freedom to select topics that appealed to them and allowed them to showcase their strengths and connections to their major disciplines.

The projects from both classes were evaluated and a few were selected for inclusion in the pre-show exhibit [17]. The students' final projects ranged from original rap songs to posters outlining the history and mathematics of juggling. Final projects were adapted to suit the needs of the exhibit space. Participating students were invited to present their displays and to help manage a juggling station to engage the audience (Figure 3). The projects were displayed in the lobby of the theater and were viewed by the audience members prior to and after the show. Over the three nights of the performances, students took turns being docents of the exhibits and interacting with the visitors. Some gave demonstrations of their own projects, and others gave mini-lessons on how to juggle. Additionally, we (two of the authors) gave a pre-performance workshop on the mathematics of juggling, featuring the Gandini company, who joined for a brief demonstration of mathematical patterns in juggling.

Students also participated in a juggling workshop led by the Gandini Juggling troupe [18]. This was an opportunity for students to learn new juggling patterns from professionals as they continued to think about the mathematics involved (Figure 4). Additionally, all students were required to attend the performance on one of the three performance dates.

5. Results – Analysis of Journals

As part of the general education course, students were required to maintain a journal to note their reflections about the course and about mathematics. On most days, students were free to choose their topic, and were even encouraged to consider atypical written forms (such as sketches). On other days, the instructor provided a prompt. The journal entries, as well

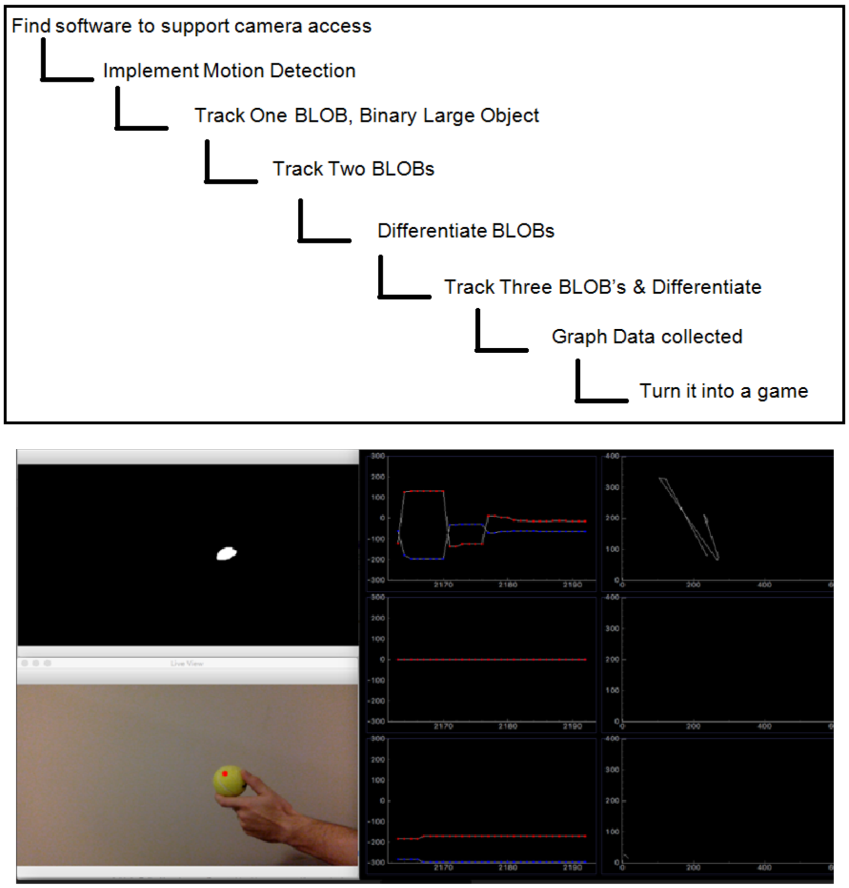

Figure 2: The flow-chart for the virtual reality project and images from the tracking software designed by students in the Classical Mechanics course. The graphs on the bottom right hand side show the x and y coordinates of the ball being tracked along with the horizontal and vertical components of the velocities [16].

as written artifacts ranging from formative assessments, such as exit tickets, to summative assessments were collected and analyzed as part of a larger research project. The data analysis for this paper was based on journal entries that pertained to juggling.

The journals were coded by two of the authors. For each journal, we first determined the module mentioned. This was our Level I journal entry. We extracted all comments related to juggling and then coded each line in the journal entries and assigned a Level II code according to the context

Figure 3: Student exhibits and juggling demonstrations at the Smashed! preshow

Figure 4: Workshop with Smashed! jugglers

that the comment pertained to. These were initial reactions to the module, class activities (including work done outside of class), workshop, final project, pre-show exhibit, and performance. Level III codes were modified from our original codebook for the entire journal and our course objectives. These were comments related to characteristics of creativity. Our Level III codes were 1.) explore connections within mathematics; 2.) explore connections to other disciplines/situations; 3.) question norms (about mathematics also); 4.) be flexible and persevere; 5.) make sense of the mathematics involved (patterns, notation, etc.); 6.) make sense of juggling; and 7.) other, referring to comments that were generic (e.g., "I really liked juggling") or did not fit the other categories.

Below we provide the frequency and description of each Level III code to represent how this module encouraged different characteristics of creativity. In the sections that follow, we provide examples of student reflections. We start with their initial reactions to the juggling module, synthesize level 3 codes pertaining to the module, and end with their reactions to the workshop, exhibit, and performance.

Level 3 codes	**Description of level 3 codes**	**Frequency (n=686)**
Explore connections within mathematics	• Connect to subdisciplines of math (geometry, patterns, mobius strip) • Connect to other course modules, such as on play, mobius strip	54 (8%)
Explore connections to other disciplines	• Connect to other courses, interests, disciplines, such as art, dance, poetry, psychology	101 (15%)
Question norms (about mathematics also)	• Considering mathematics and approaches to it differently • Considering learning differently • Considering self differently with respect to math	108 (16%)

Level 3 codes	Description of level 3 codes	Frequency (n=686)
Be flexible and persevere	• Finding ways to persist • Continuing to learn how to juggle and understand mathematics • Finding resources to help persevere • Focusing on process	71 (10%)
Make sense of juggling	• Ways of practicing juggling • Developing new understandings of juggling	83 (12%)

Level 3 codes	**Description of level 3 codes**	**Frequency (n=686)**
Make sense of the mathematics involved in juggling	• Understanding notation • Identifying patterns • Analyzing juggling in a mathematical way • Realizing that juggling depends on math	140 (20%)
Other	• Generic reactions to workshop, exhibit, or performance, unrelated to mathematics or juggling	129 (19%)

5.1. Initial Reactions to the Modules

The juggling module was introduced during the first meeting of the course (Figure 5) and also featured prominently on the syllabus. Many students were surprised and skeptical when they learned that juggling would be a focus of the course. Initial reactions ranged from surprise, "I thought it was a joke—it most certainly was not" (Julie), to being "delighted but also curious" (Andrew). Another student contrasted the course to his experiences in other mathematics courses by saying that, "It all started with the juggling unit, that's when the course stopped being a normal math class" (Ryan). These comments were made after the students engaged with the juggling unit so they indicate the change in perspective that occurred as students participated in the unit.

Figure 5: Students practicing juggling in class.

One unintended, though understandable, reaction to the unveiling of the juggling unit was shared by students who felt that the course would be easy or watered down because of the playful nature of juggling. Lysette noted, "Before I knew how it applied to math, I thought it would be a waste of time and that I would learn nothing, but it was surprisingly the opposite." Others thought that "it was going to be the easiest 'A' I ever got because I thought it was just juggling" (David). As the intent of the module became clearer, students began to try to make connections between juggling and mathematics for themselves. For example, Mark stated that "it got my mind to think more... about how math is connected to juggling." While David commented on his initial expectations for the course (above), he noted that he changed his mind about the course: "Boy was I in for a treat. I thought ... (that) there's nothing to learn. I also didn't realize this wasn't the case until way too late." These comments indicate that some students initially underestimated the depth of mathematics involved in juggling. As the students delved more deeply into the content of the modules, there was a shift in beliefs about juggling from something that is trivial, easy, and unimportant to viewing the unit as challenging and mathematical.

5.2. Explore Connections Within Mathematics

Students noted that they were able to explore connections to mathematics and connect juggling to subdisciplines within mathematics (e.g. patterns, geometry). For example, Emily connected juggling to mathematics noting, "Learning about other activities made me make more connections to juggling like fractals, speed, weight, beats." Others connected their experiences with juggling to other aspects of the course: "Many (people) would not believe that math could be applied to juggling or jump rope or hula hooping, so this unit helped open the eyes of everyone that did not think that math could almost be applied to everything... which is a pretty awesome insight and learning in life" (Thomas). Including juggling as part of the course helped students to "see mathematics differently" (Valerie) and "understand juggling as something mathematical" (Emily).

5.3. Explore Connections to Other Disciplines

Being able to connect seemingly disparate ideas is a hallmark of creativity [7]. The juggling unit seemed to spur these connections in students, not just between mathematics and juggling, but also between mathematics or juggling and other interests such as sports, music, and everyday life. Lysette, for example, pondered, "Oh well, if there's math in juggling I wonder in what else there is?" For the instructor, having students wonder about the ubiquity of mathematics in this way was rewarding. Another student, Layla, was more specific when making connections stating that "while none of these things [juggling, patterns rhythm] directly relate to my college career or major, I do have an interest in music. Much of the music I listen to relies heavily on production, beat making, meaning that patterns & rhythm play an important role in the process of making these songs, as it does in juggling." For Amanda, the module epitomized what she had heard before: "It's something that you hear in all your years of education and you simply acknowledge the idea and move on. It's when you start applying it and start dissecting it into smaller and smaller parts, and when you realize that all these small parts connect to math in some way, shape or form, it is then when you finally realize what 'math is everywhere' really means."

Students were also able to connect juggling to different disciplines, ranging from poetry to psychology. For example, Ryan commented, "When you think about it, juggling is sort of like reverse dribbling. Instead of using the ground to bounce the ball back up to you when you dribble, you rely on gravity to send the juggling ball back down to you. And the force you apply on the ball as well as the height of the drop or throw has a profound effect of the dynamic of the juggle or the dribble." Similarly, Lysette said, "I was in my perception class and we were talking about the mechanics of the eye, how the eye works. It's part, how it reacts to certain stimuli, how it focuses according to distance... I started thinking about juggling and how eye perception and focus might affect a person's ability to juggle. Because in juggling you're not supposed to look directly at the balls, you're supposed to rely on your eyes' perception of the distance and such of the balls." These student reflections reveal that the modules encouraged students to see mathematics in unexpected places and to recognize connections.

5.4. Questioning Norms/Views Related to Mathematics

One of our primary objectives for including a unit on juggling was to challenge students' conceptions about mathematics. Students' reflections showed that, indeed, they did begin to think about mathematics differently. Some students alluded to their previous notions about mathematics and how they changed by participating in the module. Lysette learned that mathematics, "doesn't have to be incredibly have to be incredibly complicated, formulaic, or like complex. There's math in the most simple of things. I feel that's something that a lot of people don't see. They see mathematics like something that's like almost out of this world but it can be pretty simple." While this comment alludes to appreciating mathematics in "simple" things, others seemed to understand the utility of mathematics as a tool for making sense of our complicated world. Specifically, Kevin wondered "if some activities would be easier to do if they were explained in a more mathematic[al] way." Both of these comments are particularly noteworthy given the initial skepticism students revealed about the potential to view juggling through a mathematical lens (or mathematics through a juggling lens).

Students' shift in views about mathematics was also noted in comments that indicated their understanding of the purpose of the juggling module. Several students expressed this from the point of view of the instructor. As Liscia put it, "The purpose of juggling for this course was to have students view math in an alternative way. It was almost like proving to the class that math can be taught in all areas, fields, [and] ideas." Another student, Kate, felt that the point of the juggling unit, "was encouraging students to think critically, discover new perspectives of mathematics, and show them that mathematics, as a science, is much more than formulas, numbers, or axioms and it can actually [have] practical use." For others, the module capitalized on different modalities of learning, especially as it contrasted to other mathematics courses. For example, Jamie reflected that "sometimes sitting in a classroom learning about numbers can be really boring, therefore having this kind of unit got me excited to go to class and felt really hands on." These comments support the notion that as students became more involved in the module, they came to understand the rationale for including juggling on a syllabus for a mathematics course.

The module also seemed to challenge students to think differently, both mathematically and in other situations. It was "a new way to teach students not to think in one way" (Liscia), a way to "get our minds to think outside the box" (Mark), and "beneficial to view things from the different perspective" (Kate). Specifically, Kate continued, "I feel like we all need to try to see more behind everything, instead of only focusing what is visible at first glance." These quotes exemplify situations that call for seeing things from various perspectives and focusing on the process over the product—important traits of creativity in mathematics.

Another student appreciated the analytic element of the module, specifically that "juggling and mathematics is not how you do things, it's the way you do things and think about it" (Mark). While the comments above suggest that this was appreciated by some, it made at least one student uncomfortable, as it deviated from the norms of "typical" classroom instruction. As Amanda put it: "The new approach towards mathematics was entertaining as well as informative, personally I like a lot of structure and content from my classes therefore I honestly was not very thrilled with having a class that was so dependent on external 'outside the box' thinking." We recognize that the type of instruction represented in the module, which provides less structure for students, is atypical in mathematics courses and can lead to frustration and confusion for students. However, this type of instruction highlights student thinking and the importance of understanding the process behind the mathematics as opposed to rote memorization of facts. Also, an open-ended instructional format that encourages manageable frustration, risk-taking, and curiosity through linking disciplines (e.g., mathematics and juggling) can be said to foster creative thinking [19].

5.5. Making Sense of Mathematics

In terms of mathematics content, most of the comments by students related to ways in which the module reinforced their understanding of mathematics rather than introducing new mathematics content. Students recognized that juggling was a natural way to connect algebra, geometry, and patterns. As Owen put it, "I never thought juggling will have math included with all the geometry, angles, beats, and movement to execute a perfect juggling pattern. All the beats have patterns... for example, it takes a ball five times until that ball lands in the original hand it went up, all depending how many balls you are juggling with.” Students also came to understand the importance of number sequences and mathematical notation as a way to communicate juggling patterns. Mark wrote about the mathematics related to “speed, rhythm, beat, tempo, and the right amount of angle you should have your arms positioned.”

Students were also able to use mathematics to reason about juggling. For example, Chad noted that “Another way would be to throw one of the balls higher than the other two to get the feel of more numbers of balls. This… just increases the speed and depends on the height.” As they learned how to juggle and how to express the patterns using siteswap notation, students began to use the language of the notations: “both these ways still follow the LRLR pattern, cross, have a rhythm and are possible… there are probably other ways to go about it but these were the first in my head” (Chad).

Although some students saw the module as reinforcing their understanding of mathematics as opposed to contributing new insights into it, some were able to express possible patterns in juggling through mathematics: “the new mathematics I learned was the equation for juggling (F+D)H=(V+D)N. F stands for how long the ball is staying in the air for. D stands for how long a ball is held in a hand. H stands for number of hands. V stands for how long a hand is empty for. N stands for the number of balls being juggled” (Mark). Another student found that a “connection I made was force and height. The height at which the balls are thrown directly affects the timing of the catch and even the amount of balls you could realistically throw. If you make short light throws you cannot throw a lot of balls, it’s not feasible. The siteswap notation wouldn’t work because you must always be catching and throwing. It is impossible to catch and throw at the same time. Juggling has hard and fast rules” (Ryan).

Some students had a newfound appreciation for how complex the mathematics of juggling can be and others were confident in their fundamental understanding, but appreciated that it could become complicated quickly. For example, Julie commented, “I definitely know about notations… I bet there is… more math involved but I doubt I’d understand.” Even after the initial connections between mathematics and juggling were made, students quickly realized the complexity of the mathematics involved. As Jamie noted, “I didn’t know we would go into such great detail about math and juggling and how it can connect in so many aspects of life.” These statements showcase an awareness that the mathematics of juggling is deep and complex.

5.6. Making Sense of Juggling

In addition to being given benchmarks for when they should be able to juggle one, two, and three balls, students were encouraged to watch "how-to" videos on juggling. Through practice, and by watching videos, some students thought deeply about juggling patterns and other possibilities. Layla conjectured that "one alternative way [of juggling three balls] is with two balls in the dominant hand, and one ball in the other hand. The two balls would be thrown up and caught in the same hand, and the single ball would just be tossed up and down in the same hand." Juggling also offered an opportunity for students to visualize, doodle, and think of possibilities. One student commented that "When I was asked to think about whether a person can juggle three balls without crossing, I was stuck. I came up with using only one hand. I believe this works because there is nowhere the balls can go but up and down, not right or left. Also, only one ball will be thrown in the air at once making it impossible to cross when going up" (Emily). The module was an opportunity to become better acquainted with an art form that few of us had given much thought to. One student appreciated that "Juggling has a lot to do with patterns and rhythm, something I have only discovered recently through my work in this class" (Layla).

5.7. Flexibility and Perseverance

Students noted several ways in which the juggling unit required perseverance, concentration, flexibility, and discipline—features that are all related to creativity [1][20]. In fact, for some students the unit helped develop these mindsets. For example, Thomas mentioned the merits of having a mindset that is comfortable with failure, a characteristic that is critical to creativity:

> I believe that with juggling comes great discipline. You, the juggler, [have] to remain super calm and expect failure when learning. I feel as if juggling and the discipline behind it relates to when I tried to learn sooner. It takes time and discipline to get better.

Thomas also seemed to appreciate that learning requires time and patience, attributes that juggling encouraged.

Several students noted the effect juggling has on cognitive activity. Kate noted that, "juggling can be great brain stimulation, because while you are juggling, [y]our brain tries to focus on different actions, which as we know is very challenging for our mental system…and will help me to sharpen my concentration and learn how to control my thoughts better." On the other hand, and perhaps related to being flexible, Jamie lamented that there was the possibility to be hindered by over analyzing juggling: "The more I understood the math behind juggling, the more I felt I could not just juggle, but felt like I actually understood math which doesn't happen for me usually." In this statement, Jamie seems to indicate that the fluidity of his juggling was hindered the more he analyzed the mechanics and mathematical patterns of juggling. Finally, Sidra, was able to use juggling to serve as a metaphor for navigating life:

> The ideal point I thought had nothing to do with math. It actually dealt with you as an individual. When I think 'what's the point of learning how to juggle?' is that we ourselves juggle a lot throughout life. We juggle school, work, family, even our social life, but to maintain all of those aspects or in juggling terms to maintain those balls it requires a lot of patience that my generation lack in.

5.8. Reactions to the Workshop, Exhibit, and Performance

As mentioned previously, the modules culminated in a student-produced exhibit that served as a pre-performance show for the audience members. Showcasing their work in a public forum was exciting for the students. In looking forward to the exhibit, Jamie exclaimed that they "are also super excited to be presenting in Kasser theater!" After the show, the students seemed to feel a sense of pride. As Christopher stated, "I was volunteering to represent my project and a lot of people really enjoy it." They also appreciated the Gandini workshop, where the entire troupe worked with students on their juggling skills: "The combinations tested each one of us and our skills in juggling. I encouraged the new teachings and appreciated them taking time out of their day to teach us some new forms of juggling" (Kevin).

Many enjoyed the multi-faceted nature of the show noting that, "I was most definitely not expecting the show that I saw. It was honestly amazing. I loved how it wasn't just juggling, as most expected. There was music involved, theatrics, dancing, acting" (Lysette). The show also served as a memorable culmination of the module: "I also liked how a lot of the juggling [in the Smashed! performance] was pretty simple and they weren't going crazy fast or difficult patterns the whole time. . . it was nice to see patterns that I know how to do. It was also cool to see some of the patterns that they taught us the other day" (Megan).

6. Conclusion

The goal of this module was to use juggling to promote mathematical ideas and to engage students in a different modality of learning. Also, because this course was a part of our larger project on creativity in science and mathematics, we strove to have students realize the role of creativity in mathematics, both historically and for themselves as learners. One point that was emphasized repeatedly throughout our course was that creativity often arises when one is able to connect seemingly disparate ideas. Students were able to make some of these connections on their own as they recognized relationships between juggling, mathematics, and other disciplines.

The skepticism the students displayed at the beginning of the semester gave way to newfound appreciation for mathematics in unexpected places, and perhaps more importantly, gave students new perspectives about mathematics as a discipline. Furthermore, although not explicitly discussed within this module, creativity was a theme throughout the course. We believe that this backdrop prepared the students to undertake the unusual task of learning how to juggle in a mathematics course, and to analyze juggling using mathematics. Elements of creativity were represented in the students' reflections about the module. In these ways, we believe that our course objectives were met and that we promoted new conceptions about mathematics.

The mathematical content that students learned was tailored to the course: the students in Contemporary Mathematics for Everyone explored fundamental patterns and notations of juggling, whereas the students in Classical Mechanics were able to delve more deeply into the physics of juggling. In both cases, juggling supported students to make new connections both within mathematics and between mathematics and their other interests. The module on juggling also encouraged the instructors in these explorations, as it was a new concept to us as well.

As described before, the development of our module on juggling was spurred by our campus performing arts center schedule. The enthusiastic response we received from Gandini Juggling and from the director of the Center boosted our own enthusiasm for developing the module and involving our students. The collaborative effort at all stages of the project contributed to an experience that was meaningful and memorable for many [18][21][22]. Through this project, we found that keeping we found that being aware of campus events can lead to fruitful collaborations and opportunities to encourage students to see mathematics as a creative pursuit. We encourage our readers to do the same and look forward to reading about the outcomes.

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A. Juggling Worksheet

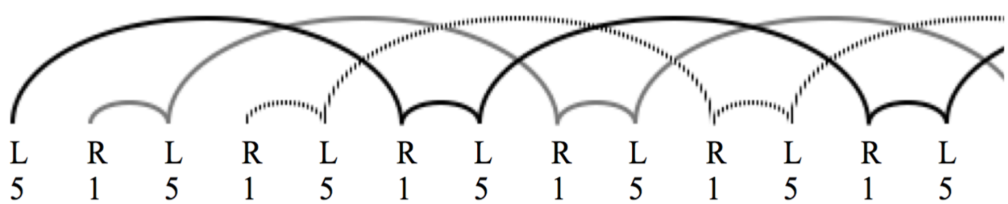
Number of balls: 0
Space-time diagram(s) with siteswap notation and verbal descriptions of what the juggling looks like. Model the juggling.

Number of balls: 1
Space-time diagram(s) with siteswap notation and verbal descriptions of what the juggling looks like, modelling the juggling that each diagram represents.

Why isn't 3,0 a possibility?

Number of balls: 2
Space-time diagram(s) with siteswap notation and verbal descriptions of what the juggling looks like:

Number of balls: 3
Here is a pattern for three ball juggling:



What's going on? Describe it and if you can, model the juggling:
Is 1,2,4 possible? How about 2, 3, 5? Why or why not?
Is 5, 4, 3 possible? If yes what does it look like? If no why does it not work and how might you fix the problem?
How can you tell from a siteswap notation that a juggling pattern is definitely **not** possible? (In other words, what is true for all notations that represent possible juggling patterns?)