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Response to Steven Colbert: Spicing Up the Exposition of Differential Equations via Engaging with Relevant History of Algebra

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Synopsis

This paper deals with some problems that can be incorporated in the exposition of ordinary differential equations in courses on Differential Equations and System Dynamics or Intermediate Strength of Materials, with a view to promote more interest and excitement by the attendees, both students and lecturers.

1. Introduction

One can safely say that our popular culture does not glorify science and engineering. Here is a quote from the "satirical bestseller" (in the words of Wikipedia) by American comedian Stephen Colbert, who is the host of CBS's *The Late Show with Stephen Colbert*.

> "Let's try a little experiment. Look at this equation:
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
>
> What you are feeling right now, is your body rejecting an idea that is trying to make you learn it. Don't fight the confusion. That's just your mind scabbing over in a desperate attempt to protect you from that unnatural commingling of numbers and letters up there. You can't add it, and you can't read it. Useless." [9, page 120]

Here is another quote from the *Colbert Report* [10]:

> "I have said it before: equations are devil's sentences. The worst one is that quadratic equation, an infernal salad of numbers, letters, and symbols."

Naturally (at least we hope so) Colbert was joking. But a reader of his book, or the viewer of his show, may think (as many do) that 90% of every joke is truth (or *truthiness*—a term ingeniously coined by Colbert himself).

In these circumstances, mathematics education needs to be spiced up, so that students will recognize that this discipline is not merely a necessary evil but a vibrant, exciting and fascinating subject.

One means of eliminating boredom and apathy is presentation of mathematics in its historical context. In the words of the immortal German poet-scientist-thinker Johann Wolfgang von Goethe (1749-1832), "the history of science is science itself" [18]. Indeed, according to Morin's Limerick [29],

> The skill to do math on a page
> Has declined to the point of outrage.
> Equations quadratica
> Are solved on a Math'matica,
> And on birthdays we don't know our age.

According to Moritz [30], in his Presidential Address to the British Association for the Advancement of Science in 1890, J.W.L. Glaisher (1848-1928) maintained that "no subject loses more than mathematics by any attempt to disassociate it from its history."[1]

What is the purpose of introducing the elements of history into the classroom? As Man-Keung Siu writes, "using the history of mathematics in the classroom does not necessarily make students obtain higher scores in the subject overnight, but it can make learning mathematics a meaningful and lively experience, so that (hopefully) learning will come easier and will go deeper" [32].

[1] Here we should also note the distinction made by Ivor Grattan-Guinness [19, 20] between history from our heritage in the context of the mathematics of the past.

It appears also that learning about mistakes made by great mathematicians makes us part of a vast human chain; if a famous mathematician made mistakes, then I (the student may think) am not hopeless! And history abounds with such mistakes over the centuries. I'm thinking of how algebraists of older times often ignored the negative solutions to quadratic equations.

Here we ought to remark that calling the work of mathematicians of the past *mistakes* or *errors* might not always be optimal. It is for example perfectly correct to neglect the negative root of a quadratic equation if you are solving for money or length, and calling it a mistake simply removes the equation from context. Of course, in the context of differential equations, the negative root(s) cannot be neglected.

This is what the English mathematician, Francis Maseres (1731-1824), a Fellow of the Royal Society, had to say about negative numbers in his 1759 book

> "They [negative roots] serve only, as far as I am able to judge, to puzzle the whole doctrine of equations, and to render obscure and mysterious things that are in their own nature exceedingly plain and simple… It was to be wished therefore that negative roots had never been admitted into algebra and were again discarded from it." [27]

According to Klein, "apparently Euler, too, was still not clear about complex numbers… He also made mistakes with the complex numbers. In his *Algebra* [of 1770] he writes: "$\sqrt{-1}\sqrt{4} = \sqrt{4} = 2$ because $\sqrt{a}\sqrt{b} = \sqrt{ab}$" [25, page 143]. Klein continues: "Though he calls complex numbers impossible numbers, Euler says they are useful."

In 1770, Euler (1707–1783) wrote that the operation of subtracting a negative quantity, say, $-b$ (where $b > 0$) was just like adding b, due to the fact that "canceling a debt signifies the same as giving a gift" [13]. This latter quote sometimes helps students master certain rules.

Mastering of the notion of complex numbers posed a major challenge for scientists. This is what Euler had to say:

> "The square roots of negative numbers are neither zero, nor less than zero, nor greater than zero. Then it is clear that the square roots of negative numbers cannot be included among the possible

> [real] numbers. Consequently, we must say that these are impossible numbers. And this circumstance leads us to the concept of such numbers, which by their nature are impossible, and ordinarily are called imaginary or fancied numbers, because they exist only in the imagination."

Analogously, Stephen Fletcher Hewson remarked, "[a]t one level, it is difficult to visualize in what way imaginary numbers are 'less' than real numbers" [22, page 31].

Such quotes speak to students' hearts. They feel that if they do understand, then this is because the stand on the shoulders of giants who, not unlike themselves, made mistakes.

Indeed, as Clifford Truesdell emphasized, "Now a mathematician has a matchless advantage over general scientists, historians, politicians, and exponents of other professions: He can be wrong. A fortiori, he can also be right" [38, page 140].

2. Connecting ODEs to Al-Khowârizmi's Quadratic

Consider the quadratic equation treated by Mohammad ibn Mûsâ Al-Khowârizmi (circa 780–850 Common Era) [3, 6]

$$r^2 + 10r = 39. \tag{1}$$

As David Burton writes, "this equation reappears frequently in later Arab and Christian texts, running 'like a thread of gold through the algebras of several centuries" [7]. In Al-Khowârizmi's original text the problem reads as follows: "One square and ten roots of the same amount to thirty-nine *dirhems*; that is to say, what must be the square which, when increased by ten of its roots, amounts to thirty-nine?" [12] John Derbyshire adds: "*Dirhem* was a unit of money. Al-Khowârizmi uses it to refer to what we nowadays call the constant term." [12]

Al-Khowârizmi provided a geometric solution of this quadratic. He found one root

$$r = 3, \tag{2}$$

but was unaware of another root, namely,

$$r_2 = -13. \tag{3}$$

One ought not engage in excessive criticism, as it were, of Al-Khowârizmi. Indeed, John Stillwell, commenting on Eq. (1) notes: "Euclid and al-Khwārizmī did not admit negative lengths, so the solution of $x = -13$ to $x^2 + 10x = 39$ does not appear. This is quite natural, since geometry admits only positive lengths" [34, page 52]. Likewise, as Derbyshire remarks, "Negative numbers, along with zero, have not yet been discovered" [12, page 29].

The associated ordinary differential equation reads

$$\ddot{y} + 10\dot{y} = 39y. \tag{4}$$

Substitution of

$$y(t) = Ce^{rt} \tag{5}$$

leads to the characteristic equation of the above Eq. (1). The solution reads

$$y(t) = C_1e^{3t} + C_2e^{-13t}. \tag{6}$$

The constants of integration are found by satisfying initial conditions. Students appreciate the fact that whereas al-Khwārizmī was happy with finding only one root (since the negative root did not make sense to him, "you can't have a negative number of apples', for example) to solve the differential equation and to be able to satisfy the initial conditions, both roots of the quadratic are needed.

When explaining the substitution (5), I quote the following limerick listed in David Morin's collection [29]:

> This is our method, essential,
> For equations we solve, differential.
> It gets the job done,
> And it's even quite fun.
> We just try a routine exponential.

One has to emphasize to the students that without recognizing negative numbers as full-fledged members of the number family, we would be unable to solve equation (4). This mathematical *democracy* or full equality of positive and negative numbers allows us to achieve our goal of solving differential equations. Indeed, according to Whitehead (as quoted by Arcavi et al [2]), "the idea of positive and negative numbers has been practically the most successful of mathematical subtleties."

As homework, students can be offered the following ODEs:

$$2\ddot{y} + 10\dot{y} = 48y \tag{7}$$

and

$$\frac{1}{2}\ddot{y} + t\dot{y} = 28y. \tag{8}$$

The associated characteristic equations are

$$2r^2 + 10r = 48 \tag{9}$$

and

$$\frac{1}{2}r^2 + 5r = 28. \tag{10}$$

Al-Khowârizmi dealt with these types of equations in Chapter 4 of his book, where "in each case only the positive answer is given" (Boyer [5]). In Chapter 6 he deals with the equation

$$3r + 4 = r^2, \tag{11}$$

and Boyer [5] notes that "again only one root is given, for the other is negative." Students can be asked to solve the associated ODE:

$$3\dot{y} + 4y = \ddot{y}. \tag{12}$$

The following quote, by Richard Courant and Herbert Robbins may come in handy when making a substitution given in Eq (5): "The natural exponential function is identical with its derivative. This is the source of all the properties of the exponential function and the basic reason for its importance in applications" [11].

3. Connecting ODEs to Bombelli's Quadratic

Consider the quadratic equation attributed to Rafael Bombelli (1526–1572)

$$r^2 + 12 = 8r. \tag{13}$$

Apparently he was not familiar with the general solution

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{14}$$

of the quadratic

$$ar^2 + br + c = 0. \tag{15}$$

If a student drops the minus sign in front of b in Eq. (14), as it happens not rarely, I give them a previously typed page with:

A young boy was filled with elation
Trying to solve a quadratic equation
But he only wrote "b"
(dropped the negative, see?)
And henceforth was merely frustration.

Bombelli was able to solve his equation in a neat manner. He rewrote it as

$$r^2 - 8r + 12 = 0. \tag{16}$$

Completing the square,

$$(r-4)^2 = 4, \tag{17}$$

and so $r = 2$.

He then rewrote Eq. (17) as follows

$$(4-r)^2 = 4, \tag{18}$$

and obtained

$$r = 6. \tag{19}$$

Thus, he derived two possible roots,

$$r_1 = 2,\ r_2 = 6. \tag{20}$$

The associated governing differential equation reads

$$\ddot{y} + 12y = 8\dot{y} \tag{21}$$

since substitution in Eq. (5) leads to the characteristic equation (13) for r. Thus, the general solution of Eq. (21) reads

$$y = C_1 e^{2t} + C_2 e^{6t} \tag{22}$$

with constants of integration C_1 and C_2 found by satisfying initial conditions, say

$$y(0) = \alpha,\ \dot{y}(0) = \beta, \tag{23}$$

where α and β are arbitrary constants.

4. Connecting Second Order ODEs to Gerolamo Cardano's Quadratic

Consider the following problem: "The perimeter of a rectangle equals 20, its area equals 21; find its sides. Then construct an ODE such that its characteristic equation is the same as the quadratic equation arising in the determination of the sides of the rectangle."

Denoting the sides q and r, we have the following equations

$$\begin{aligned} 2(q+r) &= 20 & (24)\\ qr &= 21. & (25) \end{aligned}$$

Since

$$q = 10 = r, \tag{26}$$

Eq. 25 becomes

$$r^2 - 10r + 21 = 0 \tag{27}$$

with roots

$$r_1 = 3,\ r_2 = 7. \tag{28}$$

The associated ODE is

$$\ddot{y} - 10\dot{y} + 21y = 0 \tag{29}$$

with the characteristic equation the same as Eq. (27). The solution reads

$$y(t) = C_1 e^{3t} + C_2 e^{7t} \tag{30}$$

with constants C_1 and C_2 likewise determined by satisfying initial conditions.

One can modify this problem by replacing the area by some generic value A. Then Eq. (27) is replaced by

$$r^2 - 10r + A = 0 \tag{31}$$

with roots

$$r_1 = 5 + \sqrt{25 - A},\ r_2 = 5 - \sqrt{25 - A}. \tag{32}$$

If A is less than 25, the roots are distinct and positive. The associated ODE

$$\ddot{y} - 10\dot{y} + Ay = 0 \tag{33}$$

has a solution

$$y(t) = C_1 e^{(5+\sqrt{25-A})t} + C_2 e^{(5-\sqrt{25-A})t}. \tag{34}$$

For $A = 25$ the roots are equal: $r_1 = r_2 = 5$. For $A > 25$ they become complex.

If we now pose the following problem:

> "The perimeter of a rectangle equals 20, the area equals 30; find the sides of the rectangle."

Eq. (31) must be replaced by

$$r^2 - 10r + 30 = 0 \tag{35}$$

with roots

$$r_{1,2} = 5 \pm i\sqrt{5}. \tag{36}$$

The associated differential equation reads:

$$\ddot{y} - 10\dot{y} + 30y = 0. \tag{37}$$

The characteristic equation is obtained by the substitution in Eq. (5) and is the same as Eq. (35). Hence the solution reads

$$y(t) = C_1 e^{(5+i\sqrt{5})t} + C_2 e^{(5-i\sqrt{5})t}$$

or

$$y(t) = e^{5t}\left[C_1 e^{it\sqrt{5}} + C_2 e^{-it\sqrt{t}}\right]. \tag{38}$$

Here, Euler's famous equation

$$e^{it} = \cos(t) + i\sin(t) \tag{39}$$

comes in handy to arrive at the final general equation

$$y(t) = e^{5t}\left[D_1 \cos(\sqrt{5}t) + D_2 \sin(\sqrt{5}t)\right]. \tag{40}$$

With initial conditions, say,

$$y(0) = 1, \ddot{y}(0) = 0 \tag{41}$$

the solution becomes

$$y(t) = e^{5t}\left[\cos(\sqrt{5}t) - \sin(\sqrt{5}t)\right]. \tag{42}$$

Students respond with interest to the quote by Kasner and Newman:

> "There is a famous formula—perhaps the most compact and famous of all formulas—developed by Euler. . . :
>
> $$e^{i\pi} + 1 = 0.$$
>
> It appeals equally to the mystic, the scientist, the philosopher, the mathematician" [23].

Indeed, why in the world are the five most celebrated numbers, 0, 1, e, i, and π so neatly connected? Is there some mysterious unity in the universe?

Regarding Euler's above formula, this is what Eli Maor, author of highly popular books had to say:

> "Now this is the supreme act of mathematical chutzpha [audacity], for in all our definitions of the function e^x, the variable x has always represented a real number. To replace it with an imaginary number is to play with meaningless symbols, but Euler had enough faith in his formulas to make the meaningless meaningful" [26].

Indeed, to paraphrase an anonymous author, e^{it} is a complex quantity just like life: it has both real and imaginary components!

The homework can include the following differential equations:

$$\begin{aligned}
&\ddot{y} - 12\dot{y} + 37y = 74, && y(0) = 0, && \dot{y}(0) = 1\\
&\ddot{y} - 14\dot{y} + 50y = 200, && y(0) = 1, && \dot{y}(0) = 0\\
&\ddot{y} - 18\dot{y} + 82y = 164, && y(0) = 1, && \dot{y}(0) = 1\\
&\ddot{y} - 20\dot{y} + 101y = 505, && y(0) = 0, && \dot{y}(0) = 1\\
&\ddot{y} - 40\dot{y} + 401y = 802, && y(0) = 1, && \dot{y}(0) = 0\\
&\ddot{y} - 13\dot{y} + 43.25y = 87, && y(0) = 2, && \dot{y}(0) = 5
\end{aligned} \tag{43}$$

with solutions (respectively)

$$\begin{aligned} y(t) &= 2 - e^{6t}(2\cos(t) + 13\sin(t)) \\ y(t) &= 4 - e^{7t}(-3\cos(t) + 21\sin(t)) \\ y(t) &= 2 - e^{9t}(-\cos(t) + 10\sin(t)) \\ y(t) &= 5 - e^{10t}(-5\cos(t) + 5\sin(t)) \\ y(t) &= 2 - e^{20t}(-\cos(t) + 20\sin(t)) \\ y(t) &= 2 + 5e^{6.5t}5\sin(6.5t)) \end{aligned} \tag{44}$$

Consider now a specific problem treated by Girolamo Cardano (1501–1576 Common Era), in his 1545 book *Ars Magna (The Great Art)* [8]: “The perimeter of a rectangle equals 20, the area equals 40; find the sides of the rectangle.” The associated ODE reads:

$$\ddot{y} - 10\dot{y} + 40y = 0 \tag{45}$$

whose characteristic equation

$$r^2 - 10r + 40 = 0 \tag{46}$$

was treated by Cardano (not in context of the differential equations!) He derived two roots

$$r_1 = 5 + \sqrt{-15},\ r_2 = 5 - \sqrt{-15} \tag{47}$$

calling the problem itself “manifestly impossible” and “useless”, because of negative numbers under the root sign. The interested reader can consult the entertaining book by Barry Mazur with telling title, Imagining *Numbers: Particularly the Square Root of Minus Fifteen* [28].

As we see, complex numbers are the tool that enables us to solve an ordinary differential equation of second order with its characteristic equation possessing the negative discriminant It should be noted that when discussing the “discriminant” in class, we seek alternative terms such as “distinguisher” or “demarker,” since to many students (and professors!) mathematical “discrimination” is as intolerable as the real-life one.

As homework, one can assign the problem

$$\ddot{y} + 12y = 6\dot{y} \tag{48}$$

whose characteristic equation

$$r^2 + 12r = 6r \tag{49}$$

was likewise dealt with by Cardano, who again referred to the roots "*ficta*" or fiction [33].

Students see clearly that what seemed "impossible" to great Cardano is now quite possible and straightforward for them.

On one occasion I led a discussion on the following statement by Grilly:

> ""Do imaginary numbers exist or not?" This was a question chewed over by philosophers as they focused on the word "imaginary." For mathematicians, the existence of imaginary numbers is not an issue. They are as much part of everyday life as 5 or π or a pie. They are irrelevant to shopping, but ask any aircraft designer or electrical engineer and you will find they are vitally important" [17, page 32].

Sometimes, if time is left before the end of the class, I ask students if a complex coefficient polynomial can have real roots. They appreciate to know, that the polynomial equation:

$$r^3 - 2(1+i)r^2 + (1+4i)r - 2i = 0$$

has 3 roots, 1, 1, $2i$, two of which are real. Students are assigned homework to find other polynomial equations of this sort. Most of them succeed in the assignment.

Sometimes students ask if the qualifier "imaginary" is appropriate since it frightens them. I bring a prepared quote by Carl Friedrich Gauss (1777–1855): "That this subject [imaginary numbers] has hitherto been surrounded by mysterious obscurity, is to be attributed largely to an ill-adapted notation. If for instance, $+1$, -1, $\sqrt{-1}$ had been called direct, inverse and lateral units, instead of positive, negative, and imaginary (or even impossible), such an obscurity would have been out of the question."

After some discussion students usually do not accept the suggestion by Gauss, and prefer the term "imaginary numbers" because of some attractive mystique involved in numbers being "imaginary" and thus invoking their imagination. In fact, according to great physicist Albert Einstein, "Imagination is more important than knowledge," or "Logic will get you from A to B — imagination will take you everywhere." In words of the French ruler Napoleon Bonaparte, "the human race is governed by imagination."

5. A Problem of Diophantus and Its Extension to ODEs

Consider the following problem by Diophantus of Alexandria (circa 200–284 Common Era) in his Book 6: "Given a right triangle with area A and perimeter P, find it's legs."

Denoting the legs q and r respectively, we have the following set of equations:

$$qr/2 = A \tag{50}$$

$$q + r + \sqrt{q^2 + r^2} = P \tag{51}$$

or

$$\sqrt{q^2 + r^2} = P - q - r. \tag{52}$$

Squaring this expression, and cancelling identical terms on both sides we have

$$P^2 - 2Pq - 2Pr + 2qr = 0 \tag{53}$$

and substituting

$$q = \frac{2A}{r} \tag{54}$$

we have the quadratic equation

$$2Pr^2 - (4A + P^2)r + 4PA = 0 \tag{55}$$

with the leg r satisfiying

$$r = \frac{4A + P^2 \pm \sqrt{(4A + P^2)^2 - 32P^2A}}{4P}. \tag{56}$$

For

$$A = 6, \; P = 12 \tag{57}$$

we have the quadratic equation

$$24r^2 + 168r + 288 = 0 \tag{58}$$

with roots

$$r = 3, \;\; q = 4 \tag{59}$$

resulting in an Egyptian triangle, with sides 3, 4 and 5 (Pythagorean triplet). For

$$A = 7, \;\; P = 14 \tag{60}$$

we have

$$28r^2 - 224r + 392 = 0 \tag{61}$$

with legs

$$r = 4 - \sqrt{2} \approx 2.59, \;\; q = 4 + \sqrt{2} \approx 5.41 \tag{62}$$

and for

$$A = 30, \;\; P = 30 \tag{63}$$

we get

$$60r^2 - 1020r + 3600 = 0 \tag{64}$$

with roots

$$r_1 = 5, \;\; r_2 = 12 \tag{65}$$

the hypotenuse being $r_3 = \sqrt{5^2 + 12^2} = 13$, numbers 5, 12 and 13 forming another Pythagorean triplet.

In the case considered by Diophantus

$$A = 7, P = 12 \tag{66}$$

the quadratic equation becomes

$$24r^2 - 172r + 336 = 0 \tag{67}$$

and the legs become

$$\frac{43 \pm \sqrt{-167}}{12} \tag{68}$$

which signifies that such a triangle does not exist.

Diophantus, in the words of Smith "stated that Eq. (67) cannot be solved unless the square of half the coefficient of r diminished by 24×336 is a square, not otherwise seeming to note that this equation has complex roots" [33].

It appears that we cannot negatively judge Diophantus for stating that equation $4x + 20 = 4$ is "absurd."

In Nahin's words,

> "six hundred years later (circa 850 Common Era) "the Hindu mathematician Mahavlracarya wrote on this issue, but then only to declare what Heron and Diophantus had practiced so long before: 'The square of a positive as well as of a negative (quantity) is positive; and the square roots of those (square quantities) are positive and negative. As in the nature of things a negative (quantity) is not a square (quantity), it has therefore no square root.' More centuries would elapse before opinion would change" [31].

The ODEs associated with equations (58), (61) and (64) read respectively:

$$\begin{aligned} &24\ddot{y} + 168\dot{y} + 288y = 0 \\ &28\ddot{y} - 224\dot{y} + 392y = 0 \\ &60\ddot{y} - 1020\dot{y} + 3600y = 0 \end{aligned} \tag{69}$$

with the following solutions:

$$\begin{aligned} y(t) &= C_1 e^{3t} + C_2 e^{4t} \\ y(t) &= C_1 e^{(4-\sqrt{2})t} + C_2 e^{(4+\sqrt{2})t} \\ y(t) &= e^{(42/12)t}\left[C_1 \cos(\sqrt{167}t) + C_2 \sin(\sqrt{167}t)\right]. \end{aligned} \tag{70}$$

6. Homework: A Problem That is a Cousin of That by Heron of Alexandria

In the words of Smith, "the first trace of the square root of a negative number to be found in... work is the *Stereometria* of Heron of Alexandria (circa 50 Common Era), where $\sqrt{81-144}$ is taken to be $\sqrt{144-81}$" [33]. It should be remembered that when we criticize Heron or Diophantus or others in the context of negative numbers, "only positive rational answers were admitted, and Diophantus felt satisfied when he had found a single solution" [7, page 225]. Moreover, "the idea of negative numbers was familiar to Chinese authors well before its acceptance in Europe during the fifteenth century" [7, page 262].

Interestingly, students respond with excitement when I state that it is easy to construct a differential equation whose characteristic roots lead to Heron's result. In particular, the following problem can be assigned

$$\ddot{y} + 9\dot{y} + 36y = 0 \tag{71}$$

whose characteristic equation

$$r^2 + 9r + 36 = 0 \tag{72}$$

has the roots

$$r_{1,2} = \frac{-9 \pm \sqrt{81 - 144}}{2}. \tag{73}$$

Nahin quotes in this regard Wooster Woodruff Beman, who in his 1897 lecture stated:

> "Instead of the square root of $81-144$ required by the formula, he [Heron] takes the square root of $144-81\ldots$, i.e. replaces $\sqrt{-1}$ by 1, and fails to observe that the problem as stated is impossible. Whether this mistake was due to Heron or to the ignorance of some copyist cannot be determined" [31].

If and when students make an analogous mistake, they are in a good company, committing the mistake of famous Greek Mathematician Heron. The fact that students make this mistake 2,000 year after Heron lived is not that important.

Somehow this fact consoles them as it were, and spurs them to do better. They do not feel lonely in making the mistake.

"Don't worry," I tell a student, "if Heron of Alexandria was "allowed" to make a mistake, you are entitled to make it too. The difference is that you can correct it, and try to avoid it in the future!"

7. Connection with Fibonacci's Equation

Leonardo of Pisa, better known by his other name, Fibonacci ("son of Bonaccio") lived between circa 1170 and 1250. In his treatise entitled *Flos* (meaning "blossom" or "flower") he deals with a problem posed at a mathematical disputation that took place in the presence of the Emperor Frederick II. It reads as follows:

$$r^3 + 2r^2 + 10r = 20. \tag{74}$$

Fibonacci gave one of the solutions to the equation, namely $1.3688081075\ldots$.

As Burton notes:

> "This was a remarkable estimate of the only real root of the cubic equation, correct to nine decimal places; and it was the most accurate European approximation to an irrational root of an algebraic equation that would exist for the next 300 years. But we are not told how the result was found. Although Fibonacci never revealed his sources, the possibility cannot be excluded that he had learned the solution in his travels. The same problem appears in the algebra of the great Persian poet and mathematician Omar Khayyam (circa 1050–1130 Common Era), where it was solved geometrically by intersecting a circle and a parabola." [7, page 285]

The associated ODE reads

$$\dddot{y} + 2\ddot{y} + 10\dot{y} = 20y. \tag{75}$$

Substitution (5) leads to characteristic equation (74).

The following ODEs can be assigned for homework:

$$\dddot{y} + 3\ddot{y} = 5y \tag{76}$$

and

$$\dddot{y} + 6\ddot{y} + 8\dot{y} = 1000y \tag{77}$$

whose characteristic equations are respectively,

$$r^3 + 3r^2 = 5 \tag{78}$$

and

$$r^3 + 6r^2 + 8r = 1000 \tag{79}$$

Solutions of the latter cubic equations were posed in 1530 by Niccolo Tartaglia (1500–1557) to a friend.

8. Connecting Higher Order ODEs to Bhâskara and Mahâvītra

Consider the following fourth-order ODE

$$\dddot{y} - 2\ddot{y} - 400\dot{y} = 9999y \tag{80}$$

with associated initial conditions

$$y(0) = \alpha,\ \dot{y}(0) = \beta,\ \ddot{y}(0) = \gamma,\ \dddot{y}(0) = \delta \tag{81}$$

where α, β, γ, and δ are arbitrary constants. Familiar substitution (Eq. 5) leads to the characteristic equation

$$r^4 - 2r^2 - 400r = 9999. \tag{82}$$

The solution of this quartic was derived by Bhâskara (1114–1185 Common Era) and Mahâvītra in the following interesting way. They added the term $4r^2 + 400r + 1$ to both left and right parts of the characteristic equation to get

$$r^4 + 2r^2 + 1 = 4r^2 + 400r + 10000 \tag{83}$$

or

$$(r^2 + 1)^2 = (2r + 100)^2 \tag{84}$$

which they reduced to

$$r^2 + 1 = 2r + 100 \tag{85}$$

or

$$r^2 - 2r - 99 = 0 \tag{86}$$

and obtained

$$r_1 = 11 \tag{87}$$

discarding, as was then automatically done, the negative root

$$r_2 = 1 - \sqrt{1 + 99} = -9 \tag{88}$$

However, a quartic equation (84) has four roots. Eq. (85) should actually read

$$r^2 + 1 = \pm(2r + 100). \tag{89}$$

The Indian mathematicians confined their treatment to the "plus" alternative, overlooking the fact that two other roots must be found from the "minus":

$$r^2 + 1 = -(2r + 100) \tag{90}$$

or

$$r^2 + 2r + 101 = 0 \tag{91}$$

with roots

$$r = -1 \pm \sqrt{1 - 101} = -1 \pm 10i. \tag{92}$$

Hence the solution of Eq. (80) is written as

$$xy(t) = C_1 e^{11t} + C_2 e^{-9t} + C_3 e^{-t}\cos(\sqrt{10}t) + C_4 e^{-t}\sin(\sqrt{10}t). \tag{93}$$

subject to the initial conditions in Eq. (81). Here it appears instructive to quote Steven Strogatz [35] (see also Strogatz [36]):

> “For more than 2,500 years, mathematicians have been obsessed with solving for x. The story of their struggle to find the “roots”—the solutions—of increasingly complicated equations is one of the great epics in the history of human thought. And yet, through it all, there’s been an irritant, a nagging little thing that won’t go away: the solutions often involve square roots of negative numbers. Such solutions were long derided as “sophistic” or “fictitious” because they seemed nonsensical on their face. Until the 1700s or so, mathematicians believed that square roots of negative numbers simply couldn’t exist.”

9. Discussion

It might be advisable to introduce some of the historical examples in this paper into the classroom. According to Blanco and Ginovart [4] “Fauvel [14] distinguishes between using the history of mathematics within the teaching of mathematics, and teaching the history of mathematics as a subject.” Our objective is the former. G. Hepple in 1893 set the conditions (see Fauvel [14]):

1. The history of mathematics should be strictly auxiliary and subordinate to mathematical teaching.
2. Only those portions should be dealt with which are of real assistance to the learner.
3. It is not to be made a subject of examination.

The words “subordinate” and “auxiliary” above are almost derogatory; we do not subscribe to such a hierarchy in disciplinary values. In this study we view mathematics of the past as our common heritage, in the terminology of Grattan-Guiness [19, 20] (see also Fried [16]) and value its serious study for its own sake. However when teaching mathematics, the history coming into the classroom is often supportive rather than central.

Building on the mistakes, confusions, and frustrations of mathematicians "sensitizes" both students and teachers, as it were, to borrow the term used by Arcavi: "Another potential benefit of using history of mathematics lies in sensitizing the teacher to possible difficulties of students' understanding; it may indeed yield clues on how to respond and help the student over them. History can provide a feeling, for example, for how standards of rigor evolved" [1]. (See also [39, 15, 24]).

During recent semesters I taught the courses System Dynamics, Vibration Analysis and Synthesis, and Intermediate Strength of Materials to engineering students with a Differential Equations prerequisite. During the first lesson of System Dynamics I conduct a simple test to check the knowledge that was retained from Differential Equations. It turned out that most never got the subject: only one or two students could solve simple differential equations. I continuously try to develop various materials to cure the mathematical state of the students who possibly "suffered as a result of dull and uninspired teaching," in order students to put possibly "unhappy memories behind them" [37]. Preliminary results of incorporating mathematics of the past as a heritage appear to be extremely encouraging. Indeed, in words of Reuben Hersh, "Young learners of mathematics share a common experience with the greatest creators of mathematics: "hitting a wall," meaning, first frustration, then struggle, and finally, enlightenment and elation" [21].

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