Academic Hats and Ice Cream: Two Optimization Problems

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Synopsis

This article describes the use of computer software to optimize the design of an academic hat and an ice cream cone!

Keywords: constrained optimization, symbolic and numerical computer mathematics, derivative, MATHCAD, STEM education

> *"I am a master," he grew stern and took from the pocket a completely greasy black hat with the letter "M" embroidered on it in yellow silk.*
> Mikhail Bulgakov, *The Master and Margarita* (translated from the Russian by Richard Pevear and Larissa Volokhonsky [1].)

At the beginning of each summer, students run out of the doors of educational institutions in gowns and academic hats. Let's create a special hat for mathematicians here. Mathematics is, of course, the queen of sciences. Immanuel Kant is credited with saying: "In every natural science there is as much truth as there is applied mathematics in it."[1] So it is entirely appropri-

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[1] This is from *Metaphysical Foundations of Natural Science* (*Metaphysische Anfangsgründe der Naturwissenschaft*, 1786), original in German: "Ich behaupte aber, daß

ate to want to create a special academic hat for mathematics graduates that differs from the hats of non-mathematicians. We will embroider the number π on it.

Let's create not just a hat, but *an optimized hat.* The search for optimal solutions is one of the most interesting and complicated sections of mathematics, which has important practical applications. We will design a hat in the form of a straight circular cylinder, topped with a square of side length equal to the diameter of the cylinder (Figure 1). We will minimize the surface area for a given volume, solving a simple and beautiful optimization problem, as well as saving on material!
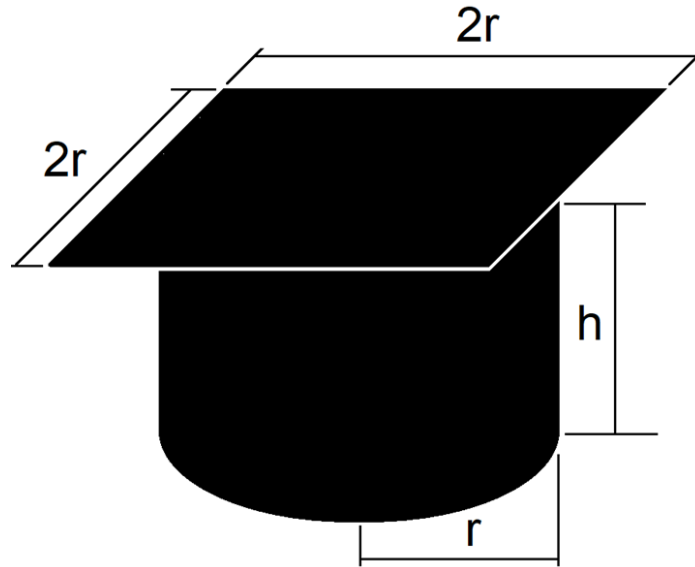
Figure 1: The problem of the mathematician's academic hat.

The surface area, S, to be minimized is given by:

$$S = 2\pi r h + (2r)^2 \text{ subject to the constraint } h = \frac{V}{\pi r^2}.$$

We replace h in the objective function, S, using the constraint equation, differentiate the resulting S with respect to r, and set the derivative to zero.

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in jeder besonderen Naturlehre nur so viel eigentliche Wissenschaft angetroffen werden könne, als darin Mathematik anzutreffen ist."

(By differentiating a second time it is a straightforward matter to show that this results in a minimum for S.)

This is not difficult to do by hand, of course, but by using an appropriate piece of computer software we can minimize not only the surface area, but also the likelihood of errors! Figure 2 shows a MATHCAD-sheet with a symbolic solution to this optimization problem. The real-valued solution for r (the other two are complex) is substituted into the ratio r/h, which is shown to be equal to $\pi/4$. This could be embroidered on our mathematician's optimized academic hat, but it can be improved further to π, as suggested below.

$$h(r,V) := V = \pi \cdot r^2 \cdot h \xrightarrow{\ solve,\,h\ } \frac{V}{r^2 \cdot \pi}$$

$$S(r,V) := 2\,\pi \cdot r \cdot h(r,V) + (2\,r)^2 \to \frac{2 \cdot V}{r} + 4 \cdot r^2$$

$$S'(r,V) := \frac{\mathrm{d}}{\mathrm{d}r} S(r,V) \to 8 \cdot r - \frac{2 \cdot V}{r^2}$$

$$Roots(V) := S'(r,V) = 0 \xrightarrow{\ solve,\,r\ } \begin{bmatrix} \dfrac{V^{\frac{1}{3}}}{4^{\frac{1}{3}}} \\[2ex] \dfrac{\left(1\mathrm{i} \cdot \sqrt{3} - 1\right) \cdot V^{\frac{1}{3}}}{2 \cdot 4^{\frac{1}{3}}} \\[2ex] \dfrac{\left(-\left(1\mathrm{i} \cdot \sqrt{3}\right) - 1\right) \cdot V^{\frac{1}{3}}}{2 \cdot 4^{\frac{1}{3}}} \end{bmatrix}$$

$$r_{opt} := Roots(V)_1 \to \frac{V^{\frac{1}{3}}}{4^{\frac{1}{3}}} \qquad \boxed{\frac{r_{opt}}{h(r_{opt},V)} \to \frac{\pi}{4}}$$

Figure 2: The analytical / symbolic solution to the problem of the mathematical academic hat (MATHCAD).

First of all, VFO, the first author, "tried on" the designed academic hat himself; i.e., he measured the size of his own head. The reader will laugh, but it turned out that with his head circumference of 58 cm, the volume of the hat turned out to be close to π liters (see Figures 3 and 8)! This reassured and pleased VFO, even though he is not a mathematician by education (he is a thermal power engineer).[2] The letter π on the facade of the hat indicates the size (or volume) of the head, and not its content.
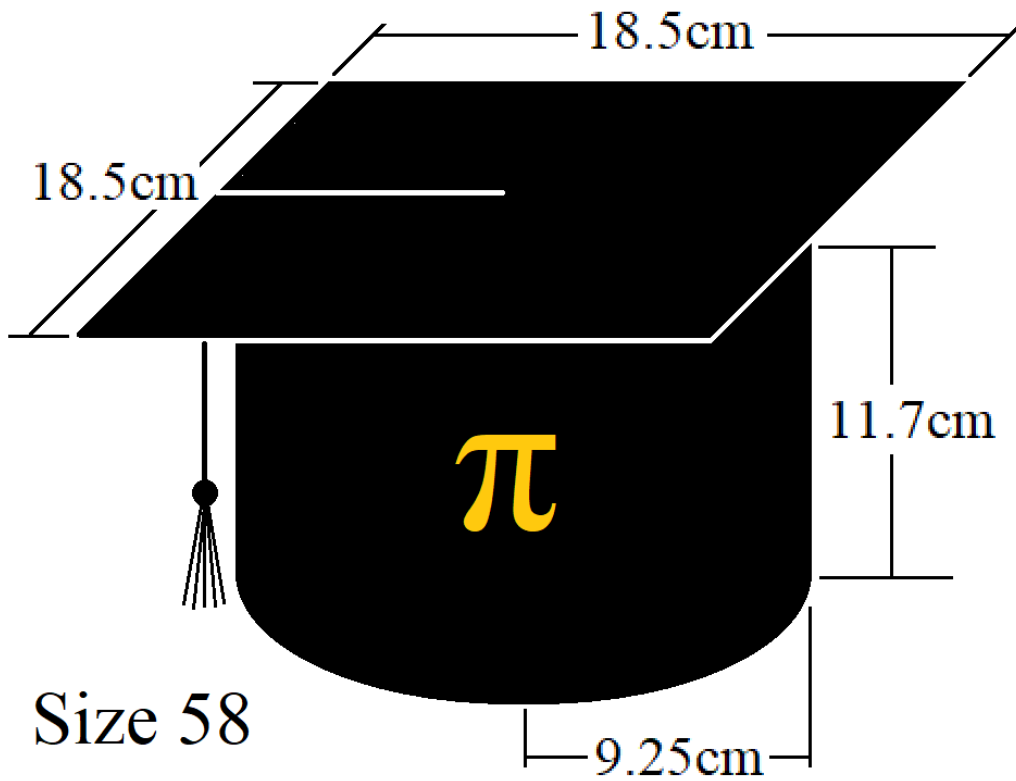
Figure 3: Sketch of the math academic hat.

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[2] In VFO's second alma mater—at the University of Stuttgart, where he underwent a scientific internship in the field of heat power engineering, a person who wore a doctoral dissertation had an academic hat with an entire thermal power station at the top. There was a boiler and a steam turbine; it all spun and whistled, generating electricity. The newly minted doctor was put on a special trolley and transported around the university campus.

In one of his lectures on computer science, VFO asks the students to write down their height, weight, and head size. The author then uses these data in a lecture about regression analysis as a kind of typical statistical sample. It turns out that the average circumferential head size of students is 57 cm. This is understandable as the professor should be somewhat "smarter" than his students! But we should expect that the heads of students during their studies at the university to grow to size 58 cm! So, this size (58 cm) can be considered the most suitable for hats.

It is possible to continue optimizing the tailoring of academic hats, for example in order to calculate how much fabric will be required with the optimal cutting of all the squares and rectangles required for hats for the entire university graduation so that there is a minimum of trimmings. This latter optimization problem belongs to the class of linear programming problems.[3] Thinking of these kinds of problems as a new class was proposed in the West in mid-1940s by George Danzig, one of the founders of linear programming, even before computers were used to solve linear optimization problems. Almost a decade earlier, the Soviet scientist Leonid Kantorovich published his work "Mathematical Methods of Organization and Production Planning" [2], which proposed similar ideas. The 1975 Nobel Memorial Prize, which he shared with Tjalling Koopmans, was given "for their contributions to the theory of optimum allocation of resources" [4].

However, let's move on to compare symbolic (analytical), numerical (approximate), and graphical computer tools for solving optimization problems, based on a slightly more complex example.

Suppose we want to find the parameters of a straight circular cone, covered with a hemisphere, for which the outer surface is minimal for a given volume. Picture a cone with ice cream (see Figure 4). Perhaps it is slowly melting in the hands of a mathematician who is dressed in a heavy academic robe and hat in the summer heat!

The objective function, S, that we are now trying to minimize is given by:

$$S = \pi r \sqrt{r^2 + h^2} + 2\pi r^2 \text{ subject to the constraint } h = (V - \frac{2}{3}\pi r^3)/\frac{1}{3}\pi r^2.$$

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[3] Here, the term *programming* must be understood in the sense of planning. See [5] for more on linear programming.

Figure 4: A cone with ice cream.

Figure 5 on the next page shows the solution to this optimization problem by the method used in Figure 2 for the first problem, namely, taking the derivative and finding zeros from it. But the attempt to find zeros in the derivative of the surface function hits a snag — see the comment (error message) in the yellow box in Figure 5. MATHCAD does not even show its "absolutely accurate and absolutely useless"[4] symbolic expression explicitly. In order to make useful progress it is necessary to start incorporating specific numerical calculations.

More specifically, it is possible to continue in a straightforward manner by simply introducing a numerical value for the volume, then selecting, by eye, the potentially relevant radii from the numerical list of mathematical roots. These can then be inserted in the expression for surface area to determine which produces the minimum area. The corresponding figure for the radius-to-height ratio indicates that the hemispherical diameter is almost 80% larger than the cone height, which would make a rather top-heavy ice cream cone!

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[4] A balloon burst from the clouds. The flying people saw a man on Earth and shouted to him: "Where are we?!" "You are in the basket of a balloon," was the answer. This answer was given by a mathematician. Only from a mathematician it is possible to hear a completely accurate and at the same time completely useless answer.

$$h(r,V) := V = \frac{1}{3}\pi \cdot r^2 \cdot h + \frac{\frac{4}{3}\pi \cdot r^3}{2} \quad \text{solve}, h \rightarrow \frac{3 \cdot V - 2 \cdot \pi \cdot r^3}{\pi \cdot r^2}$$

$$S(r,V) := \pi \cdot r \cdot \sqrt{r^2 + h(r,V)^2} + \frac{4\,\pi \cdot r^2}{2} \rightarrow 2 \cdot \pi \cdot r^2 + \pi \cdot r \cdot \sqrt{r^2 + \frac{\left(3 \cdot V - 2 \cdot \pi \cdot r^3\right)^2}{\pi^2 \cdot r^4}}$$

$$S'(r,V) := \frac{d}{dr} S(r,V) \quad \text{factor} \rightarrow \frac{10 \cdot \pi^2 \cdot r^6 - 9 \cdot V^2 - 6 \cdot \pi \cdot V \cdot r^3 + 4 \cdot \pi^2 \cdot r^5 \cdot \sqrt{\frac{9 \cdot V^2 - 12 \cdot \pi \cdot V \cdot r^3 + 5 \cdot \pi^2 \cdot r^6}{\pi^2 \cdot r^4}}}{\pi \cdot r^4 \cdot \sqrt{\frac{9 \cdot V^2 - 12 \cdot \pi \cdot V \cdot r^3 + 5 \cdot \pi^2 \cdot r^6}{\pi^2 \cdot r^4}}}$$

$r_{opt}(V) := S'(r,V) = 0 \; \text{solve}, r \rightarrow$

The symbolic result returned is too large to display, but it can be used in subsequent calculations if assigned to a function or variable.

$V := 200\text{mL} \quad r_{opt}(V) =$

	1
1	4.943
2	1.427-2.472i
3	-2.471+4.281i
4	1.427+2.472i
5	3.939
6	3.646+6.315i
7	-1.969+3.411i
8	-7.292
9	-2.471-4.281i
10	-2.854
11	-1.969-3.411i
12	3.646-6.315i

cm

$$r_{opt1} := r_{opt}(V)_1 = 4.943 \text{ cm} \qquad r_{opt2} := r_{opt}(V)_5 = 3.939 \text{ cm}$$

$$h_{opt1} := h\left(r_{opt1}, V\right) = -2.069 \text{ cm} \quad h_{opt2} := h\left(r_{opt2}, V\right) = 4.435 \text{ cm}$$

$$S\left(r_{opt1}, V\right) = 236.725 \text{ cm}^2 \qquad S\left(r_{opt2}, V\right) = 170.856 \text{ cm}^2$$

$$\frac{r_{opt2}}{h_{opt2}} = 0.8881127$$

Figure 5: Solving the ice cream cone problem analytically / symbolically (MATHCAD).

The more powerful symbolic mathematics program, MAPLE, is able to display the symbolic result for the roots; see Figure 6, which shows about one hundredth of the expression for the zeros of the derivative generated by MAPLE!

$$+ \left(- \left(3 \left(5 \sqrt{\frac{5\left(18\,I\sqrt{503}+253\right)^{2/3}+107\left(18\,I\sqrt{503}+253\right)^{1/3}+305}{\left(18\,I\sqrt{503}+253\right)^{1/3}}}\,\left(18\,I\sqrt{503}+253\right)^{2/3} \right.\right.\right.$$

$$+ 1032\sqrt{3}\,\left(18\,I\sqrt{503}+253\right)^{1/3} - 214\left(18\,I\sqrt{503}+253\right)^{1/3}$$

$$\sqrt{\frac{5\left(18\,I\sqrt{503}+253\right)^{2/3}+107\left(18\,I\sqrt{503}+253\right)^{1/3}+305}{\left(18\,I\sqrt{503}+253\right)^{1/3}}}$$

$$+ 305\sqrt{\frac{5\left(18\,I\sqrt{503}+253\right)^{2/3}+107\left(18\,I\sqrt{503}+253\right)^{1/3}+305}{\left(18\,I\sqrt{503}+253\right)^{1/3}}}\left.\right)\right) \Big/ \left(\left(18\,I\sqrt{503}+253\right)^{1/3} \right.$$

$$\left. \left(\sqrt{\frac{5\left(18\,I\sqrt{503}+253\right)^{2/3}+107\left(18\,I\sqrt{503}+253\right)^{1/3}+305}{\left(18\,I\sqrt{503}+253\right)^{1/3}}} \right) \right)^{1/2} \right)^{1/3} - 9 \right) \pi^2 \right),$$

Figure 6: A small segment of the analytical / symbolic solution to the ice cream minimization problem (MAPLE).

In other words, as we can see in this example, a purely symbolic solution to a problem often does not give an answer or gives an "absolutely accurate and absolutely useless answer". In such cases, it is necessary to move on to numerical (approximate) mathematics.

Furthermore, it is often pedagogically advantageous to incorporate a third strand of analysis to a discussion of a given optimization problem, besides the two approaches already considered (symbolic and numerical): visualization. This has become especially easy today with the graphics capabilities of modern software computing systems.

Figure 7 shows just such a hybrid solution to the problem of the optimal ice cream cone, using the symbolic and numerical mathematics of MATHCAD, as well as a graphical confirmation of the solution. The functional dependence of the surface area of the cone with the ice cream volume of 200 cm^3 is obtained symbolically in terms of the radius of the hemisphere, whose minimum is then obtained with the help of MATHCAD's numerical built-in `Minimize` function. The surface area is then plotted for a range of values of the radius, with the minimum point marked by the intersecting dashed lines.
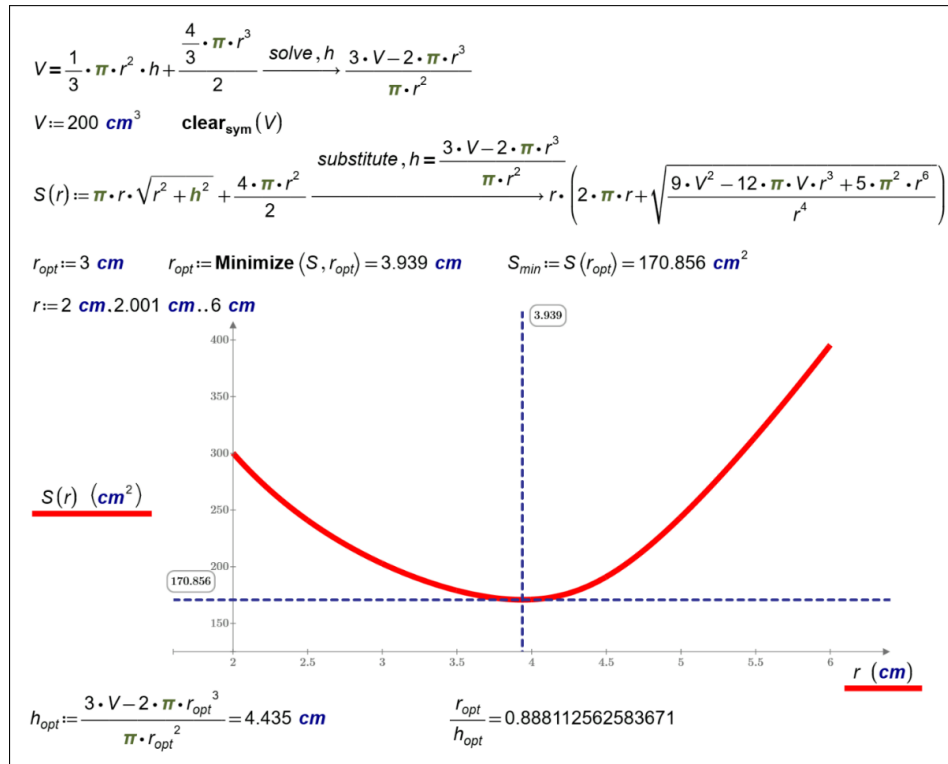
$$V = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h + \frac{\frac{4}{3} \cdot \pi \cdot r^3}{2} \xrightarrow{solve,\, h} \frac{3 \cdot V - 2 \cdot \pi \cdot r^3}{\pi \cdot r^2}$$

$$V := 200 \; \boldsymbol{cm}^3 \qquad \mathbf{clear_{sym}}(V)$$

$$S(r) := \pi \cdot r \cdot \sqrt{r^2 + h^2} + \frac{4 \cdot \pi \cdot r^2}{2} \xrightarrow{substitute,\, h = \dfrac{3 \cdot V - 2 \cdot \pi \cdot r^3}{\pi \cdot r^2}} r \cdot \left(2 \cdot \pi \cdot r + \sqrt{\frac{9 \cdot V^2 - 12 \cdot \pi \cdot V \cdot r^3 + 5 \cdot \pi^2 \cdot r^6}{r^4}} \right)$$

$$r_{opt} := 3 \; \boldsymbol{cm} \qquad r_{opt} := \mathbf{Minimize}(S, r_{opt}) = 3.939 \; \boldsymbol{cm} \qquad S_{min} := S(r_{opt}) = 170.856 \; \boldsymbol{cm}^2$$

$$r := 2 \; \boldsymbol{cm}, 2.001 \; \boldsymbol{cm} .. 6 \; \boldsymbol{cm}$$

$$h_{opt} := \frac{3 \cdot V - 2 \cdot \pi \cdot r_{opt}{}^3}{\pi \cdot r_{opt}{}^2} = 4.435 \; \boldsymbol{cm} \qquad\qquad \frac{r_{opt}}{h_{opt}} = 0.888112562583671$$

Figure 7: Hybrid solution of the ice cream cone problem (MATHCAD).

Final Remarks

Rounding off our narrative: symbolic computer solutions can provide some insight into the nature of optimization problems unless and until they provide too much detail to be easily assimilated, at which point numerical procedures must be introduced. Totally numerical procedures can also be used as independent checks on the symbolic and hybrid approaches. For example, the problem of the optimized mathematician's hat can be solved purely numerically within MATHCAD; see Figure 8.

We hope that with this activity, we were able to offer readers of *Journal of Humanistic Mathematics* a taste of what they can do with modern mathematical software in the mathematics classroom. For more details on this activity and many similar to it, readers may want to check out VFO's forthcoming book, tentatively titled "STEM Problems with MATHCAD and Python".

$$V(r,h) := \pi \cdot r^2 \cdot h \quad S(r,h) := 2\ \pi \cdot r \cdot h + (2\ r)^2$$

Solve

Guess values

$$r := 10\ \boldsymbol{cm} \qquad h := 15\ \boldsymbol{cm}$$

Constraints

$$V(r,h) = \pi \cdot L$$

Solver

$$\begin{bmatrix} r \\ h \end{bmatrix} := \mathbf{Minimize}(S,r,h) = \begin{bmatrix} 9.226 \\ 11.747 \end{bmatrix} \boldsymbol{cm}$$

$$V(r,h) = 3.142\ \boldsymbol{L} \qquad S(r,h) = 0.102\ \boldsymbol{m}^2$$

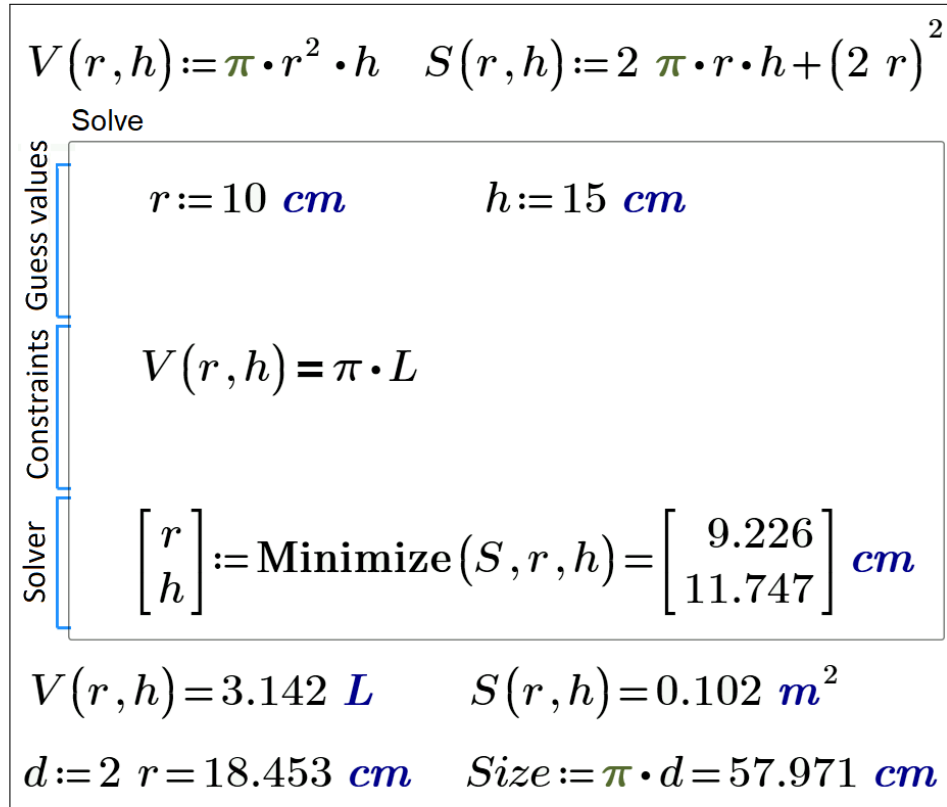$$d := 2\ r = 18.453\ \boldsymbol{cm} \qquad Size := \pi \cdot d = 57.971\ \boldsymbol{cm}$$

Figure 8: Numerical solution of the problem of the mathematician's academic hat (MATHCAD).

The book, together with [3]) introduces a new course for undergraduate STEM programs called "Physical-Mathematical Informatics", following a relatively new direction in education called STE(A)M (Science, Technology, Engineering, [Art] and Mathematics). The target audience of both books is undergraduate students (and high school students), teachers of mathematics, physics, chemistry and other disciplines (humanities), and readers who have a basic understanding of mathematics and math software.

Acknowledgement

References

[1] Mikhail Bulgakov, *The Master and Margarita*, 1967. Complete text of the novel in English, translated from the Russian by Richard Pevear and Larissa Volokhonsky in 1997, is available at `https://files.libcom.org/files/eben002_mastermargarita_pevear.pdf`, last accessed on July 21, 2022.

[2] Leonid Kantorovich, "Mathematical Methods of Organization and Production Planning," original Russian published in 1939 by Leningrad State University; English translation in *Management Science* in 1960 (Volume **6** Issue 4, pages 366–422).

[3] Valery Ochkov, 2^5 *Problems for STEM Education*, Chapman and Hall / CRC, 2020.

[4] Wikipedia authors, "Leonid Kantorovich," *Wikipedia*, 2022. Entry available at `http://en.wikipedia.org/w/index.php?title=Leonid%20Kantorovich&oldid=1095414992`, last accessed on July 21, 2022.

[5] Wikipedia authors, "Linear programming," *Wikipedia*, 2022. Entry available at `http://en.wikipedia.org/w/index.php?title=Linear%20programming&oldid=1097171406`, last accessed on July 21, 2022.