CLAREMONT McKENNA COLLEGE

INVISIBILITY: A MATHEMATICAL PERSPECTIVE

SUBMITTED TO

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Acknowledgments

To Holly, Eliseo, Christian, Scott, Jamie, and Hayley. To my extended family, teachers, and professor Sam Nelson. To all of my friends in Alta Loma, Claremont, and abroad. You have all helped influence my thoughts and actions up to this point in my life. Thank you to everyone who has dedicated their time to promote my well-being.

Abstract

The concept of rendering an object invisible, once considered unfathomable, can now be deemed achievable using artificial metamaterials. The ability for these advanced structures to refract waves in the negative direction has sparked creativity for future applications. Manipulating electromagnetic waves of all frequencies around an object requires precise and unique parameters, which are calculated from various mathematical laws and equations. We explore the possible interpretations of these parameters and how they are implemented towards the construction of a suitable metamaterial. If carried out correctly, the wave will exit the metamaterial exhibiting the same behavior as when it had entered. Thus, an outside observer will not be able to recognize any abnormal changes in wave frequency or direction. This paper will survey studies and technologies from the past 20 years to arrive at a concise mathematical examination of the possibilities and inherent issues under the umbrella of modern "cloaking."

Contents

1 Introduction

The modern era of science has brought about technologies previously seen only through popular science-fiction productions such as Harry Potter and Star Trek. The newly discovered possibilities within the field of cloaking and ”invisibility” have sparked a large interest among the scientific and mathematic communities. Although there are many methods to cloak an object, this research will focus on transformation optics, or the manipulation of electromagnetic waves and energy, using the applicable equations and derivations. However, in theory, the same concepts can be successfully implemented toward optical wavelengths as well. Since the first cloaking device was constructed in 2006, there has never been an account of perfect cloaking in which absolutely no reflectivity or light waves are evident. However, one can achieve perfect cloaking using passive objects with no internal currents. The key mathematical ideas necessary for successful cloaking include, but are not limited to, Snell's Law, Calderon's problem, Maxwell's equations, isotropic and anisotropic conduction, Dirichlet boundary, Neumann data, change of variables, Schrodinger's equation, and the wave equation.

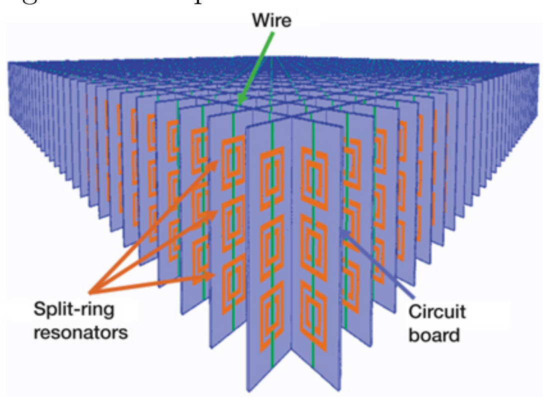
Although there will be a focus on rendering objects invisible strictly through electromagnetic waves, there are on-going studies on methods of cloaking elasticity waves, matter waves, and heat imaging. The main goal at hand is to render an object seemingly ”invisible” by passing electromagnetic waves through a unique metamaterial which bends the waves around the object and sends them out of said metamaterial in the exact path from which it entered. Although this concept appears to be quite simple, the exact wave propagation will depend largely on the uniquely artificial medium and the properties of transformation optics. According to [4], these properties are made inherent by taking advantage of the transformation rules for the material properties of optics: the index of refraction $n(x)$ for scalar optics, governed by the Helmholtz equation, and the electric permittivity $\epsilon(x)$ and magnetic permeability $\mu(x)$ for vector optics, as described by Maxwells equations. These calculations and the associated issues will be discussed in detail in order to grasp the essential aspects of transformation optics.

2 Metamaterial

We place special emphasis on the behavior of the waves near the boundary of the cloaked region $\partial\Omega$ made up of a unique metamaterial. This is crucial given that the electromagnetic parameters are singular at this cloaking surface. In mathematical terms, a singularity is a point at which the parameter is not defined or is not well-behaved. This surrounding region is made up of a unique metamaterial. Due to its recent discovery, there is still no accepted definition of a metamaterial. However, the specific functions that it serves are not attainable from the purely natural world. The combination of one, two, or three-dimensional cellular structure enables the ability to construct material parameters which are impossible to find in nature. This is based on resonances produced by the unique geometry of combining parallel and intersecting strips of fiberglass etched with copper. This leads to the key, and essential, purpose of the metamaterial. The intricate alignment specifications allow waves to split and travel around an object within Ω and reappear on the opposite side of the metamaterial as a single wave with nearly identical characteristics. This is achieved by controlling the exact degree of electric and magnetic response made possible by the highly flexible gradient-index material.

From concepts in [1] and [7], it is likely that the figure below is a microwave frequency metamaterial based on the unique allocation of the split-ring resonators. Specifically, split-ring resonators are arrays of electrically conductive loops of wire which provide inductive and capacitive capabilities for the associated metamaterial.

Figure 1: Components of a metamaterial



3 Creating a Cloak

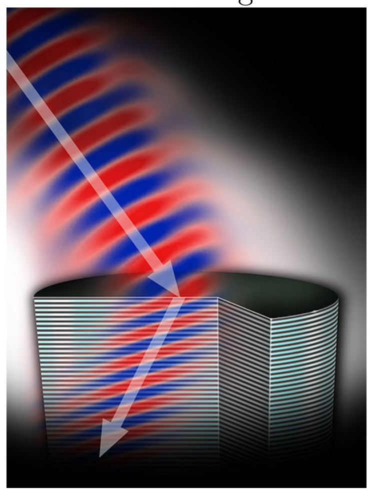
Every aspect of constructing the cloak must abide by the transformation laws of the wave propagation equations. Within the scope of wave propagation, one must proceed with either frequency domain or time domain analysis. We choose to work in the frequency domain with time-harmonic waves of frequency k since metamaterials are naturally subject to dispersion. Furthermore, the relevant electromagnetic material parameters produce the ideal values only among narrow bandwidths. The parameters at hand remain as the index of refraction, $n(x)$, $\epsilon(x)$, and $\mu(x)$. By implementing these individual values on the metamaterial, an object can be successfully cloaked.

The idea is to surround an object within Ω with a material that has an appropriate anisotropic conductivity. The necessary properties are derived from a change-of-variables example found in [3]. However, the computations are not relevant enough for the purposes of this analysis.

Furthermore, there exist multiple ways to attempt electromagnetic invisibility:

Single Coating Structure:

Figure 2: Waves hitting metamaterial



Single coating is both the most simple and most commonly attempted structure for object cloaking. In this case, the electromagnetic parameters act as the "push-forward" of an isotropic and homogenous medium through a singular transformation which "blows up" a specific point, or singularity, to the cloaking surface. Singularity implies a lack of a unique solution at some point in the in the field. The act of being a push-forward entails the differential of a smooth map between any two manifolds. Regardless of singularities and push-forwards, one must take into account the waves on all N where $N = N_1 \bigcup N_2 \bigcup N_3 .. \bigcup \Sigma$ and $\Sigma = \partial N_2 = S^2$ is the cloaking surface between N_1 and N_2. Visually, figure 2 displays N_1 as the region of the wave ray heading southeast while N_2 is the secondary ray bearing southwest with a negative refraction.

Double Coating Structure:

The only difference between this structure and the single coating structure is that the double requires placing a metamaterial around both inner and outer surfaces of Σ. This is equivalent to using an $F = (F_1, F_2)$ where both F_1 and F_2 are singular. Therefore, there is no singular Riemannian metric which approaches Σ from both sides with the same degeneration. However, in [5], we are given the singular Riemann metric metric $\tilde{g}$ defined everywhere on $Nn\Sigma = N_1 \cup N_2$ by:

$$\tilde{g} = \begin{cases} |\tilde{g}_1 := F_1 * g_1 & x \in N_1 \\ \tilde{g}_2 := F_2 * g_2 & x \in N_2 \end{cases}$$

Conceptually, one can think of this simply as a metamaterial within a metamaterial. In reality, calculations become much more difficult given the additional complete set of parameters.

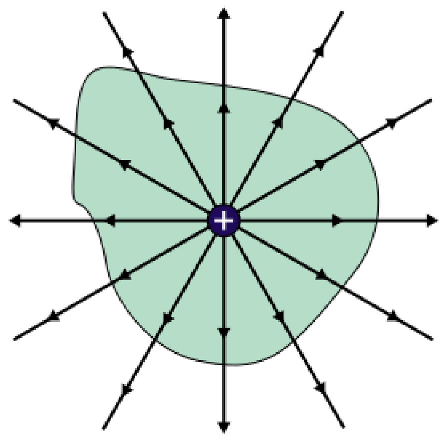
In terms of Electrostatics:

Electrostatic analysis is carried out only when dealing with a highly charged body within Ω as shown in Figure 3 . Electrostatic cloaking can be more simply thought of as optics at frequency zero [8]. Consider the following:

Theorem 1 *Gauss's law (AKA Gauss's Flux Theorem):*
The total electric flux through any closed surface of a shape in an electric field is proportional to the enclosed electric charge.

This is one of Maxwell's equations, which will be demonstrated later, and which make up the foundation for classical electrodynamics. By implementing a Riemann metric onto the electrostatic parameters, one is able to conduct minute angle and distance calculations. Therefore, given the role of surface charge in Gauss' law, we choose to create a singular Riemann manifold by collapsing a hypothetical manifold toward a limit point in the conductivity. This will appear to be a flat surface and therefore appear to have a constant conductivity in terms of boundary measurements. The mathematical construction of this singularity is produced below:

Figure 3: Gaussian region



It is appropriate to use balls and not disks in this scenario due to calculations taking place in three-dimensional Euclidean space. The idea is to create an open ball $B(0, R) \subset \mathbb{R}^3$ where $R = 2$ in the sense that N_2 is the inside of the metamaterial denoted by $N_2 = B(0, 1)$. Whereas, N_1 is the region outside of the metamaterial (close ball) denoted by $N_1 = B(0, 2) \backslash \overline{B}(0; 1)$. This is slightly counterintuitive, in that the inside region has a subscript of 2 and the outside has a subscript of 1.

Within the realm of real analysis, the open ball is a basis for the topological space, whose open sets include all possible unions of open balls. The space itself is deemed the topology induced by the metric. In this case, we denote g as the metric in N_1 and δ as the metric in N_2.

Conditions:

Let $M_1 = B(0, 2)$ and $g_{jk} = \delta_{jk} =$ Euclidean metric in M_1

We define a singular transformation as a linear transformation with no corresponding inverse transformation: $F_1 : M_1/0 \to N_1$, $F_1(x) = (\frac{x}{2} + 1)\frac{x}{|x|}$, $0 < |x| \leq 2$

According to [4], this gives rise to the singular conductivity, singular permittivity, and singular permeability in N:

$$\tilde{\sigma} = \tilde{\varepsilon}^{jk} = \tilde{\mu}^{jk} = \begin{cases} |\tilde{g}^{\frac{1}{2}} g^{jk}| & x \in N_1 \\ \delta^{jk} & x \in N_2 \end{cases}$$

The solution for $x \in N_2$ can fill in the hole left by F_1 through replacement of a diffeomorphism. A diffeomorphism is an invertible function between two smooth manifolds which maintains the given differentiable structure. Therefore, one can isolate individual singularities which can single-handedly ruin any possibility for electrostatic cloaking.

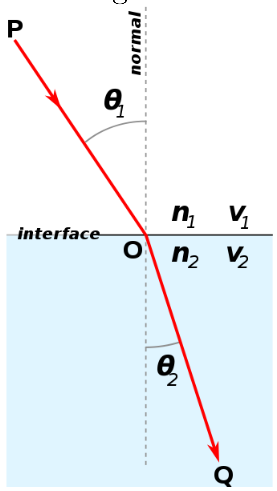
4 Snell's Law

The critical property of a metamaterial is the ability to produce the optical property of a negative refractive index ($n = \frac{c}{v}$) while natural materials only permit a positive retractive index. In this sense, the illusion of invisibility is not possible in a purely "natural" state. We assume, for the sake of simplicity, that our calculations take place within the bounded unit circle and are independent of time, which is known as "DC imaging". This negative refraction is a manipulation of Snell's Law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{1}$$

where n_1 is the refractive index of the first medium and n_2 is the refractive index of the second. Figure 4 below provides a visual model in which we let θ_1 denote the incidence angle, and θ_2 the transmission angle of (1). Therefore, this equation provides the notion that the transfer between directions, or "refraction", occurs only when light moves from one medium to another. However, if there is no transmission angle, θ_2, then it follows that the ratio of the sines of the angles of incidence and refraction is equivalent to the reciprocal ratio of the refraction indices in two different isotropic media. This is represented mathematically

Figure 4: negative refraction



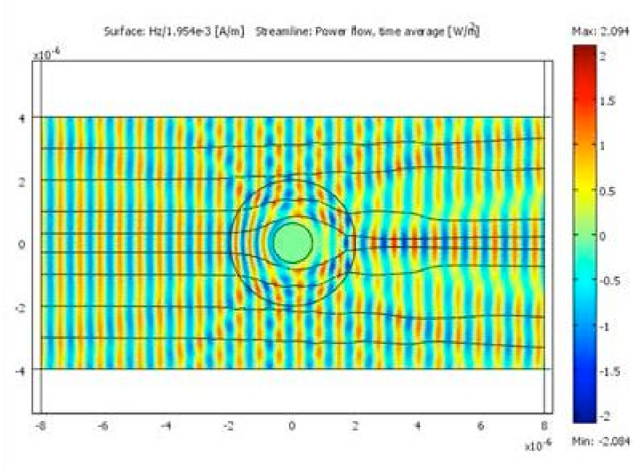
as: $\frac{n_1}{n_2}\sin\theta_1$ The isotropic media in this case refers to the type of conduction. Solving for the conduction within Ω will be proven to require some unconventional thinking.

5 Electrical Impedance Tomography

One major issue present in cloaking analysis is that an external observer can only analyze the interior conductivity of the given region based on the effects displayed on the boundary. Within the realm of mathematics, this type of issue is known as a non-linear inverse problem which is also considered ill-posed. The problem is ill-posed in the sense that it is not well-posed; meaning a solution exists, the solution is unique, and the solution depends continuously on data and parameters. However, this region must always consist of electrical currents, which makes all analysis more than just hypothetical. We will see that ill-posed problems are not as rare as one may think.

The phenomenon is referred to as Calderon's problem from the discovery of Albert Calderon that the conductivity within Ω can be measured by the current and potential

Figure 5: Electrical Impedance Tomography simulation



of the specific metamaterial on $\partial\Omega$. This is intuitive for the inherent initial-boundary problems of cloaking because small measurement errors will not lead to large prediction errors.

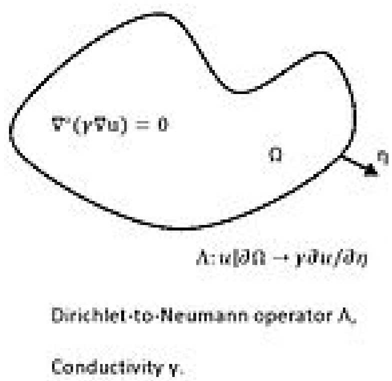
Isotropic conduction:

Despite the negative connotation as an "ill-posed" problem, Electrical Impedance Tomography (EIT) is used in the modern world for numerous medical procedures including the early detection of breast cancer. This is useful because the conductivity of a malignant breast tumor is typically 0.2 mho, significantly higher than normal tissue, which is typically measured at 0.03 mho. A simulation of this process is illustrated above in Figure 5.

According to [4], the previous information, along with other useful data, is obtained from the follwing fundamental evaluation: Within the domain Ω, coefficient $\sigma(x)$ describes the internal conductivity, which is bounded above and below by positive constants.
Let u = voltage potential, then we have the Divergence Form Equation:

$$\nabla \cdot \sigma \nabla u = 0 \tag{2}$$

Figure 6: Dirichlet-Neumann Map



where solution u has value f. Let $v =$ exterior unit normal to $\partial\Omega$, then measure all voltage distributions using $u \mid \partial\Omega = f$ and the current flux $v \cdot \sigma\nabla u$.

Obtaining all of the information from (2) implies the acquisition of the Dirichlet-Neumann (DN) map data represented as Λ_σ. Since there is respect to σ, this is effectively taking the Dirichlet boundary values, from the divergence form equation, to the corresponding Neumann boundary values:

$$\Lambda_\sigma : u \mid \partial\Omega \mapsto v \cdot \sigma\nabla u \mid \partial\Omega$$

One must come to all conclusions pertaining to the interior of Ω based solely on the DN map data portrayed in Figure 6. Therefore, the inverse problem is to find σ from Λ_σ.

Anisotropic conduction:

Anisotropic conduction, in which conductivity is direction-dependent, also has a large area of application within the physical world. As opposed to isotropic cancer detection, anisotropy is present within all muscle tissue, which makes it easy to determine the spacial aspects of the muscle fibers. In general, cardiac muscle fibers have a longitudinal conductivity of 6.3 mho and a transversal conductivity of 2.3 mho. The key principle is that some anisotropic objects create refractions, which create boundary measurements similar to that of a homogenous medium.

Extensive details of the D-N boundary data in [2] demonstrate its crucial role in achieving electromagnetic transparency. One must use the Dirichlet - Neumann boundary to understand what values must be taken for an equation to be on the boundary ∂ of the domain Ω. Often, the use of a Dirichlet - Neumann map can find the unique isotropic conductivity that is closest to an anisotropic one under changes of variables.

The authors of [4] define the anisotropic conductivity as a symmetric, positive semi-definite matrix-valued function:

$$\sigma = (\sigma^{ij}(x))$$

The domain $\Omega \subset \mathbf{R}^n$ on which we base this analysis remains consistent with the other discussed aspects of cloaking. We then assume the absence of any source or sink; a boundary where current flows from a location where it is not measured to one where it is measured. Also, let f be the voltage on the boundary. Under these conditions, the electrical potential u satisfies the equation:

$$(\nabla \cdot \sigma \nabla) u = \partial_j \sigma^{jk}(x) \partial_k u = 0 \in \Omega,$$
$$u|_{\partial\Omega} = f$$

For the sake of simplicity, Greenleaf, author of [4], later uses the Einstein summation convention to develop the voltage-to-current relationship that is the Dirichlet-to-Neumann map:

$$\Lambda_\sigma(f) = Bu|_{\partial\Omega},$$
$$Bu = v_j \sigma^{jk} \partial_k u$$

We can input the previously solved u and take v as the unit normal vector of $\partial\Omega$. Therefore, we have taken all of the steps toward the boundary data necessary to determine the conductive properties within Ω.

6 Maxwell's Equations

Maxwell's equations serve as a means to calculating the three-dimensional perfect cloak at non-zero frequencies through a singular change of variables. Three common issues arise from Maxwell's equations in the invisibilty process:

- The transformation law for Maxwell's equations is not legitimate near the boundary $\partial\Omega$ under smooth transformations
- Maxwell's equations will not hold with non-degenerate isotropic electromagnetic parameters
- Boundary conditions are overdetermined near the metamaterial surface (This will be explained at the end of the section)

Due to the nature of metamaterials, Maxwell's equations vary based on the nature of the domain. This provides another example of the various ways in which one can interpret the act of cloaking. Similar to the process taken in the "creating a cloak" section, one must choose to focus on either the frequency domain or time domain. However, for Maxwell's equations, we do not necessarily choose frequency domain.

Analysis in frequency domain:

The essence of frequency analysis lies in the direction and size of the associated wavelengths on N. The metamaterial can be homogenized because the wavelengths of interest are much larger than the scale of inhomogenities. The homogenization process is the key to dealing with varying frequencies in order to successfully refract the waves around the object in the cloaking region. This is accomplished through the use of effective permittivity and effective permeability within the lossy Drude model as observed by Li and Huang [6]:

$$\epsilon(\omega) = \epsilon_0(1\frac{\omega_{pe}^2}{\omega(\omega - j\Gamma_e)}) = \epsilon_0\epsilon_r \tag{3}$$

$$\mu(\omega) = \mu_0(1 - \frac{\omega_{pm}^2}{\omega(\omega - j\Gamma_m)}) = \mu_0\mu_r \tag{4}$$

In the general case of cloaking one must always assume $\sigma = 0$ and ε and μ are real valued. For (3) and (4), ϵ_0 and μ_0 are the vacuum permittivity and vacuum permeability respectively. For instance, $\epsilon = \epsilon_0$ in a vacuum, but in pure water $\epsilon = 80\epsilon_0$. In this domain, ω is any frequency and we denote ω_{pe} and ω_{pm} as the electric and magnetic plasma frequencies. Whereas Γ_e and Γ_m represent the electric and magnetic damping frequencies. Relating back to Snell's law, one can conveniently set $\Gamma_e = \Gamma_m = 0$ and $\omega_{pe} = \omega_{pm} = \sqrt{2}\omega$ to accomplish the negative refractive index necessary for cloaking:

$$n = -\sqrt{\epsilon_r\mu_r} = -1$$

Analysis in time domain:

Consider E and H as the electric and magnetic fields at hand and assume the material parameters, ε and μ, are singular at Σ. First, one needs to bridge the information gap between the electric field $E(x,t)$ and magnetic field $H(x,t)$ using the results of both Faraday's and Ampere's Laws:

Faradays Law:

$$\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t} \tag{5}$$

Amperes Law:

$$\nabla \times \mathbb{H} = \frac{\partial \mathbb{D}}{\partial t} \tag{6}$$

In general, Ampere's law is to magnetism as Gauss's law is to electricity. First, consider Faraday's law where $D = \varepsilon \cdot E$ and then take $B = \mu \cdot H$ to solve for Ampere's Law.

Solutions:

Definition: $(\tilde{E}, \tilde{H})$ is a finite energy solution to the following Maxwells equations on N:

$$\nabla \times \tilde{E} = ik\tilde{\mu}(x)\tilde{H}$$
$$\nabla \times \tilde{H} = -ik\tilde{\varepsilon}(x)\tilde{E} + \tilde{J}$$

These equations also hold true on a neighborhood $U \subset \overline{N}$ of ∂N.

Why cloaking active objects is difficult:

The idealized equations of this chapter show that finite energy solutions cannot exist with non-zero currents with N_2. Consider a finite energy solution [4]:

Theorem 2 *A finite energy solution must satisfy the following hidden boundary conditions on* ∂N_2 :

$$v \times \tilde{E} = 0$$
$$v \times \tilde{H} = 0$$

These boundary conditions, known as the perfect electrical conductor (PEC) and the perfect magnetic conductor (PMC) are overdetermined on N_2. For all generic external current, $J \neq 0$, there is no solution. However, these can easily be satisfied in the case of passive objects with $J = 0$, with fields that are identically zero in Ω.

We will now show the effects of linings for the following Non-homogenous Maxwell equations [4]:

$$\nabla \times \tilde{E} = ik\tilde{\mu}(x)\tilde{H} + \tilde{K}_{surf} \tag{7}$$

$$\nabla \times \tilde{H} = ik\tilde{\varepsilon}(x)\tilde{E} + \tilde{J} + \tilde{J}_{surf} \tag{8}$$

Given this non-homogeneous initial condition, $\tilde{K}_{surf}$ and $\tilde{J}_{surf}$ signify the magnetic and surface currents on Σ. One alternative to consider is placing a PEC or PMC lining on the inner side of Σ. Given a PEC lining, the solution of the boundary value f has $\tilde{K}_{surf} = 0$, whereas that of the PMC lining has $\tilde{J}_{surf} = 0$. These linings are used to help improve the mathematical behavior of metamaterials to avoid "blowing up" the field of (7) and (8) and to better facilitate approximate cloaking. Therefore, proper choice of metamaterial, based on the nature of internal currents, is crucial for successful cloaking.

7 Summary

We have proven that there are many ways in which invisibility can be examined given the various possible domains, waves, and nature of objects undergoing the cloaking process. Through manipulation of any given metamaterial structure, one can redirect electromagnetic waves at frequencies ranging from visible light to radio waves for nearly any object. However, this is only practical for small objects given current scientific technology.

The first important decision is whether the nature of the object requires a single or a double coating structure. This, along with the unique topological specifications, will dictate the ideal parameters for satisfying Snell's Law in order to achieve the ideal negative index of refraction. Furthermore, one must measure the interior conductivity of $/Omega$ by means of external testing. The resulting Dirichlet - Neumann boundary data provides the basis for calculations taken upon the relevant Maxwell's equations. Therefore, we come to a thorough understanding of the relationship between the electric and magnetic fields for the specific metamaterial.

Throughout this monumental task, one must consider both time and frequency domains separately as well as isotropic and anisotropic conductions. These decisions are made based on the nature of the object Ω and are essential in the development of a uniquely specific metamaterial. Additionally, great strides have been made recently in the construction of metamaterials for both acoustic and seismic waves. Specifically, this provides hope for limitless applications in the fields of aerospace, solar energy, military communication, and public safety.

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