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On symmetric operator ideals and s-numbers

By

Daniel Akech

Claremont Graduate University

2023

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Approval of the Dissertation Committee

This dissertation has been duly read, reviewed, and critiqued by the Committee listed below, which hereby approves the manuscript of Daniel Akech as fulfilling the scope and quality requirements for meriting the degree of Doctor of Philosophy in Mathematics.

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Abstract

On symmetric operator ideals and s-numbers

By

Daniel Akech

Claremont Graduate University: 2023

Motivated by the well-known theorem of Schauder, we study the relationship between various s-numbers of an operator T and its adjoint T^* between Banach spaces. For non-compact operator $T \in \mathcal{L}(X, Y)$, we do not have a lot of information about the relationship between n-th s-number, $s_n(T)$, with $s_n(T^*)$, however, in chapter 2, by considering X and Y, with lifting and extension properties, respectively, we were able to obtain a relationship between $s_n(T)$ with $s_n(T^*)$ for certain s-numbers. Using a certain characterization of compactness together with the Principle of Local Reflexivity, we give a different simpler proof of Hutton's theorem. In chapter 3, by considering operators which are not compact but compact with respect to certain approximation schemes Q, which we call Q-compact, we proved Hutton's Theorem for Q-compact operator T and symmetrized approximation numbers, which answers the question of comparing the degree of compactness for T and its adjoint T^* for non-compact T. Chapter 4 defines the K-functional via rearrangement-invariant function spaces, studies its effect on interpolation spaces, applies interpolation theory to some linear and non-linear partial differential equations, and also gives some criteria for the boundedness of the norms of operators arising from PDEs in some concrete Banach spaces. Under natural conditions regarding Bernstein and Jackson inequalities, interpolation spaces can be realized as approximation spaces. Consequently, the final chapter 5 defines approximation spaces for *compact H-operators* using the sequences of their eigenvalues and establishes relations among these spaces using interpolation theory, and presents an inclusion theorem and a representation theorem.

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Finally, I thank my fiancée, Abuk Awer Dau Diing for her patience and encouragement during the writing of this dissertation, and my daughter and son, Angeth and Thiong, who inspire me to wake up each day to continue the journey. I thank my entire family and friends, who have been supportive throughout the process, of whom Dr. Majak D' Agoot's encouragement has been uplifting.

This dissertation is dedicated to the memory of my beloved Uncle Ezekiel Diing Gak Thiong Diing, who was the first in our family to enter university. Diing inspired generations of Thiong to excel in education. While he was a high school student in Renk, Uncle Diing graduated from high school at the top of his class and he studied geology at the University of Juba, a program which he terminated to join a rebel

movement in 1983. His life was cut short in 1993.

Contents

Chapter 1

Compact operators

When one is given an operator on a Banach space, one may be interested in whether it is linear, bounded, or compact among other desirable features. For a smooth operator associated with a given differential equation, boundedness, for example, makes it possible to find critical points, which are solutions to the given differential equation. In other settings, the existence of a solution to a given differential equation amounts to the existence of fixed points for an associated operator, which requires such an operator to be compact. While this dissertation is not about solving PDEs, we give concrete examples to illustrate the importance of boundedness and compactness.

First, for a bounded domain $\Omega \subset \mathbb{R}^n$, we consider the following problem of searching for a solution u to the following equation:

$$-\Delta u = f(x, u), \;\; x \in \Omega. \tag{1.1}$$

The corresponding functional is defined by $G(u) = \frac{1}{2}\|\nabla u\|^2 - \int_\Omega \int_0^{u(x)} f(x, s) ds dx$, where the norm is that of L_2. Any u for which $G'(u) = 0$ (a *critical* point) solves (1.1). Global extreme can exist if the functional G is **bounded** from either below or from above. When G is not bounded, it is unclear how to look for critical points.

Second, suppose that V is a continuously differentiable function such that

$$|V(u)| \leqslant M|u|,$$

and consider the non-linear elliptic boundary value problem:

$$\begin{cases} -u''(t) = V(u(t)) \\ u(0) = 0 = u(1). \end{cases} \tag{1.2}$$

Are there any solutions of (1.2)? A solution of (1.2) would be a fixed point for a suitable linear operator $T : C([0,1]) \to C([0,1])$. There are many theorems on the existence of fixed points for this type of situation, which require T to be *compact*.

In [21], I. Fredholm created the determinant theory of integral operators in 1903, which would give rise to the abstract theory of Hilbert Spaces. In [52], F. Riesz proved in 1918 that compact operators have at most countable set of eigenvalues, which arranged in a sequence, tend to zero. At this point, the question of the rate of convergence to zero of the sequence of eigenvalues did not come into the picture. What are the conditions on T s.t $(\lambda_n(T)) \in \ell_q$? To answer this question in approximation theory, in 1966, A.S. Markus defined H -operators and in 1987, A. Pietsch developed s-numbers (closely related to singular values). More broadly, interpolation theory has been used to answer questions in approximation theory.

Interpolation theory was originally discovered by I. Schur, M. Riesz, G.O. Thorin, J. Marcinkiewicz, and A. Zygmund while J. - L. Lions, J. Peetre, A. P. Calderon, E. Stein, and E. Gagliardo made major contributions (see, [7], p. 117). The possibility of applying interpolation techniques to approximation theory was initiated by Jaak Peetre in 1963. The main realization starts with recognizing that every approximation space is a real interpolation space. This means that the K-method becomes available as a tool in approximation theory, which can then be used to obtain, for example, Bernstein and Jackson's theorems concerning the best approximation of functions in $L_p(\mathbb{R}^n)$ by entire functions of exponential type, approximation of compact operators by operators of finite rank, approximation of differential operators by difference operators (see, e.g. [6]). Interpolation techniques, such as the Trace Theorem and the parameter theorem (K-functional), were developed originally to solve partial differential equations (see, e.g., [38]). In mathematics, when one is given an object, say a group, a ring, a vector space, or the space of all continuous linear mapping from one space to another, the natural question to ask is what are some of the most interesting subspaces. Compact operators form an interesting subspace of the space of continuous linear operators from one Banach space to another. We present a characterization of compact operators results discovered in 1965 by Pietsch [46] and in 1972 by Terzioğlu [58] (and also independently in 1987 by Stephani [56]). These characterizations are the starting points for the definitions of the Kolmogorov and Gelfand numbers, respectively. It turns out that the monotonously decreasing sequences of Kolmogorov and Gelfand numbers live in certain spaces called Lorentz spaces, which can be realized as approximation spaces, which are, in turn, real interpolation spaces. The other approximation quantities related to Kolmogorov and Gelfand numbers are entropy numbers [12], and approximation numbers [31], with the entropy numbers being motivated by the definition of a compact operator. In contrast, the approximation numbers express the degree of approximability of an operator by the finite rank operator.

The dissertation consists of five chapters. Chapter 1 gives an introduction to compactness and properties. In Chapter 2, by imposing certain natural conditions on X and Y, we were able to obtain a relationship between $s_n(T)$ with $s_n(T^*)$ for certain s-numbers. Using a certain characterization of compactness together with the Principle of Local Reflexivity, we give a different simpler proof of Hutton's theorem. In chapter 3, by considering operators which are not compact but compact with respect to certain approximation schemes Q, which we call Q-compact, we proved Hutton's Theorem for Q-compact operator T and symmetrized approximation numbers, which answers the question of comparing the degree of compactness for T and its adjoint T^* for non-compact T. Chapter 4 defines the K-functional via rearrangement-invariant function spaces, studies its effect on interpolation spaces, applies interpolation theory to some linear and non-linear partial differential equations, and also gives some criteria for the boundedness of the norms of operators arising from PDEs in some concrete Banach spaces. Under certain conditions regarding Bernstein and Jackson inequalities, interpolation spaces can be realized as approximation spaces. Consequently, the final chapter 5 defines approximation spaces for *compact H-operators* using the sequences of their eigenvalues and establishes relations among these spaces using interpolation theory, and presents an inclusion theorem and a representation theorem.

1.1 Basic Notions

Let X and Y be Banach spaces and $T : X \to Y$ be an operator. We denote the closed unit ball of X by $B_X = \{x \in X : ||x|| \leqslant 1\}$ and $\mathcal{L}(X, Y)$ denotes the normed vector space of all bounded operators from X to Y and $\mathcal{L}(X)$ stands for $\mathcal{L}(X, X)$. We will use $\mathcal{K}(X, Y)$ for the collection of all compact operators from X to Y. It is also well known that $\mathcal{K}(X)$ is a two-sided ideal in $\mathcal{L}(X)$.

Definition 1.1. Let Y be a Banach space and $F \subset Y$. We say that F is **relatively compact** if one of the two following equivalent properties holds:

(i) $\forall \epsilon > 0$, there is a finite number $N \in \mathbb{N}^*$ of points $y_1, \cdots, y_N \in Y$ such that $F \subset \bigcup_{i=1}^{N} B(y_i, \epsilon)$;

(ii) for any sequence $(u_n)_{n\in\mathbb{N}}$ with values in F, there exists a subsequence $(u_{\phi(n)})_{n\in\mathbb{N}}$ which converges in Y.

We will be able to restate the following fundamental theorem on compactness as an approximation result.

Theorem 1.2 (Arzelà-Ascoli)**.** *Let K be a compact metric space and let $\mathcal{F}$ be a bounded subset of $C(K)$ whereby $C(K)$ we mean the space of continuous functions over K with values in $\mathbb{R}$ or $\mathbb{C}$. Assume that*

$$\forall \epsilon > 0 \ \exists \delta > 0 : d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon \ \forall f \in \mathcal{F}.$$

Then $\mathcal{F}$ is a relatively compact subset of $C(K)$.

Definition 1.3. Let X be a normed vector space and Y be a Banach space. A linear operator $T : X \to Y$ is **compact** if $T(B_X)$ is relatively compact in Y.

Compact operators are natural generalizations of finite rank operators, and thus, dealing with compact operators provides us with the closest analogy to the usual theorems of finite-dimensional spaces. In particular, the spectral properties of compact operators resemble those of square matrices.

Proposition 1.4. *(1) Every compact operator is bounded; however, the identity operator on an infinite-dimensional space is bounded, but not compact.*

(2) Every bounded linear operator with a finite-dimensional range is compact. If Y is a Hilbert space, then any close map $T : X \to Y$ is a limit of a sequence of finite rank operators.

(3) A theorem due to Schauder says that a bounded linear operator $T : X \to Y$ between Banach spaces is compact if and only if its adjoint $T^* : Y^* \to X^*$ is compact.

We can use (1) and (3) in the preceding proposition to find a non-trivial example of a bounded but not compact operator.

Example 1.5. Let S be a shift operator defined by: $S : \ell_2 \to \ell_2 : S(x_0, x_1, x_2, \cdots) = (0, x_0, x_1, x_2, \cdots)$. Then the adjoint of the shift S is the backward shift given by $S^*(f_0, f_1, f_3, \cdots) = (f_1, f_2, f_3, \cdots)$. It follows that $S^*S = I$. By (3), S and S^* must be compact together or not. If they were compact together, their composition would be compact, contradicting (1). Hence, the shift S is an example of a non-compact operator.

Compact operators play a significant role in studying differential and integral equations.

Example 1.6. Let $I = [0, 1]$ and suppose that $k : I \times I \to \mathbb{C}$ is continuous on $I \times I$; define $(Kx)(s) = \int_0^1 k(s, t)x(t)dt$ for all $s \in I$ and for all x in the Banach space $C(I)$ of all continuous complex-valued functions on I with the norm

$$\|x\| = \max\{|x(s)| : s \in I\}.$$

Then $K : C(I) \to C(I)$ is compact operator.

The standard proof shows that $K(B_{C(I)})$ is bounded, closed, and equicontinuous and invokes the Arzelà-Ascoli Theorem to conclude that $K(B_{C(I)})$ is relatively compact so that K is compact (cf. [17], p. 2).

Remark 1.7. The notions of relatively compact and compact coincide in a complete metric space (for example, in a Banach space) because a set is totally bounded if and only if its closure is compact.

In the setting where X and Y are Banach spaces, we have the following definition, which is the motivation for introducing the dyadic entropy numbers.

Definition 1.8. $T \in \mathcal{L}(X, Y)$ is compact if and only if for every $\epsilon > 0$, there exists elements $y_1, y_2, \cdots, y_n \in Y$ such that

$$T(B_X) \subset \bigcup_{k=1}^{n} \{y_k + \epsilon B_Y\}.$$

The set of all compact operators, denoted by $\mathcal{K}(X, Y)$, is an example of *operator ideals* of Banach spaces defined below.

Definition 1.9. An *operator ideal* $\mathcal{U} := \{\mathcal{U}(X, Y),$ where X and Y are Banach spaces $\}$ is a subclass of $\mathcal{L}(X, Y)$ such that its components $\mathcal{U}(X, Y) := \mathcal{U} \cap \mathcal{L}(X, Y)$ satisfy the following conditions:

(i) $I_{\mathbb{K}} \in \mathcal{U}$, where $\mathbb{K}$ indicates a one-dimensional Banach space.

(ii) If $S_1, S_2 \in \mathcal{U}(X, Y)$ then $\lambda_1 S_1 + \lambda_2 S_2 \in \mathcal{U}(X, Y)$ for any scalars λ_1, λ_2.

(iii) If $T \in \mathcal{L}(X_0, X), S \in \mathcal{U}(X, Y)$ and $R \in \mathcal{L}(Y, Y_0)$, then $RST \in \mathcal{U}(X_0, Y_0)$.

1.2 Dyadic entropy numbers and compact operators

Looking for the smallest ϵ for which the image of the unit ball under T is covered in the fashion of the preceding definition 1.8 has led to the following definition of entropy numbers, which quantify the degree of compactness of an operator $T \in \mathcal{L}(X, Y)$ (cf. [12], pp. 7 - 12).

Definition 1.10. Let M be a bounded subset of X. Then for $n \in \mathbb{N}$ define *the nth entropy number* of M as follows:

$$\epsilon_n(M) = \inf\{\epsilon > 0 : \text{ there exists } q \leqslant n \text{ points } x_1, x_2, \cdots, x_q \text{ in } X \text{ such that } M \subset \bigcup_{i=1}^{q} B(x_i, \epsilon)\}$$

and the corresponding nth entropy number for $T \in \mathcal{L}(X, Y)$:

$$\epsilon_n(T) := \inf\{\epsilon > 0 : T(B_X) \subset \bigcup_{j=1}^{q} (y_j + \epsilon B_Y)\}$$

for all $\epsilon > 0$ and for some $y_j \in Y$.

Definition 1.11. A bounded subset $M \subset X$ is relatively compact if and only if

$$\lim_{n\to\infty} \epsilon_n(M) = 0$$

and correspondingly, $T \in \mathcal{L}(X, Y)$ is compact if and only if

$$\lim_{n\to\infty} \epsilon_n(T) = 0$$

This definition provides a concise way to write a shorter proof for the compactness of a closed interval of real numbers.

Example 1.12. Let $X = \mathbb{R}$. Then X is complete and so a relatively compact subset of X is compact. Let $M = [a, b]$. Then $\epsilon_n([a, b]) = \frac{b-a}{2n}$. Thus $\epsilon_n([a, b]) \to 0$ as $n \to \infty$. Hence, $[a, b]$ is relatively compact and hence it is compact.

The sequence $\{\epsilon_n(M)\}$ is monotonously decreasing, and the rate of decrease may be regarded as a measure of the degree of pre-compactness of the set M.

Further properties of the entropy numbers $\epsilon_n(T)$ (cf. [12], pp. 7 - 12):

1. $\epsilon_n(T) \geqslant \epsilon_{n+1}(T)$ with $\epsilon_1(T) = ||T||$

2. $\epsilon_{kn}(T+S) \leqslant \epsilon_k(T) + \epsilon_n(S)$ for $T, S \in \mathcal{L}(X,Y)$

3. $\epsilon_{kn}(TS) \leqslant \epsilon_k(T)\epsilon_n(S)$ for $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Z,W)$.

Remark 1.13. The well-known fact that $\mathcal{K}(X)$ is a two-sided ideal in $\mathcal{L}(X)$ follows from property (3) of entropy numbers. Indeed, if T or S is relatively compact, then TS is relatively compact since

$$\lim_{n\to\infty} \epsilon_n(TS) = 0$$

Definition 1.14. Let $f(x)$ be defined on a compact set K. For arbitrary $\delta \geqslant 0$, the function

$$\omega(f,\delta) = \sup_{x,y\in K : d(x,y)\leqslant\delta} |f(x) - f(y)|$$

is called the *modulus of continuity* of f on K.

Remark 1.15. i) An equicontinuous family, $\mathcal{F}$, as required in the classical Arzelà-Ascoli, can be seen as a set of functions sharing the same modulus of continuity. For example, the M-Lipschitz functions all have $\omega(f,\delta) \leqslant M\delta$. The equicontinuity condition is equivalent to

$$\lim_{\delta\to 0^+} \sup_{f\in\mathcal{F}} \omega(f,\delta) = 0.$$

ii) The boundedness of the family of functions that also appears in the classical Arzelà-Ascoli allows us to define the nth entropy number of that set.

From remark 1.14, we may now restate the classical Arzelà-Ascoli as an approximation theorem:

Theorem 1.16 (refined Arzelà-Ascoli). *Let K be a compact metric space and let $\mathcal{F}$ be a bounded subset of $C(K)$. Then*

$$\lim_{\delta\to 0^+} \omega(f,\delta) = 0 \ \ \textit{for all } f \in \mathcal{F} \implies \lim_{n\to\infty} \epsilon_n(\mathcal{F}) = 0.$$

As a corollary, we can characterize the compactness of the operator $T : E \to C(X)$ in terms of its modulus of continuity defined as $\omega(T,\delta) = \sup_{||x||<1} \omega(Tx,\delta)$ and taking $\mathcal{F} = T(B_E)$ in the theorem.

Corollary 1.17 ([12], Prop. 5.5.1). *An operator $T : E \to C(X)$ mapping an arbitrary Banach space E into the space $C(X)$ of continuous functions on a compact metric space (X, d) is compact if and only if*

$$\lim_{\delta\to 0^+} \omega(T,\delta) = 0.$$

Example 1.18. Consider the homogenous linear ordinary differential equation:

$$f' = f, \text{ where } f \in C^1([0,1]).$$

Define $T : C([0,1]) \to C([0,1])$ by $Tf(x) = \int_0^x f(t)dt$. Then u is a solution of the differential equation if and only if u is a fixed point of T. It turns out that T is compact. We will first show that T is bounded. Set $k(x,t) = \chi_{[0,x]}(t)$. We have:

$$||Tf||^2_{L^2} = \langle Tf, Tf\rangle = \int_0^1 \left[\int_0^1 k(x,t)f(t)dt\right]^2 dx = \int_0^1 \left[\int_0^1 |k(x,t)||f(t)|dt\right]^2 dx$$

$\leqslant \int_0^1 \left[\left(\int_0^1 |k(x,t)|^2 dt\right)^{\frac{1}{2}} \cdot \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}}\right]^2 dx = \int_0^1 \int_0^1 |k(x,t)|^2 dt dx \cdot ||f||_{L^2}^2 = \frac{1}{2}||f||_{L^2}^2$.

We get $||Tf||_{L^2} \leqslant \sqrt{\frac{1}{2}}||f||_{L^2}$. Since T is linear, our calculation shows that T is Lipschitz with Lipschitz constant $M = \sqrt{\frac{1}{2}}$. It follows that $\omega(T,\delta) \leqslant \sqrt{\frac{1}{2}}\delta$ so that $\lim_{\delta\to 0^+} \omega(T,\delta) = 0$. By Corollary 1.17, T must be compact.

1.3 Relatively p-compact sets

In 1955, Grothendieck [28] characterized the compact subsets of a Banach space as those sets sitting inside the closed convex hull of a norm null sequence: A subset K of a Banach space X is relatively compact if and only if there exists a sequence $\{x_n\}$ with

$$\lim_{n\to\infty} ||x_n|| \to 0 : K \subset \left\{\sum_n \alpha_n x_n : \sum_n |\alpha_n| \leqslant 1\right\}$$

It turned out that Grothendieck's characterization holds for more general sequences in ℓ_p for $1 \leqslant p \leqslant \infty$ as the following definition of relatively p-compact sets shows.

Definition 1.19. (1) A subset K of a Banach space X is called relatively p-compact if there exists a sequence (x_n) in X such that

$$K \subset \left\{\sum_{n=1}^{\infty} \alpha_n x_n : \sum_{n=1}^{\infty} |\alpha_n|^q \leqslant 1\right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\sum_{n=1}^{\infty} ||x_n||^p < \infty,$$

which makes sense for $1 < p < \infty$.

(2) In the case where $p = 1$, we replace the characterization with

$$K \subset \Big\{ \sum_{n=1}^{\infty} \alpha_n x_n : |\alpha_n| \leqslant 1 \Big\} \text{ and } \sum_{n=1}^{\infty} ||x_n|| < \infty$$

(3) In the case where $p = \infty$, we replace the characterization with

$$K \subset \Big\{ \sum_{n=1}^{\infty} \alpha_n x_n : \sum_{n=1}^{\infty} |\alpha_n| \leqslant 1 \Big\}$$

and

$$\lim_{n\to\infty} ||x_n|| = 0$$

Propositions 1.20 and 1.24 have known proofs, but we give new proofs using the entropy numbers.

Proposition 1.20. *[22], Ch.3, p.77*

Suppose $\{T_N\}$ is a sequence of relatively compact operators in $\mathcal{L}(X, Y)$. If $T_N \to T \in \mathcal{L}(X, Y)$, then T is relatively compact.

Here, we give a different proof from the usual $\frac{\epsilon}{3}$ argument:

Proof. It suffices to show that

$$\lim_{n\to\infty} \epsilon_n(T) = 0.$$

Let $\delta > 0$. Then by the hypothesis, there exists M such that $||T - T_M|| < \frac{\delta}{2}$. We also know that $\epsilon_n(T_M) \to 0$ as $n \to \infty$ implies that there exists K such that $\epsilon_n(T_M) < \frac{\delta}{2}$ for all $n > K$. Take $k = 1$ in this inequality: $\epsilon_{kn}(T + S) \leqslant \epsilon_k(T) + \epsilon_n(S)$ for $T, S \in \mathcal{L}(X, Y)$, we get

$$\epsilon_n(T) = \epsilon_n(T - T_M + T_M) \leqslant \epsilon_1(T - T_M) + \epsilon_n(T_M) = ||T - T_M|| + \epsilon_n(T_M) < \frac{\delta}{2} + \epsilon_n(T_M)$$

It follows that $\epsilon_n(T) < \delta$ for all $n > K$. Hence,

$$\lim_{n\to\infty} \epsilon_n(T) = 0, \text{ as it was promised .}$$

∎

The preceding proposition does not hold if we replace the notion of relatively compact with compact unless we also demand that Y be complete. We give an application of the proposition.

Example 1.21. Let $I = [0,1]$ and let $1 < p, q < \infty$, with p' and q' conjugate to p and q, respectively. Suppose $k(s,t)$ is in $L_r(I \times I)$, where r is the larger of p' and q. Then the linear operator K defined by $(Kx)(t) = \int_0^1 k(s,t)x(s)ds$ is compact as a map from $L_p(I) \to L_q(I)$.

By the density of continuous functions with compact support in L_r, one extracts a sequence of continuous functions $\{k_n(s,t)\}$ on $I \times I$ that converges to $k(s,t)$ and then the sequence $(K_n x)(t) = \int_0^1 k_n(s,t)x(s)ds$ is a sequence of compact (relatively compact) operators that converges to K and by the preceding proposition, K must be relatively compact (hence compact since $L_q(I)$ is complete) (cf. [22], p. 79).

Proposition 1.22. *[[22], Ch.3, p. 82]*

A bounded linear operator is relatively compact if its conjugate is compact.

Remark 1.23. If the conjugate of a bounded linear operator is compact, it does not necessarily follow that the operator is compact (cf. [22], example III.1.7)

However, for Hilbert spaces, we provide new proof for the following result.

Proposition 1.24. *Suppose X and Y are arbitrary Hilbert spaces. Then $T \in \mathcal{L}(X,Y)$ is relatively compact if and only if its conjugate is compact.*

Proof. First observe that if H is a Hilbert space, we consider the isometry $E_H \in \mathcal{L}(X, X')$ defined by $E_H(x) = x'$, where $x'z = \langle z, x\rangle$, $z \in H$.

With this notation, the adjoint T^* of T may be defined as $T^* = E_X^{-1} T' E_Y$.

Now we have

$\epsilon_n(T^*) \leqslant \epsilon_1(E_X^{-1})\epsilon_n(T')\epsilon_1(E_Y) = ||E_X^{-1}||\epsilon_n(T')||E_Y|| = \epsilon_n(T')$.

Thus, if the conjugate T' is compact, then T^* is relatively compact and hence also compact since the underlying spaces are complete, and by Schauder's Theorem, it follows that T is compact.

Conversely, $T' = E_X T^* E_Y^{-1}$. Now, if T is relatively compact and hence compact as the underlying spaces are complete, then by Schauder's Theorem T^* is compact, and by applying the multiplicativity of the entropy numbers, it follows that T' is relatively compact and hence compact.

∎

1.4 Factorization of compact operators

The range of an operator $T \in \mathcal{L}(X, Y)$ denoted $R(T)$ is a linear subspace of Y which is not necessarily closed. Let $Y_0 = \overline{R(T)}$ and consider the operator induced by T, $T_0 \in \mathcal{L}(X, Y_0)$, which is given by $T_0 x = Tx$ for $x \in X$ and the natural embedding $I_{Y_0} \in \mathcal{L}(Y_0, Y)$. Then we get a canonical factorization $T = I_{Y_0} T_0$

$$\begin{tikzcd} X \arrow[r, "T_0"] \arrow[dr, "T"'] & Y_0 \arrow[d, "I_{Y_0}"] \\ & Y \end{tikzcd}$$

Remark 1.25.

i Question: Under what conditions on X, Y, T can we have $\overline{R(T)} \in \{c_0, \ell_p\}$?

ii If I_{Y_0} or T_0 is compact, then T would be compact.

iii A Banach space X is compactly embedded in Y if $X \subset Y$ and the inclusion $i : X \to Y$ is compact. So in particular, since $Y_0 = \overline{R(T)} \subset Y$. If I_{Y_0} is compact, then we have compact embedding.

Analogously, the $\ker T$ of an operator $T \in \mathcal{L}(X, Y)$ is a closed linear subspace of X. We can factor T through the quotient space $X/\ker T$. Consider the operator $Q_{\ker T} \in \mathcal{L}(X, X/\ker T)$ defined as $Q_{\ker T}(x) = x + \ker T$ and the $T_0 \in \mathcal{L}(X/\ker T, Y)$ defined by $T_0(x + \ker T) = Tx$. Then we get a canonical factorization $T = T_0 Q_{\ker T}$

$$\begin{tikzcd} X \arrow[r, "Q_{\ker T}"] \arrow[dr, "T"'] & X/\ker T \arrow[d, "T_0"] \\ & Y \end{tikzcd}$$

In the 1970s, Terzioğlu [58] gave a factorization of compact maps through a closed subspace of c_0.

Theorem 1.26 ([58])**.** *Let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator between Banach spaces X and Y. Then the following are equivalent:*

1. *$T \in \mathcal{K}(X, Y)$,*

2. *There exists a norm-null sequence (x_n^*) in X^* such that*

$$||Tx|| \leqslant \sup_n |\langle x, x_n^* \rangle| \quad \forall x \in X,$$

3. *For some closed subspace Z of c_0 there are compact operators $P \in \mathcal{K}(X, Z)$ and $Q \in \mathcal{K}(Z, Y)$ such that $T = Q \circ P$.*

The following characterization of compact operators due to Pietsch, which appeared in 1965, is closely connected with the definition of the so-called Gelfand numbers.

Theorem 1.27 ([12], Prop. 2.3.1)**.** *An operator $T \in \mathcal{L}(X,Y)$ between arbitrary Banach spaces X and Y is compact if and only if for $\epsilon > 0$ there are finitely many functionals $a_i \in X^*, 1 \leqslant i \leqslant n_\epsilon$, such that*

$$||Tx|| \leqslant \sup_{1 \leqslant i \leqslant n_\epsilon} |\langle x, a_i \rangle| + \epsilon ||x|| \textit{ for all } x \in X.$$

Definition 1.28. The nth Gelfand number $c_n(T)$ is defined as:

$$c_n(T) = \inf\{\epsilon > 0 : ||Tx|| \leqslant \sup_{1 \leqslant i \leqslant k} |\langle x, a_i \rangle| + \epsilon ||x||, \text{where } a_i \in X^{'}, 1 \leqslant i \leqslant k \text{ with } k < n\}$$

By Theorem 1.27 an operator T is compact if and only if $c_n(T) \to 0$ as $n \to \infty$.

Analogous results to Terzioğlu's theorem have been obtained, which factor a compact operator through ℓ_p spaces, $(1 \leqslant p < \infty)$ using p-compact operators. These operators and their properties as well as injective hulls of p-compact operators are also studied in [19] and [20], [1] studied Q-compact operators which are a generalization of compact operators, in particular, they are compact with respect to a given approximation scheme Q on Y.

Chapter 2

Schauder's theorem and s-numbers

This chapter investigates an extension of Schauder's theorem by studying the relationship between various s-numbers of an operator T and its adjoint T^*. There are two main results. First, we present new proof that the approximation number of T and T^* are equal for compact operators. Second, for non-compact, bounded linear operators from X to Y, we obtain a relationship between certain s-numbers of T and T^* under natural conditions on X and Y.

Recall that $\mathcal{L}(X,Y)$ denotes the normed vector space of all continuous operators from X to Y, X^* is the dual space of X, and $\mathcal{K}(X,Y)$ denotes the collection of all compact operators from X to Y. Denote by $T^* \in \mathcal{L}(Y^*, X^*)$ the adjoint operator of $T \in \mathcal{L}(X,Y)$. The well known theorem of Schauder states that $T \in \mathcal{K}(X,Y) \iff T^* \in \mathcal{K}(Y^*, X^*)$. The proof of Schauder's theorem that uses Arzelà-Ascoli Theorem is presented in most textbooks on functional analysis (see, e.g. [54]). A new and simple proof, which does not depend on Arzelà-Ascoli can be found in [53].

Define rank-1 operator $a \otimes y \in \mathcal{L}(X,Y)$ as $(a \otimes y)(x) := a(x)y$ where $a \in X^*,\ y \in Y$.

An operator $T \in \mathcal{L}(X,Y)$ has finite rank if $rank(T) := \dim\{Tx : \ x \in X\}$ is finite.

Such an operator can be represented by

$$T = \sum_{k=1}^{n} a_k \otimes y_k \quad \text{with } a_1, \ldots, a_n \in X^* \text{ and } y_1, \ldots, y_n \in Y.$$

For two arbitrary normed spaces X and Y, let

$$\mathcal{F}_n(X, Y) := \{A \in \mathcal{L}(X, Y) : \operatorname{rank}(A) \leqslant n - 1\},$$

and define the collection of the finite-rank operators as follows:

$$\mathcal{F} := \mathcal{F}(X, Y) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X, Y),$$

which forms the smallest operator ideal.

The concept of *s-numbers* $s_n(T)$ is introduced axiomatically in [47], and there are several possibilities of assigning to every operator $T : X \to Y$ a certain sequence of numbers $\{s_n(T)\}$ such that

$$s_1(T) \geqslant s_2(T) \geqslant \cdots \geqslant 0$$

which characterizes its degree of approximability or compactness of T. The main examples of s-numbers are approximation numbers, Gelfand numbers, and Kolmogorov numbers, which we define below.

Definition 2.1. The *nth approximation number*

$$a_n(T) = \inf\{||T - A|| : A \in \mathcal{F}_n(X, Y)\}, \quad n = 0, 1, \ldots$$

$\alpha_n(T)$ provides a measure of how well T can be approximated by finite mappings whose range is at most n-dimensional. Approximation numbers of an operator have the following properties [47]:

1. $a_0(T) = ||T||$

2. $a_n(T) \geqslant a_{n+1}(T)$ for all n

3. $a_n(S+T) \leqslant a_k(S) + a_j(T)$ where $k+j=n$

4. $a_n(\lambda T) = |\lambda| a_n(T)$ for all n and scalars λ

5. $|a_n(S) - a_n(T)| \leqslant ||S-T||$ for all n

Definition 2.2. We say that $T \in \mathcal{L}(X,Y)$ is of *type* l^p where $0 < p < \infty$ if $(a_n(T)) \in \ell^p$ or $(a_n(T)) \in c_0$ in case $p = \infty$. For $0 < p < \infty$ in case $\sum_n^{\infty}(a_n(T))^p < \infty$ and we denote such collection by $\ell^p(X,Y)$, which is again a linear subspace of $\mathcal{L}(X,Y)$ and it is the space of all linear operators of type ℓ^p.

s-numbers are used to define quasi-Banach operator ideals

$$\mathcal{L}_w^{(\rho)} := \{T \in \mathcal{L}(X,Y) : \quad (n^{\rho - 1/w} s_n(T)) \in \ell_w\}$$

and their properties and the relationship between s-numbers and the eigenvalue distribution are studied by many. See for example [47] and [45] and the references therein.

2.1 Symmetric operator ideals

Definition 2.3. A class of operators $\mathcal{A}(X,Y) \subset \mathcal{L}(X,Y)$ is called *symmetric* if $T \in \mathcal{A}(X,Y) \implies T^* \in \mathcal{A}(Y^*,X^*)$. To be able to compare the degree of non-compactness of $T \in \mathcal{A}(X,Y)$ with that of $T^* \in \mathcal{A}(Y^*,X^*)$ requires $\mathcal{A}$ to be a *symmetric operator ideal.*

The class $\mathcal{K}(X, Y)$ of compact operators between arbitrary Banach spaces X and Y is an example of a symmetric ideal of operators in $\mathcal{L}(X, Y)$.

Using the Principle of Local Reflexivity, Hutton ([31] , Theorem 2.1) proved that $T \in \mathcal{K}(X, Y)$ implies that $a_n(T) = a_n(T^*)$ for all n. However for non-compact operators $a_n(T) \neq a_n(T^*)$ as shown in the following example:

Example 2.4. [31] Consider $T = I : \ell_1 \to c_0$ canonical injection and $T^* : \ell_1 \to \ell_\infty$ natural injection. Then, one has $a_n(T) = 1$ for each n and $a_n(T^*) = \frac{1}{2}$.

For non-compact operator $T \in \mathcal{L}(X, Y)$, we do not have a lot of information about the relationship between $s_n(T)$ with $s_n(T^*)$, however by imposing certain natural conditions on X and Y we were able to obtain a relationship between $s_n(T)$ with $s_n(T^*)$ for certain s-numbers.

The following theorem follows as a consequence of Hutton's above-mentioned theorem.

Theorem 2.5. *The operator ideal $\overline{\mathcal{F}}$ is symmetric.*

Pietsch has shown that the space of finite-rank linear operators is a dense subset of the space of all linear operators of type ℓ^p between Banach spaces (see, [49], Prop. 8.2.5). This can be used to prove that every operator of type ℓ^p is relatively compact (see, [49], Prop. 8.2.6) and hence compact since the notions coincide for Banach spaces.

As a corollary, we have:

Corollary 2.6. *If $T \in \overline{\mathcal{F}(X, Y)} = \ell^p(X, Y)$, then $a_n(T) = a_n(T^*)$ for all n.*

The corollary implies that for $0 < p \leqslant \infty$, $T \in \ell^p(X, Y) \iff T^* \in \ell^p(Y^*, X^*)$, which shows that $\ell^p(X, Y)$ is an example of a symmetric ideal of operators in $\mathcal{L}(X, Y)$.

2.2 Hutton's Theorem Revisited

In this section, we re-state a version of Hutton's theorem and give a different proof that uses the fundamental theorems of functional analysis and the Principle of Local Reflexivity. Lindenstrass and Rosenthal [37] discovered a principle that shows that all Banach spaces are "locally reflexive" or, said in another way every bidual X^{**} is finitely representable in the original space X. The following is a stronger version of this property called *Principle of Local Reflexivity* (PLR) due to Johson, Rosental, and Zippin [32]:

Definition 2.7. Let X be a Banach space regarded as a subspace of X^{**}, let E and F be finite dimensional subspaces of X^{**} and X^* respectively and let $\epsilon > 0$. Then there exist a one-to-one operator $T : E \to X$ such that

1. $T(x) = x$ for all $x \in X \cap E$
2. $f(Te) = e(f)$ for all $e \in E$ and $f \in F$
3. $||T||||T^{-1}|| < 1 + \epsilon$.

PLR is an effective tool in Banach space theory. More recently, Oja and Silja in [42] investigated versions of the principle of local reflexivity for nets of subspaces of a Banach space and gave some applications to some duality and lifting theorems. Next, we define Kolmogorov diameter of $T \in \mathcal{L}(X)$ and observe an alternate way of characterizing compact operators using the Kolmogorov diameter of T.

Definition 2.8 ([12], Prop. 2.2.2)**.** The *nth -Kolmogorov diameter* of $T \in \mathcal{L}(X)$ is defined by

$$\delta_n(T) = \inf\{||Q_G T|| : \dim G \leqslant n\}$$

where the infimum is over all subspaces $G \subset X$ and Q_G denotes the canonical quotient map $Q_G : X \to X/G$.

Lemma 2.9 (Lemma 1 in [53])**.** *Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then $T \in \mathcal{K}(X)$ if and only if, for each $\epsilon > 0$, there is a finite-dimensional subspace F_ϵ of X such that $||Q_{F_\epsilon} T|| < \epsilon$, where $Q_{F_\epsilon} : X \to X/F_\epsilon$.*

In the following, we restate Hutton's theorem and give a different proof that uses the basic theorems of functional analysis, together with PLR.

Theorem 2.10. *Let $T \in \mathcal{K}(X)$. Then $a_n(T) = a_n(T^*)$.*

Proof. Since one always has $a_n(T^*) \leqslant a_n(T)$, if we have $a_n(T) \leqslant a_n(T^{**})$, then $a_n(T^{**}) \leqslant a_n(T^*)$ would imply $a_n(T) \leqslant a_n(T^*)$. Thus we must verify $a_n(T) \leqslant a_n(T^{**})$. To this end, suppose $T \in \mathcal{K}(X)$, by Schauder's theorem, T^* and T^{**} are compact. Let $\epsilon > 0$, then by definition, there exists $A \in \mathcal{F}_n(X^{**})$ such that $||T^{**} - A|| < a_n(T^{**}) + \epsilon$.

By Lemma 3.3, there are finite-dimensional subspaces E_ϵ of X^{**} and F_ϵ of X^* such that $||Q_{E_\epsilon} T^{**}|| < \epsilon$, where $Q_{E_\epsilon} : X^{**} \to X^{**}/E_\epsilon$ and $||Q_{F_\epsilon} T^*|| < \epsilon$, where $Q_{F_\epsilon} : X^* \to X^*/F_\epsilon$.

By the Principle of Local Reflexivity (PLR), there exists a one-to-one linear operator $S : E_\epsilon \to X$ such that $||S||||S^{-1}|| < 1 + \epsilon$, $y^*(Sx^{**}) = x^{**}(y^*)$ for all $x^{**} \in E_\epsilon$ and all $y^* \in F_\epsilon$, and $S_{|E_\epsilon \cap X} = I$.

Let $J : X \to X^{**}$ be the canonical map. By the Hahn-Banach theorem, since E_ϵ is a subspace of X^{**}, $S : E_\epsilon \to X$ can be extended to a linear operator $\overline{S} : X^{**} \to X$.

We now have $T \in \mathcal{L}(X)$ and $\overline{S}AJ \in \mathcal{L}(X)$ and rank $(\overline{S}AJ) = \text{rank}(A) < n$, and therefore

$$a_n(T) \leqslant ||T - \overline{S}AJ||.$$

To get an upper bound for $||T - \overline{S}AJ||$ we estimate $||Tx - \overline{S}AJ(x)||$ for $x \in B_X$ using an appropriate element z_j of the covering of the set $T(B_X)$.

Indeed, the compactness of T implies that $T(B_X)$ is relatively compact so that one can extract a finite-dimensional subset $Y_\epsilon \subset T(B_X) \subset X$ and let $z_j = Tx_j$ be the n elements forming a basis.

Let $x \in B_X$. Then we have

$$a_n(T) \leqslant |Tx - \overline{S}AJ(x)|| \leqslant ||Tx - z_j|| + ||z_j - \overline{S}AJ(x)||$$

$$\leqslant \epsilon + ||z_j - \overline{S}AJ(x)|| = \epsilon + ||\overline{S}z_j - \overline{S}AJ(x)|| \leqslant \epsilon + (1+\epsilon)||z_j - AJ(x)|| < \epsilon + (1+\epsilon)(a_n(T^*) + \epsilon),$$

since

$$||z_j - AJ(x)|| = ||Jz_j - AJ(x)|| \leqslant ||Jz_j - JTx|| + ||JTx - AJ(x)|| \leqslant \epsilon + ||JTx - AJx||$$

$$= \epsilon + ||T^{**}Jx - AJx|| \leqslant ||T^{**} - A|| < \alpha_n(T^*) + \epsilon.$$

It follows that $a_n(T) \leqslant a_n(T^{**})$, as promised.

∎

Remark 2.11. Since a nuclear operator is compact for which a trace may be defined (nuclear operators on Hilbert spaces are called trace-class operators), it is natural to ask how nuclearity of T and T^* are related. Recall that if $T \in \mathcal{L}(X, Y)$ is a nuclear operator with the nuclear representation of $T = \sum_{n=1}^{\infty} \phi_n \otimes y_n$ then its adjoint defined as $T^*(\psi) = \sum_{n=1}^{\infty} \psi(y_n)\phi_n$ and its nuclear norm defined as :

$$||T||_{\mathcal{N}} = \inf \left\{ \sum_{n=1}^{\infty} ||\phi_n|| ||y_n|| : \quad T(x) = \sum_{n=1}^{\infty} \phi_n(x) y_n \right\}$$

where the infimum is taken over all representations of T of the form $T(x) = \sum_{n=1}^{\infty} \phi_n(x) y_n$ and (ϕ_n) and (y_n) are bounded sequences in X^* and Y respectively satisfying $\sum_{n=1}^{\infty} ||\phi_n|| ||y_n|| < \infty$. It is known that in case X^* has the approximation property and if the operator $T \in \mathcal{L}(X, Y)$ has a nuclear adjoint, then T is nuclear as well and $||T||_{\mathcal{N}} = ||T^*||_{\mathcal{N}}$ (see Proposition 4.10 in [55]).

2.3 Comparing various approximation quantities

Recall the definition of the nth entropy number for $T \in \mathcal{L}(X, Y)$:

$$e_n(T) := \inf\{\epsilon > 0 : T(B_X) \subset \bigcup_{j=1}^{n} B(y_j, \epsilon)\}$$

for all $\epsilon > 0$ and for some $y_j \in Y$. A variant of the nth entropy number is defined as follows (known as the Kuratowski measure of non-compactness):

$$\gamma(T) := \inf\{\epsilon > 0 : T(B_X) \subset \bigcup_{k=1}^{n} A_k, \ \text{diam } (A_k) < \epsilon\}.$$

Remark 2.12. a) Since the diameter of $B(y_j, \epsilon)$ is at most 2ϵ, it follows that $\gamma(T) \leqslant e_n(T) \leqslant 2\gamma(T)$.

b) While γ is invariant under isometry, e_n is not.

It is possible to compare various s-numbers such as $a_n(T), \delta_n(T), c_n(T)$ if one imposes some mild restrictions on X and Y.

Definition 2.13. We say that a Banach space X has *the lifting property* if for every $T \in \mathcal{L}(X, Y/F)$ and every $\epsilon > 0$ there exists an operator $S \in \mathcal{L}(X, Y)$ such that $||S|| \leqslant (1+\epsilon)||T||$ and $T = Q_F S$, where F is a closed subspace of the Banach space Y and $Q_F : Y \to Y/F$ denotes the canonical projection.

Definition 2.14. A Banach space Y is said to have *the extension property* if for each $T \in \mathcal{L}(M, Y)$ there exists an operator $S \in \mathcal{L}(X, Y)$ such that $T = SJ_M$ and $||T|| = ||S||$, where M is a closed subspace of an arbitrary Banach space X and $J_M : M \to Y$ the canonical injection.

Next, we need to consider two universally important Banach spaces.

The Banach space $\ell_1(\Gamma)$ of *summable number families* $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ over an arbitrary index set Γ, whose elements $\{\lambda_{\gamma\in\Gamma}\}$ are characterized by $\sum_{\gamma\in\Gamma} |\lambda_\gamma| < \infty$, has the metric lifting property.

If T is any map from a Banach space with metric lifting property to an arbitrary Banach space, then $a_n(T) = \delta_n(T)$ (cf. [12], Prop. 2.2.3). It is known that every Banach space X appears as a quotient space of an appropriate space $\ell_1(\Gamma)$ (for proof of this, see [12], p. 52).

The Banach space $\ell_\infty(\Gamma)$ of *bounded number families* $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ over an arbitrary index set Γ has the metric extension property.

If T is any map from an arbitrary Banach space into a Banach space with metric extension property, then $a_n(T) = c_n(T)$ (cf. [12], Prop. 2.3.3). It is known that every Banach space Y can be regarded as a subspace of an appropriate space $\ell_\infty(\Gamma)$ (for proof of this, see [12], p. 60).

Remark 2.15. If $T \in \mathcal{L}(X, Y)$, where X and Y are arbitrary Banach spaces with metric lifting and extension property, respectively, then $\delta_n(T) = a_n(T) = c_n(T)$. It is also known that if $T \in \mathcal{L}(X, Y)$, where X and Y are arbitrary Banach spaces, then $\delta_n(T^*) = c_n(T)$ (cf. [12], Prop. 2.5.5). Hence, $\delta_n(T^*) = c_n(T) = a_n(T) = \delta_n(T)$, which we summarize as a theorem below.

Theorem 2.16. *If $T \in \mathcal{L}(X, Y)$, where X and Y are arbitrary Banach spaces with metric lifting and extension property, respectively, then $\delta_n(T^*) = \delta_n(T)$ for all n.*

Definition 2.17. Let $\mathcal{A}$ be an operator ideal, X a Banach space, and $D \subset X$ bounded. Then $\gamma_{\mathcal{A}}$, the $\mathcal{A}$ - variation, is defined by

$$\gamma_{\mathcal{A}}(D) := \inf\{\epsilon > 0 : D \subset T(B_Y) + \epsilon B_X, T \in \mathcal{A}(X, Y)\}.$$

As usual, $\gamma_{\mathcal{A}}(T) = \gamma_{\mathcal{A}}(T(B_X))$ and set $||T||_{\mathcal{A}} = \inf\{||T - S|| : S \in \mathcal{A}(X, Y)\}$.

Theorem 2.18 ([5], Theorem 5.3)**.** *Suppose $\mathcal{A}$ is a symmetric operator ideal and $T \in \mathcal{L}(X, Y)$. If Y has the extension property and X has the lifting property, then*

$$\gamma_{\mathcal{A}}(T^*) = ||T^*||_{\mathcal{A}} = ||T||_{\mathcal{A}} = \gamma_{\mathcal{A}}(T).$$

This theorem holds true for the ideals of ℓ^p type, compact, and nuclear operators since they are all symmetric ideals of operators.

Remark 2.19. In a particular case where $\mathcal{A} = \mathcal{K}$, Astala in [5] proved that if $T \in \mathcal{L}(X, Y)$, where X and Y are arbitrary Banach spaces with metric lifting and extension property, respectively, then $\gamma(T) = \gamma(T^*)$, where $\gamma(T)$ denotes the measure of non-compactness of T. In [1], it is shown that $\lim_{n\to\infty} \delta_n(T) = \gamma(T)$. This relationship between Kolmogorov diameters and the measure of non-compactness together with theorem 2.16 provide an alternative proof for theorem 2.18

If $T \in \mathcal{K}(X, Y)$, then it is known that $\delta_n(T) = c_n(T^*)$ (cf. [12], Prop. 2.5.6). If X and Y are Banach spaces with metric lifting and extension property, respectively, then we have $\delta_n(T) = a_n(T) = c_n(T)$. Thus, we have the following theorem.

Theorem 2.20. *If $T \in \mathcal{K}(X, Y)$, where X and Y are arbitrary Banach spaces with metric lifting and extension property, respectively, then $c_n(T^*) = c_n(T)$ for all n.*

Remark 2.21 ([30])**.** It is known that if X has the lifting property, then X^* has the extension property. However, if Y has the extension property, then Y^* has the lifting property if and only if Y is finite-dimensional.

Remark 2.22. If X has the lifting property and Y is finite-dimensional with the extension property, then by remark 2.20, Y^* has the lifting property and X^* has the extension property, so that we have $\delta_n(T^*) = a_n(T^*) = c_n(T^*)$.

For our needs in the next chapter, we choose the closed unit ball B_Z of the Banach space Z as an index set Γ. Our proof of Schauder's theorem for Q-compact operators in the next section will depend on the fact that $\ell_1(B_Z)$ has the lifting property and $\ell_\infty(B_Z)$ has the extension property.

Chapter 3

Approximation schemes and Q-compact operators

In this chapter, we consider operators which are not compact but compact with respect to certain approximation schemes, we call such operators Q-compact and prove a version of Schauder's theorem for Q-compact operators. In the case of non-compact operators, this answers the question of comparing the degree of compactness for T and its adjoint T^*.

Approximation schemes were introduced in Banach space theory by Butzer and Scherer in 1968 [11] and independently by Y. Brudnyi and N. Kruglyak under the name of "approximation families" in 1978 [9]. They were popularized by Pietsch in his 1981 paper [48], for later developments we refer the reader to [1, 2, 4].

The following theorem may be viewed as a motivation for the definition of the Kolmogorov diameters.

Definition 3.1 (Generalized Approximation Scheme)**.** Let X be a Banach space. For each $n \in \mathbb{N}$, let $Q_n = Q_n(X)$ be a family of subsets of X satisfying the following conditions:

$(GA1)$ $\{0\} = Q_0 \subset Q_1 \subset \cdots \subset Q_n \subset \ldots.$

$(GA2)$ $\lambda Q_n \subset Q_n$ for all $n \in N$ and all scalars λ.

$(GA3)$ $Q_n + Q_m \subseteq Q_{n+m}$ for every $n, m \in N$.

Then $Q(X) = (Q_n(X))_{n\in N}$ is called a *generalized approximation scheme* on X. We shall simply use Q_n to denote $Q_n(X)$ if the context is clear.

We use here the term "generalized" because the elements of Q_n may be subsets of X. Let us now give a few important examples of generalized approximation schemes.

Example 3.2.

1. $Q_n =$ the set of all at-most-n-dimensional subspaces of any given Banach space X.

2. Let E be a Banach space and $X = L(E)$; let $Q_n = N_n(E)$, where $N_n(E) =$ the set of all n-nuclear maps on E. [47]

3. Let $a^k = (a_n)^{1+\frac{1}{k}}$, where (a_n) is a nuclear exponent sequence. Then Q_n on $X = L(E)$ can be defined as the set of all $\Lambda_\infty(a^k)$-nuclear maps on E.[16]

Definition 3.3 (Generalized Kolmogorov Number)**.** Let B_X be the closed unit ball of X, $Q(X) = (Q_n(X))_{n\in N}$ be a *generalized approximation scheme* on X, and D be a bounded subset of X. Then the n^{th} *generalized Kolmogorov number* $\delta_n(D;Q)$ of D with respect to B_X is defined by

$$\delta_n(D;Q) = \inf\{r > 0 : D \subset rB_X + A \text{ for some } A \in Q_n(X)\}. \tag{3.1}$$

Assume that Y is a Banach space and $T \in \mathcal{L}(Y,X)$. The n^{th} Kolmogorov number $\delta_n(T;Q)$ of T is defined as $\delta_n(T(B_Y);Q)$.

It follows that $\delta_n(T;Q)$ forms a non-increasing sequence of non-negative numbers:

$$\|T\| = \delta_0(T;Q) \geqslant \delta_1(T;Q) \geqslant \cdots \geqslant \delta_n(T;Q) \geqslant 0. \tag{3.2}$$

We are now able to introduce Q-compact sets and operators:

3.1 Q-compact sets and maps

Definition 3.4 (Q-compact set)**.** Let D be a bounded subset of X. We say that D is *Q-compact* if $\lim_n \delta_n(D;Q) = 0$.

Definition 3.5 (Q-Compact map)**.** We say that $T \in L(Y,X)$ is a *Q-compact map* if $\lim_n \delta_n(T;Q) = 0$, i.e., $T(B_Y)$ is a Q-compact set.

There are examples of Q-compact maps that are not compact, first, such map involves projections $P : L_p[0,1] \to R_p$ where R_p denotes the closure of the span of the space of Rademacher functions (see [3] for details), another example is the weighted backward shift operator B_w on $c_0(\mathbb{N})$ with $w = \{w_n\}$ not converging to 0 is Q-compact but not compact.

Definition 3.6. The *nth symmetrized approximation number* $\tau_n(T)$ for operator T between arbitrary Banach spaces X and Y is defined as follows:

$$\tau_n(T) = \delta_n(J_Y T),$$

where $J_Y : Y \to \ell_\infty(B_{Y^*})$ is an embedding map

Remark 3.7. Definition 3.6 is equivalent to

$$\tau_n(T) = a_n(J_Y T Q_X)$$

as well as to

$$\tau_n(T) = c_n(TQ_X),$$

where $Q_X : \ell_1(B_X) \to X$.

Proposition 3.8 (Refined version of Schauder's theorem [12], p. 84)**.** *An operator T between arbitrary Banach spaces X and Y is compact if and only if*

$$\lim_{n\to\infty} \tau_n(T) = 0$$

and moreover,

$$\tau_n(T) = \tau_n(T^*).$$

Motivated by this, we define Q-compact using the symmetrized approximation numbers.

3.2 Schauder's type theorem for Q-compact maps

Definition 3.9. We say T is Q- symmetric compact if and only if

$$\lim_{n\to\infty} \tau_n(T, Q) = 0.$$

Remark 3.10 ([12], Prop. 2.5.4-6)**.**

a) From remark 3.7, we have $\tau_n(T, Q) = c_n(TQ_X, Q)$, where $Q_X : \ell_1(B_X) \to X$.

b) We will also abbreviate the canonical embedding $K_{\ell_1(B_{Y^*})} : \ell_1(B_{Y^*}) \to \ell_\infty(B_{Y^*})^*$ by K so that $Q_{Y^*} = J_Y^* K$.

c) Denote by $P_0 : \ell_\infty(B_{X^{**}}) \to \ell_\infty(B_X)$ the operator which restricts any bounded function on $B_{X^{**}}$ to the subset $K_X(B_X) \subset B_{X^{**}}$ so that $Q_X^* = P_0 J_{X^*}$.

d) The relations (b) and (c) are crucial facts for the estimates of $\delta_n(T^*, Q^*)$ and $c_n(T^*, Q^*)$. In particular, we have $c_n(T^*, Q^*) \leqslant \delta_n(T, Q)$.

We now state and prove (adopting similar proof due to Pietsch and reproduced in [12], Prop. 2.6) the following.

Theorem 3.11 (Schauder's theorem for Q-compact operators)**.** *An operator T between arbitrary Banach spaces X and Y is Q- symmetric compact if and only if*

$$\lim_{n\to\infty} \tau_n(T, Q) = 0$$

and moreover,

$$\tau_n(T^*, Q^*) = \tau_n(T, Q),$$

that is to say the degree of Q-compactness of T and T^* is the same in so far as it is measured by the symmetrized approximation numbers τ_n.

Proof. The first part is the definition. So it suffices to show $\tau_n(T^*, Q^*) = \tau_n(T, Q)$. By Remark 3.10 (a) and (b) we have the following estimates:

$$\tau_n(T^*, Q^*) = c_n(T^* Q_{Y^*}, Q^*) = c_n(T^* J_Y^* K, Q^*) \leqslant c_n((J_Y T)^*, Q^*) \leqslant \delta_n(J_Y T, Q) = \tau_n(T, Q)$$

Conversely, we have by using Remark 3.10 (c) and (d):

$$\tau_n(T, Q) = c_n(TQ_X, Q) = \delta_n(TQ_X)^*, Q^*) = \delta_n(Q_X^* T^*, Q^*)$$

$$= \delta_n(P_0 J_{X^*} T^*, Q^*) \leqslant \delta_n(J_{X^*} T^*, Q^*) = \tau_n(T^*, Q^*)$$

∎

Chapter 4

Applications of interpolation techniques to PDEs

This chapter defines the K-functional via rearrangement-invariant function spaces, studies its effect on interpolation spaces, applies interpolation theory to some linear and non-linear partial differential equations, and also gives some criteria for the boundedness of the norms of operators arising from PDEs in some concrete Banach spaces. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space: X denotes the underlying space, $\mathcal{A}$ the σ-algebra of measurable sets, and μ the measure. Denote by $\mathfrak{M}(X)$ the set of μ-measurable complex-valued functions on X. If $f \in \mathfrak{M}(\mathbb{R}^n)$ and $1 \leqslant p < \infty$, we define

$$L_p(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : ||f||_{L_p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f|^p d\mu\right)^{\frac{1}{p}} < \infty\}.$$

For $0 < p < 1$, $||f||_p$ does not define a norm because it does not satisfy the triangle inequality. We will restrict ourselves to the case $1 < p < \infty$ because the duality theory is easier when $1 < p < \infty$. For example, If $1 < p < \infty$, where q is the dual exponent of p: $\frac{1}{p} + \frac{1}{q}$, then $L_q = (L_p)^*$. Also, many problems in Fourier Analysis and partial differential equations concern the boundedness of operators on Lebesgue spaces L_p and we have many operators that are only bounded in intermediate spaces

L_p for $1 < p < \infty$ and not for $p = 1$ or $p = \infty$. For example, consider the Hardy operator $H : L_p \to L_p$ given by

$$Hf(x) = \frac{1}{x} \int_0^x f(s)ds,$$

where f is an integrable function with non-negative values. In [29] it is shown that $||Hf||_{L_p} \leqslant \frac{p}{p-1}||f||_{L_p}$ and that the constant is the best possible one. Therefore, $||H||_{L_p} = \frac{p}{p-1}$, which shows that H is unbounded on L_1.

Interpolation techniques are powerful tools in the study of the boundedness of operators. Before we define K-functional, we need the concept of re-arrangement-invariant function spaces.

4.1 Rearrangement-invariant function spaces

The sequence (b_n) is said to be a re-arrangement of (a_n) if there exists a permutation of $\mathbb{N}$ such that $b_n = a_{\sigma(n)}$ for all n.

For example,

$$(b_n) = 1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{8}, \frac{1}{5}, -\frac{1}{10}, -\frac{1}{12}, \cdots,$$

is a rearrangement of

$$a_n = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \frac{1}{7}, -\frac{1}{8}, \frac{1}{9}, -\frac{1}{10}, -\frac{1}{11}, -\frac{1}{12}, \frac{1}{13}, -\frac{1}{14}, \cdots,$$

where

$$\sigma(k) = \begin{cases} 4n \text{ for } k = 3n \\ 2(2n-1) \text{ for } k = 3n-1 \\ 2n-1 \text{ for } k = 3n-2 \end{cases}.$$

Note that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n + 1}{n} = \log(2)$$

It can be shown that

$$\sum_{n=1}^{\infty} b_n = \frac{1}{2}\log(2).$$

This scary situation does not happen for sequences with non-negative terms. In fact, suppose that $a_n \geqslant 0$ for all n. Then, if (b_n) is a rearrangement of (a_n), we have $\sum b_n = \sum a_n$.

In more general measure spaces, we say that non-negative functions f and g are rearrangements of one another if their *distribution functions* coincide. For each measurable function f, this notion enables the construction of a decreasing right-continuous function f^* on $(0, \infty)$ called the *decreasing rearrangement* of f, which is analogous to rearranging the terms of a non-negative sequence in decreasing order.

Definition 4.1. For each $f \in \mathfrak{M}(X)$, we define the *distribution function* $\mu_f : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\mu_f(t) = \mu\left(\{x \in X : |f(x)| > t\}\right).$$

Definition 4.2. For each f on X we define its *decreasing rearrangement* f^* by

$$f^*(s) = \inf\{t : \mu_f(t) \leqslant s\}, s \geqslant 0.$$

The function f^* is locally integrable if and only if $f \in L_1(X) + L_\infty(X)$. The action of the Hardy operator on f^* is usually denoted by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, t > 0.$$

Let (X, μ) be a measure space. $L_{p,q}(X)$ denotes the space of measurable functions f which satisfy

$$||f||_{L_{p,q}} = \left(\frac{q}{p}\int_{(0,\infty)} [t^{\frac{1}{p}} f^*(t)]^q\right)^{\frac{1}{q}} < \infty$$

when $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, and

$$||f||_{L_{p,\infty}} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty$$

when $1 \leqslant p \leqslant \infty$. When $p = q$,

$$||f||_{L_{p,p}} = ||f^*||_p = ||f||_p$$

and we recover L_p.

In general, however, $||.||_{L_{p,q}}$ is not a norm since the triangle inequality only holds when $1 \leqslant q \leqslant p < \infty$ or $p = q = \infty$. But when $1 < p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$, if we replace f^* in the definition of $||f||_{L_{p,q}}$ with f^{**}, then we get a quantity which is equivalent to $||f||_{L_{p,q}}$ and which defines a norm. The reason the triangle inequality does not fail when we use f^{**} is that f^{**} is sub-additive.

Example 4.3. Take $X = [0, 2\pi]$ and consider $f : X \to \mathbb{R}$ given by $f(x) = \sin x$.

Then the distribution of f is given by

$$\mu_f(t) = 2\pi - 4 \arcsin t, t \in [0, 1]$$

Also, the decreasing rearrangement of f is given by

$$f^*(t) = \cos \frac{t}{4}$$

We have

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds = \frac{\sin \frac{t}{4}}{4t}.$$

In measure theory, to prove a fact about a measurable function one tries it on much simpler functions and then invokes the standard limit theorems. Measurable functions can be approximated by simple functions. More precisely, for $1 \leqslant p < \infty$, the set of *simple functions* $f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$, where $\mu(E_j) < \infty$ for all j, is dense in L_p.

Example 4.4. In this example, we compute the distribution function and the decreasing rearrangement function of a nonnegative simple function f.

Let

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$$

be such that E_j are pairwise disjoint and all a_i distinct and such that $a_1 > a_2 > \cdots > a_n > 0$. Then for $a_1 \leqslant t$, we have $\mu_f(t) = 0$, but for $a_2 \leqslant t < a_1$ we have that $\mu_f(t) = \mu(E_1)$. Similarly we find for $a_3 \leqslant t < a_2$ that $\mu_f(t) = \mu(E_1) + \mu(E_2)$. Then we have

$$\mu_f(t) = \sum_{j=1}^{n} \left(\sum_{i=1}^{j} \mu(E_i) \right) \chi_{[a_{j+1}, a_j)}(t) = \sum_{j=1}^{n} \mu(E_j) \chi_{[0,a_j)}(t)$$

where $a_{n+1} = 0$.

Let $m_j = \sum_{i=1}^{j} \mu(E_i)$. By definition we find $f^*(t) = 0$ when $t \geqslant m_n$, $f^*(t) = a_n$ when $m_n \geqslant t > m_{n-1}$. We find

$$f^*(t) = \sum_{j=1}^{n} a_j \chi_{[m_{j-1}, m_j)}(t).$$

We have

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds = \frac{1}{t} \sum_{j=1}^{n} a_j \mu([0, t] \cap [m_{j-1}, m_j)).$$

We are now ready to define rearrangement-invariant spaces.

Definition 4.5. A Banach function space F of measurable functions on X which satisfies $F \subset L_1(X) + L_\infty(X)$ and contains characteristic functions of subsets of X of finite measure is a *rearrangement-invariant space* if $f \in F$ and there are $C, D \in (0, \infty)$ such that $C|f| \leqslant |g| \leqslant D|f|$ for $g \in \mathfrak{M}(X)$ implies that $g \in F$ and that $||g||_F = ||f||_F$.

Basic examples of rearrangement-invariant spaces are $L_1 \cap L_\infty$ and $L_1 + L_\infty$.

For $\theta > 0, 0 < p \leqslant \infty$, and $w(t)$, a non-negative monotone function on $\mathbb{R}_+$, here is a functional Φ which generates useful norms and quasi-norms:

$$||w||_{\theta,q} := \Phi_{\theta,q}(w) := \begin{cases} \left(\int_0^\infty \left[t^{-\theta}w(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, \ 0 < p < \infty \\ \text{ess } \sup_{0 \leqslant t < \infty} \left(t^{-\theta}w(t)\right), \ q = \infty \end{cases} \tag{4.1}$$

If we take $\theta = 1 - \frac{1}{q}$, $w(t) = \int_0^t f^*(s)ds = tf^{**}(t)$, then the norm (1.1) gives *Calderon's definition* of the spaces $L_{p,q}$, which are also known as the *Lorentz spaces*, and it becomes

$$||f||_{L_{p,q}} := \begin{cases} \left(\int_0^\infty \left[t^{\frac{1}{p}} f^{**}(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, 1 \leqslant q < \infty \\ \text{ess } \sup_{t>0} \left(t^{\frac{1}{p}} f^{**}(t)\right), q = \infty \end{cases} \tag{4.2}$$

The Lorentz spaces are examples of rearrangement-invariant spaces.

Now, we replace $w(t)$ with a function we will call the K-functional and replace $L_1(X)$ and $L_\infty(X)$ with general Banach spaces X_0 and X_1, respectively, so that they are continuously embedded into a topological vector space V so that $X_0 \cap X_1$ and $X_0 + X_1$ are defined. This motivates the following definition.

Definition 4.6. A pair $\overline{X} = (X_0, X_1)$ of Banach spaces is called a *Banach couple* if X_0 and X_1 are both continuously embedded in some Hausdorff topological vector space V.

4.2 Real interpolation spaces

Definition 4.7. An intermediate space between X_0 and X_1 is any normed space X such that $X_0 \cap X_1 \subset X \subset X_0 + X_1$ (with continuous embedding).

Example 4.8. If $0 < p < q < r \leqslant \infty$, then $X = L_q$ is an intermediate space between $X_0 = L_r$ and $X_1 = L_p$, that is, $L_r \cap L_p \subset L_q \subset L_r + L_p$ and $||f||_q \leqslant ||f||_p^{\lambda}||f||_r^{1-\lambda}$, where $\lambda \in (0,1)$ is defined by $\lambda = \frac{q^{-1}-r^{-1}}{p^{-1}-r^{-1}}$.

Proof. Suppose $f \in L_q$ and let $E = \{x : |f(x)| > 1\}$, $g = f\chi_E$, and $h = f\chi_{E^c}$. Then $|g|^p = |f|^p\chi_E \leqslant |f|^q\chi_E$, so $g \in L_p$, and $|h|^r = |f|^r\chi_{E^c} \leqslant |f|^q\chi_{E^c}$, so $h \in L_r$ (For $r = \infty$, $||h||_\infty \leqslant 1$). Hence, we have $L_q \subset L_r + L_p$.

On the other hand, if $r = \infty$, we have $|f|^q \leqslant ||f||_\infty^{q-p}|f|^p$ and $\lambda = \frac{p}{q}$, so $||f||_q \leqslant ||f||_p^{\frac{p}{q}}||f||_\infty^{1-\frac{p}{q}} = ||f||_p^{\lambda}||f||_\infty^{1-\lambda}$.

If $r < \infty$, we use Hölder's inequality, taking the pair of conjugate exponents to be $\frac{p}{\lambda q}$ and $\frac{r}{(1-\lambda)q}$, we have $\int |f|^q = \int |f|^{\lambda q}|f|^{(1-\lambda)q} \leqslant ||f||_{\frac{p}{\lambda q}}^{\lambda q}||f||_{\frac{r}{(1-\lambda)q}}^{(1-\lambda)q}$
$= \left[\int |f|^p\right]^{\frac{\lambda q}{p}} \left[\int |f|^r\right]^{\frac{(1-\lambda)q}{r}} = ||f||_p^{\lambda q}||f||_r^{(1-\lambda)q}$. The result follows as it was promised.

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Definition 4.9. An interpolation space between X_0 and X_1 is any intermediate space X such that every linear mapping from $X_0 + X_1$ into itself which is continuous from X_0 into itself and from X_1 into itself is automatically continuous from X into itself.

An interpolation space is said to be of exponent θ $(0 < \theta < 1)$, if there exists a constant C such that one has

$$||A||_{L(X)} \leqslant C||A||_{L(X_0)}^{1-\theta}||A||_{L(X_1)}^{\theta} \text{ for all } A \in L(X_0) \cap L(X_1).$$

Definition 4.10. Let $X_i, i = 0, 1$ be two normed spaces, continuously embedded into a topological vector space V so that $X_0 \cap X_1$ and $X_0 + X_1$ are defined with $X_0 \cap X_1$ equipped with the norm

$$||f||_{X_0 \cap X_1} = \max\{||f||_{X_0}, ||f||_{X_1}\}$$

and $X_0 + X_1$ is equipped with the norm

$$||f||_{X_0+X_1} = \inf_{f=f_0+f_1} (||f_0||_{X_0} + ||f_1||_{X_1}).$$

Definition 4.11. For $f \in X_0 + X_1$ and $t > 0$ one defines

$$K(f,t) = \inf_{f=f_0+f_1} (||f_0||_{X_0} + t||f_1||_{X_1}),$$

and for $0 < \theta < 1$ and $1 \leqslant p \leqslant \infty$ (or for $\theta = 0, 1$ with $p = \infty$), one defines the *real interpolation space* as follows:

$$(X_0, X_1)_{\theta,p} := \left\{ f \in X_0 + X_1 : t^{-\theta} K(f,t) \in L_p\left([0,\infty), \frac{dt}{t} \right) \right\}$$

with the norm $||f||_{(X_0,X_1)_{\theta,p}} := \begin{cases} \left(\int_0^\infty \left(t^{-\theta} K(f,t) \right)^p \frac{dt}{t} \right)^{\frac{1}{p}}, \; 0 < p < \infty \\ \sup_{0 \leqslant t < \infty} t^{-\theta} K(f,t), \; p = \infty \end{cases}$

$K(f,t)$ is continuous and monotone decreasing in t, with $K(f,t) \to 0$ as $t \to 0+$.

Example 4.12 ([6], Theorem 5.2.1)**.** Let $X_0 = L_1(\mathbb{R}^n)$ and $X_1 = L_\infty(\mathbb{R}^n)$. Then $(X_0, X_1)_{\theta,p} = L_p(\mathbb{R}^n)$, where $\theta = 1 - \frac{1}{p}$, and $K(f,t) = \int_0^t f^*(s)ds$.

The preceding example shows that the L_p spaces are examples of interpolation spaces.

Definition 4.13. A *quasi-norm* is a non-negative function $||.||_X$ defined on a real or complex linear space X for which the following conditions are satisfied:

(1) If $||f||_X = 0$ for some $f \in X$, then $f = 0$.

(2) $||\lambda f||_X = |\lambda| ||f||_X$ for $f \in X$ and all scalars λ.

(3) There exists a constant $c_X \geqslant 1$ such that

$$||f + g||_X \leqslant c_X [||f||_X + ||g||_X]$$

for $f, g \in X$.

One advantage of the real interpolation method is the fact that it generalizes to any *quasi-Banach space*, a linear space X equipped with a quasi-norm $||.||_X$ such that every Cauchy sequence is convergent.

4.3 An Application of the interpolation theory to the heat equation

We will need the convolution of two functions:

Definition 4.14. The convolution of two measurable functions f and g is defined as:

$$f * g\,(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$$

In our application of interpolation theory to partial differential equations, we will need the Riesz-Thorin theorem.

Theorem 4.15 (The Riesz-Thorin interpolation theorem, [6], Theorem 1.1.1)**.** *Let (X,μ) and (Y,ν) be σ-finite measure spaces. Assume that $p_0 \neq p_1$, and $q_0 \neq q_1$ and that*

$$T\colon L_{p_0}(X,\mu) \to L_{q_0}(Y,\nu)$$

is a bounded linear map with norm M_0, and that

$$T\colon L_{p_1}(X,\mu) \to L_{q_1}(Y,\nu)$$

is a bounded linear map with norm M_1.

Then

$$T\colon L_{p_\theta}(X,\mu) \to L_{q_\theta}(Y,\nu)$$

is a bounded linear map with the norm $M \leqslant M_0^{1-\theta}M_1^{\theta}$ provided that $\theta \in (0,1)$ and $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

As an application of the Riesz-Thorin interpolation theorem, we have Young's Inequality.

Theorem 4.16 (Young's Inequality, [26], Example 1.3.6)**.** *If* $f \in L_p(\mathbb{R}^n)$ *and* $g \in L_q(\mathbb{R}^n), p, q, r \geqslant 1,$ *and* $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$*, then* $f * g \in L_r(\mathbb{R}^n)$ *and:*

$$||f * g||_r \leqslant ||f||_p ||g||_q.$$

Next, we need the concept of a *semigroup*.

Consider a differential equation $f'(x) = af(x)$. If f is real-valued, then a solution is the exponential function $f(x) = e^{ax}$. If f is matrix-valued, then a solution will be given by a matrix exponential $f(x) = e^{Ax}$. If we want to go one step further (say we take A to be a bounded or unbounded linear operator on some concrete Banach space), we will need a *semigroup* of operators.

Definition 4.17. Let X be a Banach space, and for $t > 0$, the family $\{T_t \in \mathcal{L}(X)\}_{t \geqslant 0}$ is said to be a *strongly continuous semigroup*, denoted C_0, if it satisfies the following conditions:

(i) $T_0 = \mathrm{Id}$

(ii) $T_{t+s} = T_t \circ T_s$ for all $t, s \geqslant 0$.

(iii) $\lim_{t \to t_0} T_t f = T_{t_0} f$ for all $t_0 \geqslant 0$ and all $f \in X$.

A semigroup approach is a tool for solving initial boundary value problems. In particular, the boundedness of the norm of a semigroup of linear operators is important in this respect and we appeal to interpolation theory to either show that the solution makes sense in intermediate spaces or that the solution can be extended uniquely to a much larger space.

Example 4.18. Let $X = L_p(\mathbb{R})$. Recall that the heat equation as given by

$$\begin{cases} u_t = u_{xx}, \ x \in \mathbb{R}, \ t > 0 \\ u(x,0) = f(x) \end{cases} \tag{4.3}$$

Using Fourier Transform methods, the solution to (4.3) can be written as

$$u(x,t) = (4\pi t)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)dy.$$

The heat kernel is given by $K_t(s) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{s^2}{4t}}, t > 0$

We can write the solution using the convolution as $u(x,t) = K_t * f$. It can be shown that the solution to (4.3) is a continuous semigroup on X written as

$$T_t f(x) = \begin{cases} f(x), \ \text{for t} = 0 \\ (K_t * f)(x), \ \text{for } t > 0, x \in \mathbb{R}, f \in X \end{cases}$$

Physically, the function $T_t f(x)$ represents the temperature at position x and time t in a homogeneous isotropic medium(one whose electromagnetic properties are the same in all directions) $\mathbb{R}$ with the unit coefficient of thermal diffusivity, given that the temperature at position x at time 0 is $f(x)$.

Now, consider the linear operator $T_t : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ given by $T_t f(x) = u(x,t)$, then to what extent does the following inequality hold: $||T_t f||_p \leqslant C_p(t)||f||_p$?

We examine the heat kernel $K_t : \mathbb{R} \to \mathbb{R}$, $t > 0$, given by $K_t(s) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{s^2}{4t}}$.

Since $\int_{\mathbb{R}} e^{-a|s|^2} dx = \left(\frac{\pi}{a}\right)^{1/2}$, we can see that $K_t \in L_1(\mathbb{R})$ and $\|K_t\|_{L_1(\mathbb{R})} = 1$.

Applying Young's inequality to the convolution functional $T_t f = K_t * f$, we get

$$\|T_t f\|_{L_p(\mathbb{R})} \leqslant \|K_t * f\|_{L_p(\mathbb{R})} \leqslant \|K_t\|_{L_r(\mathbb{R})} \|f\|_{L_q(\mathbb{R})}, \tag{4.4}$$

where $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{q}$.

Now, we just have to estimate $\|K_t\|_{L_r(\mathbb{R})}$. We have

$$\|K_t\|_{L_r(\mathbb{R})} = (4\pi t)^{-1/2} \left(\int_{\mathbb{R}} e^{-\frac{r}{4t}|s|^2} ds \right)^{1/r} = (4\pi t)^{-1/2} \left(\frac{\pi}{\frac{r}{4t}} \right)^{1/2r} = C_r t^{-\frac{1}{2}\left(1 - \frac{1}{r}\right)} = C_r t^{-\frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)}.$$

$$ \tag{4.5} $$

Combining (4.4) and (4.5), we get

$$\|T_t f\|_{L_p(\mathbb{R})} \leqslant \|K_t * f\|_{L_p(\mathbb{R})} \leqslant \|K_t\|_{L_r(\mathbb{R})} \|f\|_{L_q(\mathbb{R})} \leqslant C_r t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{L_q}. \tag{4.6}$$

Now assume $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$. For $i \in \{0, 1\}$, set $\alpha(i) = -\frac{1}{2}\left(\frac{1}{q_i} - \frac{1}{p_i}\right)$ and $1 + \frac{1}{p_i} = \frac{1}{r_i} + \frac{1}{q_i}$.

Then (4.6) gives us

$$\|T_t f\|_{L_{p_i}(\mathbb{R})} \leqslant C_{r_i} t^{\alpha_i} \|f\|_{L_{q_i}} \tag{4.7}$$

Now set $X_i = L_{p_i}, Y_i = L_{q_i}$. It is known that $(X_0, X_1)_{\theta, p_\theta} = L_{p_\theta}$ and $(Y_0, Y_1)_{\theta, q_\theta} = L_{q_\theta}$, where $\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p_\theta}$ and $\frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1}{q_\theta}$.

By applying the Riesz-Thorin to (4.7), we have

$$\|T_t f\|_{L_{p_\theta}(\mathbb{R})} \leqslant C_{r_0}^{1-\theta} C_{r_1}^{\theta} t^{\alpha_0(1-\theta)+\alpha_1\theta} \|f\|_{L_{q_\theta}(\mathbb{R})}. \tag{4.8}$$

So we have applied the interpolation techniques to bound the norm of the semigroup $\{T_t\}_{t \geqslant 0}$ associated with the heat equation.

What we have learned in this example can be generalized as a theorem.

4.4 Theorems on boundedness of linear operators on concrete Banach Spaces

Theorem 4.19. *Let $\{T_t\}_{t \geqslant 0} : L_p \to L_q$ be a semigroup associated with a PDE, $u(x,t)$ a solution of the PDE, $u(x, 0) = f(x)$, such that $T_t f(x) = u(x,t) = K_t * f$, where K_t is a kernel, and $K_t \in L_p$, $f \in L_q$. Then we can control the norm of the semigroup $\{T_t\}_{t \geqslant 0}$ in all interpolation spaces.*

Proof. By the generalized Minkowski's inequality applied to Young's Inequality for the case $q = 1$ and $r = p$, we have

$$||K_t * f||_p \leqslant ||K_t||_p||f||_1.$$

and, by Hölder's inequality,

$$||Tf||_\infty \leqslant ||K_t||_p||f||_{\frac{p}{p-1}}$$

Thus,

$T : L_1 \to L_p$,

$T : L_{\frac{p}{p-1}} \to L_\infty$

are bounded, linear operators,

and therefore by the Riesz-Thorin interpolation theorem, we have

$$T : L_p \to L_q$$

is a bounded linear map provided $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{\frac{p-1}{p}}$, $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{\infty} = \frac{1-\theta}{p}$.

∎

If we take $K(x, y) = K_t(x - y)$ in the following Theorem 4.20, then Theorem 4.19 is a special case.

Theorem 4.20 ([18], Theorem 6.18)**.** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let K be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$. Suppose that there exists $C > 0$ such that $\int |K(x, y)|d\mu(x) \leqslant C$ for a.e. $y \in Y$ and $\int |K(x, y)|d\nu(y) \leqslant C$ for a.e. $x \in X$, and that $1 \leqslant p \leqslant \infty$. If $f \in L_p(Y, \nu)$, then the integral*

$$Tf(x) = \int K(x, y)f(y)d\nu(y)$$

converges absolutely for a.e. $x \in X$*, the function* Tf *thus defined is in* $L_p(X,\mu)$*, and* $||Tf||_p \leqslant C||f||_p$.

Using the real interpolation theory, we will now prove that theorem 4.20 holds true in larger spaces called Lorentz spaces.

We need the following general interpolation theorem for quasi-Banach spaces, which generalizes Riesz-Thorin theorem. For the definition of quasi-normed spaces, the reader should see Definition 5.10 on page 58.

Theorem 4.21 ([6], Theorem 3.11.8)**.** *Let* (X_0, X_1) *and* (Y_0, Y_1) *be interpolation couples of quasi-normed spaces. Let* T *be defined on* $X_0 + X_1$ *such that* $T : X_i \to Y_i$ *be sub-linear with quasi-norm* $M_i (i = 0, 1)$*. Then for any* $\theta \in (0, 1)$ *and* $q \in [1, \infty]$ *we have*

$$T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$$

is sub-linear with quasi-norm M *bounded by*

$$M \leqslant M_0^{1-\theta} M_1^{\theta}.$$

Now we are ready to state the general theorem about the boundedness of integral operators: $T : L_{p,r} \to L_{q,r} : ||Tf||_{L_{q,r}} \leqslant C||f||_{L_{p,r}}$ on Lorentz spaces.

Theorem 4.22. *Let* $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *be* σ*-finite measure spaces, and let* K *be an* $(\mathcal{M} \otimes \mathcal{N})$*-measurable function on* $X \times Y$*. Suppose that there exists* $C > 0$ *such that* $\int |K(x, y)|d\mu(x) \leqslant C$ *for a.e.* $y \in Y$ *and* $\int |K(x, y)|d\nu(y) \leqslant C$ *for a.e.* $x \in X$*. Let* $1 < q < \infty$ *and* $1 \leqslant r \leqslant \infty$*. Then for* $1 < p \leqslant q < \infty$*, if* $f \in L_{p,r}(Y, \nu)$*, then the integral*

$$Tf(x) = \int K(x, y)f(y)d\nu(y)$$

converges absolutely for a.e. $x \in X$*, the function* Tf *thus defined is in* $L_{q,r}(X, \mu)$*, and* $||Tf||_{L_{q,r}} \leqslant C||f||_{L_{p,r}}$.

Proof. Showing that the integral $Tf(x) = \int K(x,y)f(y)d\nu(y)$ converges absolutely for a.e. $x \in X$ follows from Fubini's theorem in a similar manner in which it is shown in [18], Theorem 6.18. It only remains to show that Tf is in $L_{q,r}(X, \mu)$, and $||Tf||_{L_{q,r}} \leqslant C||f||_{L_{p,r}}$.

For $1 < p < q$ there are numbers $1 < p_1 < p < p_2 < q, p < p_2 \leqslant q_1 < q < q_2$ and $\eta \in (0,1)$ such that $\frac{1}{p} = \frac{1-\eta}{p_1} + \frac{\eta}{p_2}$ and $\frac{1}{q} = \frac{1-\eta}{q_1} + \frac{\eta}{q_2}$. To see that this is always satisfied, we can take $\eta = 2^{-1}, p_1 = (2q)^{-1}p(p+q), p_2 = 2^{-1}(p+q), q_2 = (2p)^{-1}(q(p+q))$. Now take $X_i = L_{p_i}$ and $Y_i = L_{q_i}$. Then by real-interpolation method we have $(X_0, X_1)_{\eta,r} = L_{p,r}$ and $(Y_0, Y_1)_{\eta,r} = L_{q,r}$. Since $||Tf||_{L_q} \leqslant C||f||_{L_p}$ for $1 < p \leqslant q < \infty$, the hypothesis of Theorem 4.21 are satisfied and we have, $T : L_{p,r} \to L_{q,r}$ is sub-linear with

$$||Tf||_{L_{q,r}} \leqslant M_0^{1-\eta} M_1^{\eta} ||f||_{L_{p,r}}.$$

∎

Now if we take $p = q = r$ in Theorem 4.22, then we have Theorem 4.20.

4.5 Application of interpolation theory to a non-linear Schrödinger equation

While the semigroup approach gives a unique solution in a certain Banach space context, there may be other solutions (if one widens one's notion of solution). For example, it is well-known that another solution to the heat equation treated above is given by

$$v(x,t) = u(x,t) + \sum_{k=0}^{\infty} \frac{g^{(k)} x^{2k}}{(2k)!}$$

where u is the semigroup solution given above and $g(t) = e^{\frac{-1}{t^2}}$ (cf. [23], p. 7).

Since the heat equation is an example of a linear PDE, we will add a second application of interpolation theory to a non-linear PDE called the *Schrödinger equation*. This demonstrates that the theory is applicable to non-linear PDEs.

Before we state the problem and tools for solving it, we need to describe the concrete Banach spaces suited for the problem. An n-tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is called a multi-index and we define

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

and

$$x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i} \text{ for } x \in \mathbb{R}^n.$$

Denoting $D_k = \frac{\partial}{\partial x_k}$ and $D = (D_1, D_2, \cdots, D_n)$ we have $D^\alpha = \prod_{i=1}^{n} D_i^{\alpha_i}$.

Let $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$. $C^m(\Omega)$ denotes the set of all m-times continuously differentiable real-valued functions in Ω. $C_0^m(\Omega)$ denotes the subspace of $C^m(\Omega)$ consisting of those functions with compact support in Ω. For $u \in C^m(\Omega)$ and $1 \leqslant p < \infty$, we define

$$||u||_{m,p} = \left(\int_\Omega \sum_{|\alpha| \leqslant m} |D^\alpha u|^p dx \right)^{\frac{1}{p}}.$$

Denoting by $\tilde{C}_p^m(\Omega)$ the subset of $C^m(\Omega)$ consisting of those functions u for which $||u||_{m,p} < \infty$, we define $W^{m,p}(\Omega)$ to be the completion in the norm $||.||_{m,p}$, turning it into a Banach space.

For $p = 2$, we denote $W^{m,2}(\Omega) = H^m(\Omega)$. In this case, for $u, v \in H^m$, we define $\langle u, v\rangle_m = \int_\Omega \sum_{|\alpha|\leqslant m} D^\alpha u \overline{D^\alpha v} dx$.

The spaces $W^{m,p}(\Omega)$ consist of functions $u \in L_p(\Omega)$ whose derivatives $D^\alpha u$, in the sense of distributions, of order $k \leqslant m$ are in $L_p(\Omega)$.

We will need the following theorem.

Theorem 4.23 ([43], Theorem 10.8)**.** *A is the infinitesimal generator of a C_0 group of unitary operators on a Hilbert space H if and only if iA is self-adjoint.*

Example 4.24. We consider the following non-linear Schrödinger equation in $\mathbb{R}^2$

$$\begin{cases} \frac{1}{i}\frac{\partial u}{\partial t} - \Delta u + k|u|^2 u = 0, \ (t, x) \text{ in } [0, \infty] \times \mathbb{R}^2 \\ u(x, 0) = f(x) \text{ in } \mathbb{R}^2 \end{cases} \tag{4.9}$$

where u is complex-valued function and k is a real constant. The space in which this problem will be considered is $L_2(\mathbb{R}^2)$. Defining the linear operator A_0 by $D(A_0) = H^2(\mathbb{R}^2)$ and $A_0 u = -i\Delta u$ for $u \in D(A_0)$, the initial value problem (4.9) can be rewritten as

$$\begin{cases} \frac{du}{dt} + A_0 u + F(u) = 0, \quad \text{for } t > 0 \\ u(0) = f \end{cases} \tag{4.10}$$

where $F(u) = ik|u|^2 u$.

Using the scalar product $\langle u, v\rangle_m = \int_\Omega \sum_{|\alpha|\leqslant m} D^\alpha u \overline{D^\alpha v} dx$, for $m = 0$, integration by parts, and Fourier transforms method, we have the following lemma.

Lemma 4.25 ([43], Lemma 5.2)**.** *The operator iA_0 is self-adjoint in $L_2(\mathbb{R}^2)$.*

Since iA_0 is self-adjoint, by Theorem 4.23, we know that $-A_0$ is the infinitesimal generator of a C_0 group of unitary operators, $T_t, -\infty < t < \infty$, on $L_2(\mathbb{R}^2)$. An application of the Fourier Transform gives the explicit formula for T_t:

$$T_t u(x) = \frac{1}{4\pi i t} \int_{\mathbb{R}^2} e^{i\frac{|x-y|^2}{4t}} u(y) dy. \tag{4.11}$$

Theorem 4.26. *Let $\{T_t\}_{t\geqslant 0}$ be the semigroup given by equation (4.11). If $2 \leqslant p \leqslant \infty$ and $q^{-1}+p^{-1}=1$ then T_t can be extended in a unique way to an operator from $L_q(\mathbb{R}^2)$ into $L_p(\mathbb{R}^2)$ and*

$$||T_t u||_{0,p} \leqslant (4\pi t)^{-(\frac{2}{q}-1)}||u||_{0,q}. \tag{4.12}$$

Proof. Since T_t is a unitary operator on $L_2(\mathbb{R}^2)$, $T_t : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ is a bounded linear operator with $||T_t u||_{0,2} = ||u||_{0,2}$ and norm $M_0 = 1$. From equation (4.11), $T_t : L_1(\mathbb{R}^2) \to L_\infty(\mathbb{R}^2)$ is a bounded linear operator with $||T_t u||_{0,\infty} \leqslant (4\pi t)^{-1}||u||_{0,1}$ and norm $M_1 = (4\pi t)^{-1}$

Since $2 \leqslant p \leqslant \infty$ and $q^{-1} + p^{-1} = 1$, we have $\theta = 1 - \frac{2}{q}$, and by the Riesz-Thorin interpolation theorem, $T_t : L_q(\mathbb{R}^2) \to L_p(\mathbb{R}^2)$ is a bounded linear operator with $||T_t u||_{0,p} \leqslant (4\pi t)^{-(\frac{2}{q}-1)}||u||_{0,q}$.

∎

Chapter 5

Compact H-Operators

This chapter defines and establishes relations among approximation spaces of certain operators called *H-operators*, which generalize the notion of self-adjoint to Banach spaces. The problem of creating interpolation spaces is at the core of interpolation theory [6].That is, given a pair (X_0, X_1) of Banach (or quasi-Banach) spaces, called a *Banach couple*, with X_0 and X_1 both continuously embedded in some Hausdorff topological vector space, how can one construct and describe interpolation spaces $(X_0, X_1)_{\theta,q}$ for the pair (X_0, X_1), where θ and q are some parameters. Such spaces $(X_0, X_1)_{\theta,q}$ should have the interpolation property that a linear operator T, which is bounded on X_i for $i = 0, 1$ is automatically bounded on $(X_0, X_1)_{\theta,q}$. A natural question to ask is what properties of T as a linear operator on X_i still hold true when T is viewed as a linear operator on $(X_0, X_1)_{\theta,q}$. The answer to this classical question depends on the details of the method used to construct $(X_0, X_1)_{\theta,q}$. Two of the main methods used are real and complex methods, but there are others. In [13], it is shown using the real interpolation method that if $T \in \mathcal{K}(X_0, Y_0)$ and $T \in \mathcal{L}(X_1, Y_1)$, then $T \in \mathcal{K}((X_0, X_1)_{\theta,q}, (Y_0, Y_1)_{\theta,q})$. In this chapter, we construct and describe interpolation spaces when T is a compact H-operator. Under certain conditions regarding Bernstein and Jackson inequalities, interpolation spaces can be realized as approximation spaces, see [14], Theorem 9.1 on page 235. Thus, we are able to define approximation spaces for *compact H-operators* using the sequences of their eigenvalues and establish relations among these spaces using interpolation theory. In section 5.1, we start with a motivational example leading to the definition of H-operators. Section 5.2 discusses approximation spaces. Section 5.3 briefly presents interpolations spaces and illustrates how approximation spaces can be realized as examples of interpolation spaces under some conditions. Section 5.4 defines approximation spaces for compact H-operators contrasts them with the general approximation spaces and presents an inclusion theorem and a representation theorem. Section 5.5 points to a connection with Bernstein's Lethargy problem.

5.1 Defining H-Operators

A fundamental result about linear operators on Hilbert spaces is the spectral theorem, which says that for a compact self-adjoint operator T acting on a separable Hilbert space H, one can choose a system of orthonormal eigenvectors $\{v_n\}_{n\geqslant 1}$ of T and corresponding eigenvalues $\{\lambda_n\}_{n\geqslant 1}$ such that

$$Tv = \sum_{n=1}^{\infty} \lambda_n \langle v, v_n \rangle v_n, \text{ for all } x \in H. \tag{5.1}$$

The sequence $\{\lambda_n\}$ is decreasing and, if it is infinite, converges to 0.

To investigate the spectral properties of an arbitrary $T \in \mathcal{K}(H)$, where H is a Hilbert space, it is useful to study the eigenvalues of the compact positive self-adjoint operator T^*T associated with T. If

$$\lambda_1(T^*T) \geqslant \lambda_2(T^*T) \geqslant \cdots > 0$$

denote the positive eigenvalues of T^*T, where each eigenvalue is repeated as many times as the value of its multiplicity, then the *singular values* of T are defined to be

$$s_n(T) := \sqrt{\lambda_n(T^*T)}, n \geqslant 1.$$

Using the representation (5.1) for T^*T one can prove the following *Schmidt representation* for T

$$Tx = \sum_{n=1}^{\infty} s_n(T)\langle \psi_n, x\rangle \phi_n, x \in H, \tag{5.2}$$

where $\{\psi_n\}_{n\geqslant 1}$ and $\{\phi_n\}_{n\geqslant 1}$ are orthonormal systems in H (see [34]).

It is known that the concept of H-operators is the generalization in a Banach space of the concept of self-adjoint operators. We start with a motivation that will lead to a concrete definition.

Definition 5.1. A norm $\|\cdot\|$ on the $n \times n$ matrices is called a *unitarily invariant norm* if

$$\|UXV\| = \|X\|$$

for all X and for all unitary matrices U and V.

Example 5.2. For $d_i \in \mathbb{R}, i = 1, 2, \cdots n$, let's consider

$$T = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

We will examine the operator norm of the resolvent of T, $(T - \lambda I)^{-1}$, where $\lambda \in \mathbb{C}$ is in the spectral set of T. First, note that

$$|d_j-\lambda| = |d_j-(a+bi)| = \sqrt{(d_j-a)^2+b^2} \geqslant \sqrt{b^2} = |b| \implies |d_j-\lambda|^{-1} \leqslant |b|^{-1} = |\text{ Im } \lambda|^{-1}.$$

Therefore, we have

$$\|(T-\lambda I)^{-1}\| = \max\{|d_1-\lambda|^{-1}, \cdots, |d_n-\lambda|^{-1}\} \leqslant |\text{Im}(\lambda)|^{-1} \tag{5.3}$$

If T is self-adjoint, then one can choose an orthonormal basis such that there exists a unitary matrix P and a diagonal matrix such that $T = PDP^{-1}$.

Since the operator norm or any norm defined in terms of singular values is unitarily invariant, we have the norm of T equal to the norm of D. So that it suffices to compute the operator norm of a diagonal matrix with real diagonal entries as we have done above.

Thus if T is a self-adjoint operator, which must have real eigenvalues, between complex Banach spaces, then

$$\|(T - \lambda I)^{-1}\| \leqslant |\mathrm{Im}(\lambda)|^{-1} \tag{5.4}$$

which motivates the following definition.

Definition 5.3. Let $T \in \mathcal{L}(X, Y)$ be a linear operator between arbitrary complex Banach spaces X and Y. Then T is an *H-operator* if and only if its spectrum is real and its resolvent satisfies

$$||(T - \lambda I)^{-1}|| \leqslant C|\ \mathrm{Im}\ \lambda|^{-1},$$

where $\mathrm{Im}\ \lambda \neq 0$.

Here C is independent of the points of the resolvent. An operator in Hilbert space is an H-operator with constant $C = 1$ if and only if it is a self-adjoint operator. In [40], it is proved that closed operators with real eigenvalues are an example of H-operators.

If T is a compact H-operator, then $\{\lambda_k(T)\}$ denotes the sequence of eigenvalues of T, and each eigenvalue is repeated according to its multiplicity. We also assume that $\{\lambda_k(T)\}$ is ordered by magnitude, so that $|\lambda_1(T)| \geqslant |\lambda_2(T)| \geqslant \cdots$.

In 1918, F. Riesz proved compact operators have at most countable set of eigenvalues, $\lambda_n(T)$, which arranged in a sequence, tend to zero. This result raises the following question:

What are the conditions on $T \in \mathcal{L}(X, Y)$ *such that* $(\lambda_n(T)) \in \ell_q$*?*

Having $(\lambda_n(T)) \in \ell_1$ is the precise condition one needs in order to generalize the *Schmidt representation* (5.2) given above for compact operators on Banach spaces.

The question can be recast more specifically, what is the rate of convergence to zero of the sequence $(\lambda_n(T))$?

Here is an example that shows the importance of the preceding question.

Example 5.4. Consider the diagonal operator

$$T = \text{ diag } (a_1, a_2, a_3, \cdots,) \quad \text{ where } \quad a_n = \frac{1}{\log(n+1)}, \quad n = 1, 2, \cdots .$$

Note that T is compact and its eigenvalues are $\lambda_n(T) = a_n$. In [34], it is shown that for each $q > 0$, the number a_n^q goes to zero slower than $\frac{1}{n}$ when $n \to \infty$. It follows that $\lambda_n(T) \notin \ell_q$.

To answer the question on the rate of convergence, in [45], A. Pietsch developed the theory of s-numbers, $s_n(T)$, which characterize the degree of compactness of T. There are several possibilities of assigning to every operator $T : X \to Y$ a certain sequence of numbers $\{s_n(T)\}$ such that

$$s_1(T) \geqslant s_2(T) \geqslant \cdots \geqslant 0.$$

Definition 5.5. We recall the definition of the following s-numbers:

1. The *nth approximation number*

$$\alpha_n(T) = \inf\{||T - A|| : A \in \mathcal{F}(X,Y)\}, \quad n = 0, 1, \ldots$$

   Note that $\alpha_n(T)$ provides a measure of how well T can be approximated by finite mappings whose range is at most n-dimensional. The largest s-number is the approximation number.

2. The *nth Kolmogorov diameter* of $T \in \mathcal{L}(X)$ is defined by

$$\delta_n(T) = \inf\{||Q_G T|| : \dim G \leqslant n\}$$

   where the infimum is over all subspaces $G \subset X$ and Q_G denotes the canonical quotient map $Q_G : X \to X/G$.

It is clear that $\alpha_n(T)$ and $\delta_n(T)$ are monotone decreasing sequences and that

$$\lim_{n\to\infty} \alpha_n(T) = 0 \quad \text{if and only if} \quad T \in \mathcal{F}(X,Y)$$

and

$$\lim_{n\to\infty} \delta_n(T) = 0 \quad \text{if and only if} \quad T \in \mathcal{K}(X,Y).$$

In [25], it is shown that for any compact operator T on a Hilbert space H the n-th singular value $s_n(T)$ coincides with the n-th approximation number $\alpha_n(T)$. This allows us to compute $\alpha_n(T)$.

Example 5.6. Consider the non-self-adjoint $T = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

The characteristic polynomial of T is $\Delta(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4$ and the eigenvalues of T are $\lambda_1(T) = 2$, $\lambda_2(T) = 2$ and $\lambda_3(T) = 1$.

We have $T^*T = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 6 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

The characteristic polynomial of T^*T is $\Delta(\lambda) = \lambda^3 - 12\lambda^2 + 30\lambda - 16$ and the eigenvalues are approximately $\lambda_1(T^*T) = 8.796$, $\lambda_2(T^*T) = 2.466$, and $\lambda_3(T^*T) = 0.738$

Note that we have $\alpha_1(T) = s_1(T) > |\lambda_1(T)|$, $\alpha_2(T) = s_2(T) < |\lambda_2(T)|$, and $\alpha_3(T) = s_3(T) < |\lambda_3(T)|$, which means we cannot compare $\alpha_n(T)$ with $|\lambda_n(T)|$.

Note also that in the preceding example, $T \neq T^*$ and $T^*T \neq TT^*$, that is, T is neither self-adjoint nor normal. We should give an example that is not self-adjoint, but normal and see how it compares.

Example 5.7. Consider $T = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$. Then $T^* = T^t = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

We have $T^*T = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} = TT^*$.

The eigenvalues of T are $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$ where it follows that $|\lambda_1| = \sqrt{13} = |\lambda_2|$. The singular values of T are $s_1(T) = \sqrt{13} = s_2(T)$.

In this case, we have $\alpha_n(T) = s_n(T) = |\lambda_n(T)|$, for $n \in \{1, 2\}$.

For compact self-adjoint operators on Hilbert spaces and more broadly for compact H- operators we will be able to always compare the preceding approximation quantities.

Indeed, the importance of H-operators comes from a result of Markus [40], which shows that for a compact H-operator T with eigenvalues (λ_n) (numbered in order of decreasing modulus and taking into account their multiplicity), the sequence $(|\lambda_n(T)|)$ is equivalent to *approximation numbers* and *Kolmogorov diameters*. Specifically, in [40] Markus proved the following theorem.

Theorem 5.8. *If T is a compact H-operator on a Banach space X, then*

$$\delta_{n-1}(T) \leqslant \alpha_n(T) \leqslant 2\sqrt{2}C|\lambda_n(T)| \leqslant 8C(C+1)\delta_{n-1}(T), \tag{5.5}$$

where C is a constant from the definition of H-operator, and $\delta_n(T)$ and $\alpha_n(T)$ are the n-th Kolmogorov diameter and n-th approximation numbers of T respectively.

The following corollary follows from the preceding theorem.

Corollary 5.9. *If T is a compact H-operator on a Banach space X, then for any $0 < \mu \leqslant \infty$, $|\lambda_n(T)| \in \ell_\mu \iff \delta_n(T) \in \ell_\mu \iff \alpha_n(T) \in \ell_\mu$.*

This equivalence allows us to construct approximation spaces for H- operators using sequences of eigenvalues, but before we do this we will first introduce approximation spaces.

5.2 Approximation spaces

For ease of flow, we recall the following definitions already encountered in the preceding pages.

Definition 5.10. A *quasi-norm* is a non-negative function $||.||_X$ defined on a real or complex linear space X for which the following conditions are satisfied:

(1) If $||f||_X = 0$ for some $f \in X$, then $f = 0$.

(2) $||\lambda f||_X = |\lambda| ||f||_X$ for $f \in X$ and all scalars λ.

(3) There exists a constant $c_X \geqslant 1$ such that

$$||f + g||_X \leqslant c_X[||f||_X + ||g||_X]$$

for $f, g \in X$.

A *quasi-Banach space* is any linear space X equipped with a quasi-norm $||.||_X$ such that every Cauchy sequence is convergent.

Definition 5.11. An *approximation scheme* (X, A_n) is a quasi-Banach space X together with a sequence of subsets A_n satisfying the following:

$(A1)$ there exists a map $K : \mathbb{N} \to \mathbb{N}$ such that $K(n) \geqslant n$ and $A_n + A_n \subseteq A_{K(n)}$ for all $n \in \mathbb{N}$,

$(A2)$ $\lambda A_n \subset A_n$ for all $n \in \mathbb{N}$ and all scalars λ,

$(A3)$ $\bigcup_{n\in\mathbb{N}} A_n$ is a dense subset of X.

Approximation schemes were introduced in Banach space theory by Butzer and Scherer in 1968 [11] and independently by Y. Brudnyi and N. Kruglyak under the name of "approximation families" in [9]. They were popularized by Pietsch in his 1981 paper [48], for later developments we refer the reader to [1, 2, 4].

Let (X, A_n) be an approximation scheme. For $f \in X$ and $n = 1, 2, \cdots$, the *nth approximation number* is defined by

$$\alpha_n(f, X) := \inf\{||f - a|||_X : a \in A_{n-1}\}.$$

$\alpha_n(f, X)$ is the error of best approximation to f by the elements of A_{n-1}.

Definition 5.12. Let $0 < \rho < \infty$ and $0 < \mu \leqslant \infty$. Then the *approximation space* X^ρ_μ, or more precisely $(X, A_n)^\rho_\mu$ consists of all elements $f \in X$ such that

$$(n^{\rho-\mu^{-1}}\alpha_n(f, X)) \in \ell_\mu,$$

where $n = 1, 2, \cdots$. We put $||f||_{X^\rho_\mu} = ||n^{\rho-\mu^{-1}}\alpha_n(f, X))||_{\ell_\mu}$ for $f \in X^\rho_\mu$.

Now, we define and present Lorentz sequences as examples of approximation spaces.

Definition 5.13. A null sequence $x = (\zeta_k)$ is said to belong to the *Lorentz sequence space* $\ell_{p,q}$ if the non-increasing re-arrangement $(s_k(x))$ of its absolute values $|\zeta_k|$ satisfies

$$\left(k^{\frac{1}{p}-\frac{1}{q}} s_k(x)\right) \in \ell_q, \tag{5.6}$$

so that

$$\lambda_{p,q}(x) = \begin{cases} \left(\sum_{k=1}^{\infty} \left(k^{\frac{1}{p}-\frac{1}{q}} s_k(x)\right)^q\right)^{\frac{1}{q}} & \text{for } 0 < p < \infty \text{ and } \quad 0 < q < \infty \\ \sup_{1 \leqslant k < \infty} k^{\frac{1}{p}} s_k(x) & \text{for } 0 < p < \infty \text{ and } \quad q = \infty \end{cases} \tag{5.7}$$

is finite.

Example 5.14. Let $0 < \rho < \infty$ and $0 < \mu \leqslant \infty$. Consider the approximation scheme (X, A_n), where $X = \ell_\infty$ and A_n:= the subset of sequences having at most n coordinates different from 0. For any $\eta \in \ell_\infty$, the sequence $\alpha_n(\eta; \ell_\infty)$ is the non-increasing rearrangement of the sequences η and $X_\mu^\rho = \ell_{\rho^{-1},\mu}$ (see[48], page 123).

5.3 Realizing approximation spaces as interpolation spaces

For completeness, we briefly recall a number of definitions in interpolation theory already presented in chapter 4.

Definition 5.15. An *intermediate space* between X_0 and X_1 is any normed space X such that $X_0 \cap X_1 \subset X \subset X_0 + X_1$ (with continuous embedding).

Definition 5.16. An *interpolation space* between X_0 and X_1 is any intermediate space X such that every linear mapping from $X_0 + X_1$ into itself which is continuous from X_0 into itself and from X_1 into itself is automatically continuous from X into itself. An interpolation space is said to be of exponent θ $(0 < \theta < 1)$, if there exists a constant C such that one has

$$||A||_{L(X)} \leqslant C||A||_{L(X_0)}^{1-\theta}||A||_{L(X_1)}^{\theta} \text{ for all } A \in L(X_0) \cap L(X_1).$$

Definition 5.17. Let $X_i, i = 0, 1$ be two normed spaces, continuously embedded into a topological vector space V so that $X_0 \cap X_1$ and $X_0 + X_1$ are defined with $X_0 \cap X_1$ equipped with the norm

$$||f||_{X_0 \cap X_1} = \max\{||f||_{X_0}, ||f||_{X_1}\}$$

and $X_0 + X_1$ is equipped with the norm

$$||f||_{X_0+X_1} = \inf_{f=f_0+f_1}(||f_0||_{X_0} + ||f_1||_{X_1}).$$

Definition 5.18. For $f \in X_0 + X_1$ and $t > 0$ one defines

$$K(f,t) = \inf_{f=f_0+f_1}(||f_0||_{X_0} + t||f_1||_{X_1}),$$

and for $0 < \theta < 1$ and $1 \leqslant p \leqslant \infty$ (or for $\theta = 0, 1$ with $p = \infty$), one defines the *real interpolation space* as follows:

$$(X_0, X_1)_{\theta,p} := \left\{ f \in X_0 + X_1 : \quad t^{-\theta}K(f,t) \in L_p\left([0,\infty), \frac{dt}{t}\right) \right\}$$

with the norm $||f||_{(X_0,X_1)_{\theta,p}} := \begin{cases} \left(\int_0^\infty \left(t^{-\theta}K(f,t)\right)^p \frac{dt}{t}\right)^{\frac{1}{p}}, & 0 < p < \infty \\ \sup_{0 \leqslant t < \infty} t^{-\theta}K(f,t), & p = \infty \end{cases}$

$K(f,t)$ is continuous and monotone decreasing in t, with $K(f,t) \to 0$ as $t \to 0+$. K-functional provides a relationship between interpolation and approximation spaces.

Once again, the Lorentz sequences are examples of interpolation spaces as can be seen from this classical example.

Example 5.19. The Lorentz sequence space $l_{p,q}$ can be created as an approximation space using a real interpolation space. Take $X = \ell_r$ and $Y = \ell_s$. Then $(\ell_r, \ell_s)_{\theta,q} = \ell_{p,q}$ for $\frac{1}{p} := \frac{(1-\theta)}{r} + \frac{\theta}{s}$ and $0 < q \leqslant \infty$.

Now, if we take $p = q$, then $(\ell_r, \ell_s)_{\theta,p} = \ell_{p,p} = \ell_p$ for $\frac{1}{p} := \frac{(1-\theta)}{r} + \frac{\theta}{s}$ and $0 < p \leqslant \infty$, so that every ℓ_p may be realized as an interpolation space.

The real interpolation method provides the connection between interpolation spaces and approximation spaces. To state this connection, we need the following fundamental inequalities [10].

Jackson's inequality, which measures the rate of decrease of $\alpha_n(f; X)$ is given by

$$\alpha_n(f, X) \leqslant C(n+1)^{-\sigma}\alpha_n(f; Y), \text{ where } C \text{ is a constant.} \tag{5.8}$$

Berstein's inequality, which measures the rate of increase of $\|p_n\|_X$, where $p_n \in A_n$, is given by

$$\|p_n\|_X \geqslant C(n+1)^{-\sigma}\|p_n\|_Y, \text{ where } C \text{ is a constant.} \tag{5.9}$$

If Jackson's and Bernstein's inequalities are valid for the pair X and Y, then we can characterize completely the approximation spaces X^ρ_μ using the real interpolation spaces $(X, Y)_{\theta,q}$.

Proposition 5.20 ([14], Theorem 9.1)**.** *If both the Jackson and the Bernstein inequalities hold for the spaces X and Y, then for $0 < \rho < r$ and $0 < \mu \leqslant \infty$ we have*

$$X^\rho_\mu = (X, Y)_{\frac{\rho}{r},\mu}. \tag{5.10}$$

According to the preceding proposition, to identify for a given X and approximation scheme Q, the approximation spaces X_μ^ρ, $0 < \mu < r$, it is enough to find a space Y for which the Jackson and Bernstein inequalities are valid.

There is a way to find such spaces Y:

Proposition 5.21 ([14], Theorem 9.3)**.** *Consider an approximation scheme (X, A_n). Then for $0 < \mu \leqslant \infty$, $0 < \rho < \infty$, the space $Y := Y_\rho := X_\mu^\rho$ satisfies the Jackson and the Bernstein inequalities. Moreover, for $0 < \alpha < r$ and $0 < \mu_1 \leqslant \infty$,*

$$(X, Y)_{\frac{\alpha}{r},\mu_1} = X_{\mu_1}^\alpha. \tag{5.11}$$

We will use the preceding theorem to establish relations among approximation spaces of *H-operators*.

5.4 Inclusion and Representation Theorems for Approximation Spaces

Now, we define an approximation space for compact H-operator by using $|\lambda_n(T)|$.

Definition 5.22. Let $0 < \rho < \infty$ and $0 < \mu \leqslant \infty$. Set $X :=$ the set of all compact H-operators between two arbitrary Banach spaces.Consider an approximation scheme (X, A_n). We define an approximation space for H-compact operators by nd $A_\mu^\rho := \{T \in X : (n^{\rho-\mu^{-1}}|\lambda_n(T)|) \in \ell_\mu\}$. We put $||T||_{A_\mu^\rho} = ||n^{\rho-\mu^{-1}}|\lambda_n(T)|)||_{\ell_\mu}$ for $T \in A_\mu^\rho$.

To realize the importance of constructing approximation spaces for compact H-operators, let's recall the following question: What are the conditions on $T \in \mathcal{L}(X, Y)$ such that $(\lambda_n(T)) \in \ell_q$?

Answer: If T is a compact H-operator, then $A_q^{\frac{1}{q}}$ consists of all elements $T \in \mathcal{L}(X, Y)$ such that $(|\lambda_n(T))| \in \ell_q$, which implies $(\lambda_n(T)) \in \ell_q$.

Lemma 5.23. *If* $0 < \theta < 1$ *and* $1 \leqslant \mu_1 \leqslant \mu_2 \leqslant \infty$*, one has* $(X_0, X_1)_{\theta,\mu_1} \subset (X_0, X_1)_{\theta,\mu_2}$ *(with continuous embedding).*

Proof. Note that if $1 \leqslant \mu < \infty$, and $t_0 > 0$, then by the monotonicity of the K-functional, $K(f,t)$, we have $K(f,t) \geqslant K(f,t_0)$ for $t > t_0$, so that

$$\left(||f||_{(X_0,X_1)_{\theta,\mu}}\right)^\mu = \int_0^\infty \left(t^{-\theta}K(f,t)\right)^\mu \frac{dt}{t} \geqslant [K(f,t_0)]^\mu \int_{t_0}^\infty t^{-\theta\mu}\frac{dt}{t} = [K(f,t_0)]^\mu \frac{t_0^{-\theta\mu}}{\theta\mu}$$

which implies

$$t_0^{-\theta}K(f,t_0) \leqslant C||f||_{(X_0,X_1)_{\theta,\mu}}$$

and thus we have $||t^{-\theta}K(f,t)||_{L_\infty((0,\infty),\frac{dt}{t})} \leqslant C||f||_{(X_0,X_1)_{\theta,\mu}}$ and now using the Hölder's inequality, one obtains:

$$||f||_{(X_0,X_1)_{\theta,\mu_2}} = ||t^{-\theta}K(f,t)||_{L_{\mu_2}((0,\infty),\frac{dt}{t})} \leqslant C'||f||_{(X_0,X_1)_{\theta,\mu_1}} \text{ for } 1 \leqslant \mu_1 \leqslant \mu_2 \leqslant \infty.$$

We are done as promised. ∎

We will also use the following lemma, which gives the basic relation between Kolmogorov numbers and approximation numbers.

Lemma 5.24 ([12])**.** $\delta_n(T) \leqslant \alpha_n(T)$ *for all* $T \in \mathcal{L}(X,Y)$.

Theorem 5.25 (Inclusion Theorem)**.** *Let* $0 < \rho < \infty$ *and* $0 < \mu_1 \leqslant \mu_2 \leqslant \infty$*. If* T *is a compact H-operator between arbitrary Banach spaces* X *and* Y*, then* $A^\rho_{\mu_1} \subset A^\rho_{\mu_2}$*.*

Proof. From (5.5) we have $\alpha_n(T) \leqslant 2\sqrt{2}C\,|\lambda_n(T)|$, which implies

$$n^{\rho-\mu^{-1}}\alpha_n(T) \leqslant 2\sqrt{2}Cn^{\rho-\mu^{-1}}\,|\lambda_n(T)|\,.$$

Thus, if $n^{\rho-\mu^{-1}}|\lambda_n(T)| \in \ell_\mu$, then $2\sqrt{2}Cn^{\rho-\mu^{-1}}|\lambda_n(T)| \in \ell_\mu$, which implies that $n^{\rho-\mu^{-1}}\alpha_n(T) \in \ell_\mu$. It follows that if $T \in A_\mu^\rho$, then $T \in X_\mu^\rho$. Therefore, $A_\mu^\rho \subset X_\mu^\rho$.

By Lemma 5.24, we have $\delta_{n-1}(T) \leqslant \alpha_{n+1}(T)$, (5.5) implies that

$$2\sqrt{2}C|\lambda_n(T)| \leqslant 8C(C+1)\delta_{n-1}(T) \leqslant 8C(C+1)\alpha_n(T).$$

Hence,

$$\frac{2\sqrt{2}C}{8C(C+1)}n^{\rho-\mu^{-1}}|\lambda_n(T)| \leqslant n^{\rho-\mu^{-1}}\alpha_n(T).$$

Thus, if $n^{\rho-\mu^{-1}}\alpha_n(T) \in \ell_\mu$, then $\frac{2\sqrt{2}C}{8C(C+1)}n^{\rho-\mu^{-1}}|\lambda_n(T)| \in \ell_\mu$. It follows that $T \in X_\mu^\rho \implies T \in A_\mu^\rho$. Hence, $X_\mu^\rho \subset A_\mu^\rho$.

We have $A_\mu^\rho = X_\mu^\rho$. By Proposition 5.20, we also know $A_\mu^\rho = X_\mu^\rho = (X, Y)_{\frac{\rho}{r},\mu}$.

By Lemma 5.23, we have $A_{\mu_1}^\rho \subset A_{\mu_2}^\rho$ as was promised.

∎

The proof of the following theorem relies on Markus' inequality (5.5), Hölder's inequality and proof of an analogous representation theorem in [48].

Theorem 5.26 (Representation Theorem)**.** *Let $0 < \rho < \infty$ and $0 < \mu \leqslant \infty$. Set $X :=$ the set of all compact H-operators between two arbitrary Banach spaces and $A_\mu^\rho := \{T \in X : (n^{\rho-\mu^{-1}}|\lambda_n(T)|) \in \ell_\mu\}$. Consider an approximation scheme (X, A_n). Then $T \in X$ belongs to A_μ^ρ if and only if there exists $g_n \in A_{2^n}$ such that $T = \sum_{n=0}^\infty g_n$ and $(2^{n\rho}||g_n||_X) \in \ell_\mu$. Moreover, $||T||_{A_\mu^\rho}^{rep} := \inf ||(2^{n\rho}||g_n||)||_{\ell_\mu}$, where the infimum is taken over all possible representations, defines an equivalent quasi-norm on A_μ^ρ.*

Proof. Suppose $T \in A_\mu^\rho$. We wish to find $g_n \in A_{2^n}$ such that $T = \sum_{n=0}^\infty g_n$ and $(2^{n\rho}||g_n||_X) \in \ell_\mu$. Choose $g_n^\star \in A_{2^n-1}$ such that

$$||T - g_n^\star||_X \leqslant 2\alpha_{2^n}(T) \leqslant 4\sqrt{2}C|\lambda_{2^n}(T)|.$$

Set $g_0 = 0 = g_1$, and $g_{n+2} = g^\star_{n+1} - g^\star_n$ for $n = 0, 1, \cdots$.. We have $g_n \in A_{2^n}$, and

$$T = \lim_{n\to\infty} g^\star_n = \sum_{n=0}^{\infty} g_n.$$

Moreover, it follows from

$$||g_{n+2}||_X \leqslant c_X[||T - g^\star_{n+1}||_X + ||T - g^\star_n||_X] \leqslant 4c_X\alpha_{2^n}(T) \leqslant 16\sqrt{2}c_X C|\lambda_{2^n}(T)|$$

that $(2^{n\rho}||g_n||_X) \in \ell_\mu$.

Next, suppose there exists $g_n \in A_{2^n}$ such that $T = \sum_{n=0}^{\infty} g_n$ and $(2^{n\rho}||g_n||_X) \in \ell_\mu$. We must show that $T \in A^\rho_\mu$. Although $\alpha_n(T, X)$ is in general not a continuous function of T, we can also find an equivalent quasi-norm X, p-norm, that is always continuous. Thus, we can assume that $||.||_X$ is a p-norm with $0 < p < \mu$. If $T \in X$ can be written in the form $T = \sum_{n=0}^{\infty} g_n$ such that $g_n \in A_{2^n}$ and $(2^{n\rho}||g_n||_X) \in \ell_\mu$, then it follows from $\sum_{n=0}^{N-1} g_n \in A_{2^N-1}$ that

$$|\lambda_{2^N}(T)| \leqslant 2\sqrt{2}(C+1)\alpha_{2^N}(T) \leqslant 2\sqrt{2}(C+1)||T - \sum_{n=0}^{N-1} g_n||^p_X \leqslant 2\sqrt{2}(C+1)\sum_{n=N}^{\infty} ||g_n||^p_X.$$

In the case $0 < \mu < \infty$ we put $q = \frac{\mu}{p}$, and choose γ such that $\rho p > \gamma > 0$. Then

$$\sum_{N=0}^{\infty}[2^{N\rho}\lambda_{2^N}(T)]^\mu \leqslant 2\sqrt{2}(C+1)\sum_{N=0}^{\infty}[2^{N\rho}\alpha_{2^N}(T)]^\mu \leqslant 2\sqrt{2}(C+1)\sum_{N=0}^{\infty} 2^{N\rho\mu}\left(\sum_{n=N}^{\infty} 2^{-n\gamma}2^{n\gamma}||g_n||^p_X\right)^q$$

$$\leqslant 2\sqrt{2}(C+1)\sum_{N=0}^{\infty} 2^{N\rho\mu}\left(\sum_{n=N}^{\infty} 2^{-n\gamma q'}\right)^{\frac{q}{q'}}\left(\sum_{n=N}^{\infty} 2^{n\gamma q}||g_n||^p_X\right)$$

$$\leqslant c_1 2\sqrt{2}(C+1)\sum_{N=0}^{\infty} 2^{N(\rho\mu-\gamma q)}\sum_{n=N}^{\infty} 2^{n\gamma q}||g_n||^\mu_X$$

$$\leqslant c_1 2\sqrt{2}(C+1)\sum_{n=0}^{\infty} 2^{n\gamma q}||g_n||^\mu_X\sum_{N=1}^{n} 2^{N(\rho\mu-\gamma q)}$$

$$c_2 2\sqrt{2}(C+1)\sum_{n=0}^{\infty}[2^{n\rho}||g_n||_X]^{\mu} < \infty.$$

The desired result follows as was promised.

∎

5.5 Connection to Bernstein's Lethargy Theorem

The question of the rate of convergence of $\lambda_n(T)$ provides some connection to the classical Bernstein's Lethargy problem.

Now, we consider the *Bernstein lethargy problem* for linear approximation: given a nested system $A_1 \subset A_2 \subset \cdots$ of linear subspaces of a Banach space X and a strictly decreasing sequence $d_0 > d_1 > \cdots > d_n \to 0$, does there exist an element $x \in X$ such that for all $n = 0, 1, 2, \cdots$, $\alpha_n(x) = \alpha_n(x, A_n) = d_n$?

The answer is *yes* in many particular cases: if X is a Hilbert space; if all A_n are finite-dimensional; if $d_n > \sum_{k=n+1}^{\infty} d_k$ for all n. However, the Bernstein problem is still unsolved in its general setting.

In the remaining part of this section, we investigate for infinite-dimensional Banach spaces X and Y the existence of an operator $T \in \mathcal{L}(X, Y)$ whose sequence of approximation numbers $\{\alpha_n(T)\}$ behaves like the prescribed sequence $\{d_n\}$ given above in the Bernstein lethargy problem. If $\mathcal{A}_n$ denotes the space of all bounded linear operators from X into Y with rank at most n, then $\alpha_n(T) = \rho(T, \mathcal{A}_n)$.

Definition 5.27. The operator $T \in \mathcal{K}(X, Y)$ where X and Y are complex Banach spaces is said to be a *kernel operator* if it can be represented in the form

$$T = \sum_{j=1}^{\infty} \alpha_j f_j(\cdot) y_j \tag{5.12}$$

$(f_j \in X^*, y_j \in X, ||f_j|| = ||y_j|| = 1, j = 1, 2, \cdots)$, where $\sum_{j=1}^{\infty} |\alpha_j| < \infty$.

Proposition 5.28. *[40] For any sequence of non-negative numbers* (d_n) *that tends to zero, a kernel operator* T *exists such that* $\delta_n(T) \geqslant d_n$ *(*$n = 0, 1, 2, \cdots$*).*

By Lemma 5.24, we always have $\alpha_n(T) \geqslant \delta_n(T)$ for every $T \in \mathcal{L}(X, Y)$. Then by the preceding proposition, we have for a strictly decreasing sequence $d_0 > d_1 > \cdots > d_n \to 0$, there is always an element in $\mathcal{L}(X, Y)$, namely a kernel operator T such that one has $\alpha_n(T) \geqslant \delta_n(T) \geqslant d_n$.

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