“You Will Remain Unwavering in Your Determination to Succeed No Matter How Long It Takes”: A Mathematico-Emotional Analysis of A Guide to Higher Learning

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“You Will Remain Unwavering in Your Determination
to Succeed No Matter How Long It Takes”:
A Mathematico-Emotional Analysis
of *A Guide to Higher Learning*

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Synopsis

In this article we present an analysis of the multi-fold artist's book, *A Guide to Higher Learning* [2]. In our analysis, we propose different perspectives from which to view the mathematical and philosophical aspects found within the book. We explore aspects of “higher learning” by analyzing the complex mathematical equations and textual elements that readers meet in the book. Chen challenges the reader with: “This is a test. You will not be given any assistance or instructions on how to proceed. You will not be told when to begin or when to stop” [2]. Thus, in our exploration of the book, we consider Chen's challenge from the perspectives of the “person on the street” and the “mathematician in the academy.” By doing so, we formulate answers to philosophical questions regarding the process of doing mathematics and the role emotions play in this process (which we call the mathematico-emotional journey). Through this analysis, and the establishment of the distinct perspectives, we were able to identify diverging purposes that *A Guide to Higher Learning* serves, depending upon the perspective the reader chooses.

1. Introduction

Mathematics might be considered by some to be one of the most elusive of disciplines to understand. In fact, you may have often heard the lament, “Oh! I am not a math person; I can’t even balance my checkbook!” Yet for many, “balancing checkbooks” may be an obsolete task given modern online and electronic banking and, to be transparent, we consider such “math” really to be arithmetic. Whereas this notion of “mathematics” is not the discipline of mathematics we will explore in this article, it speaks to a population that we will refer to when we examine a component of the discipline from two different perspectives (a “person on the street” (POTS) and a “mathematician in the academy” (MITA)).

In this article, we introduce readers to the lovely artist’s book, *A Guide to Higher Learning*, published by Julie Chen in 2009 [2]. The book is described in the following way on the publisher’s website (<https://flyingfishpress.com/>):

A Guide to Higher Learning examines the experiential process of acquiring knowledge, on both academic and personal levels. The piece [comprises] 8 sections of rigid square pages that are hinged together in unexpected ways, giving the reader a physical reading experience that mirrors the complex meaning of the content. The book in its fully unfolded form reveals an intricate and fascinating visual pattern of information.

From our own perspective, this artist’s book is a multi-fold art book that reflects the complex and beautifully self-contained world of mathematics. It consists of square panels that unfold into eight sections (Figure 1): four sections of mathematical content (which we refer to as either loops or quadrants) and four connectors. The unfolding results in a unique and unexpected journey to reveal four branches of mathematics: Analysis, Geometry, Abstract Algebra (this is how Chen identified the third branch, but we consider the content to be Linear Algebra), and Number Theory. The squares (or tiles) which comprise the book are composed of complex mathematical equations, textual elements, and origami crease fold patterns that are used to create a challenging experience for the reader as they embark on a journey of acquiring knowledge.

Figure 1: *A Guide to Higher Learning* in its fully unfolded state (Flying Fish Press, <https://flyingfishpress.com/>).

When asked about the intended audience for the book, Chen stated:

The book was meant for a general reading audience. I wanted people with no particular specialized knowledge to be able to see that there is beauty in the meaning presented in various technical, visual, and written languages, even if they themselves have no idea how to decipher that language. (personal communication, 2020)

In this article, we keep in mind the author's orientation for a general reading audience and investigate the potential and actual reading experience from two perspectives of readers. We also explore aspects of “higher learning” by analyzing the mathematics that readers meet in the book.

Though Chen stated that “this book doesn’t have anything to do with math... [the] math is there sort of as a proxy for any inaccessible knowledge” (personal communication, 2020), as authors of this article, we both entered this project as a result of being attracted to the mathematics, as well as a desire to search for its connection and representation within the guide to higher learning created by Chen.

2. Meeting the Book: Each Author’s Perspective

2.1. Kathleen’s Perspective

My desire to explore and analyze the content of *A Guide to Higher Learning* (from here on, we use GTHL as an abbreviation of the book’s full title) has existed since 2012, when I was first introduced to the copy in Special Collections at the Florida State University Libraries. While working on another project focused on the history of mathematics, the Associate Dean of Special Collections brought out GTHL, and on that day, I was hooked. Unfortunately, with many projects on my plate, I knew that I could not dedicate time to digging into this mathematical specimen until I had some assistance. Over the next five years or so, I would ask to see GTHL on various visits to Special Collections, but I would quickly become perplexed by how I might approach an investigation of what was represented in the artist’s book, and how I might make sense of it to ultimately share results of an analysis. After several attempts to interest students participating in a university-wide undergraduate research program at my university, I had all but given up in finding someone with as enthusiastic of a response as I had to GTHL. In Spring 2019, as part of an interview for an unrelated research study, I met Oksana and somehow GTHL came up in our conversation. Weeks later, Oksana and I designed a research opportunity which would enable her to collaborate on research in which we could explore and analyze the artist’s book and we began our collaboration in August 2019.

2.2. Oksana’s Perspective

As research is a fundamental component in the organization in which I participate (Women in Math, Science & Engineering, or WIMSE), I was eager to find a project that would excite me and align with my interests of study.

Out of pure chance, I had the great pleasure of meeting Dr. Clark, who had been seeking a student with whom to collaborate on this magnificent work of art. And from the second she introduced GTHL to me in Spring 2019, I knew there was nothing else I wanted to study more; it was as if the book was speaking to me, illustrating my passion for mathematics in such an elegant, yet perplexing way. Through WIMSE's Research Experience Program, I was awarded the incredible opportunity to work with Dr. Clark beginning in Fall of 2019, in which we eagerly dove into our research of the multi-fold art book.

Through my initial investigation, I began by solving or proving the mathematical results found in the first two quadrants (Quadrant I: Analysis, Quadrant II: Geometry; see Figure 2), in order to begin forming ideas about why Chen used the specific equations in her book, and how they were used to convey her purpose, particularly in representing higher learning. I also began to formulate conjectures about how the physical format of the book reflects philosophical ideas in mathematics, the purpose of GTHL mirroring a complex mathematics exam, and how engagement with the book reflects the experience of doing mathematics. By analyzing the mathematical content in each quadrant, as well as how the equations are related, I began to formulate ideas about the purposes this book serves, and the ways in which the book reflects the art and beauty of doing mathematics. It is important to note that I am not a research mathematician yet, so I am not "discovering new mathematics" as part of my current work. However, the crux of my work thus far has been to explore what it means to do mathematics. By researching GTHL, I have begun to see mathematics in a new light, as a form of art that requires imagination and intuition, along with the creative struggle to inspire new directions and ingenious breakthroughs.

3. Identification of Two Perspectives: The "MITA" and the "POTS"

After an initial analysis of the first two quadrants of GTHL, a challenge arose in attempting to understand how a reader might engage with the mathematical components, textual elements, and the philosophical and emotional journey the reader may embark upon. We consistently questioned who the primary audience of GTHL was, thus it was necessary to establish context for the audience that Chen is addressing. That is, who is this "reader" taking the "exam" and following this "emotional journey"?

Figure 2: Image of both the Analysis loop (Quadrant I) and the Geometry loop (Quadrant II). (Image provided by the Florida State University Libraries; used with permission from Julie Chen.)

It is not appropriate for us to assume that the only reader is a mathematician; as well, we cannot assume the reader to be only a person with limited or no mathematical experience. Therefore, we established two different lenses for our conception of who could potentially be an audience for GTHL: The MITA and the POTS. We define the reader categorized as the MITA to be a “Mathematician in the Academy” and the POTS to be a non-mathematician, “Person on the Street.” By defining these two different lenses, we can better understand the philosophical/emotional journey through which Chen takes the reader. As well, these lenses aid in providing a better interpretation to how the textual elements relate to the equations presented in each quadrant, depending upon who the reader is. As neither of us consider ourselves a MITA or POTS, we have tried our best to accurately put ourselves in the position of both lenses, in order to offer a better analysis of how GTHL might serve multiple purposes.

By defining these two distinct lenses, we can next analyze how the reader will engage with the book, depending upon which lens they are most closely associated with. For example, familiarity with the equations alone affect how a reader will engage with GTHL; therefore, it is safe to assume that the way a mathematician will engage with the equations and text in Quadrant II is much different than that of a person who has little to no experience with the equations and concepts Chen presents in GTHL. Presenting the journey as an “exam” allows the mathematician to put their knowledge to the test and create their own method of going through each loop, as Chen provides no explicit instructions on how to proceed (see an example of textual elements found in Quadrant II in Figure 3).

Figure 3: Examples of what we refer to as textual elements appear as the quadrant unfolds, as seen in the middle square tiles of the left- and right-hand sides in Quadrant II. (Image provided by the Florida State University Libraries; used with permission from Julie Chen.)

As mathematicians are great critical thinkers, their curious nature allows them to use both inductive and deductive reasoning to not only solve problems but to create connections between thinking with logic and order, as well as intuition and inventive manipulation that develops new ideas and thoughts. Non-mathematicians tend to view mathematics as a strict and ordered process in which one follows a set of rules and procedures to produce an expected outcome. While mathematicians rely on certain axioms in mathematics to be true, most often it is the bending or reinterpreting of these rules that inspire new mathematical concepts and conjectures to be formed, and a perfect example of this is the discovery of non-Euclidean geometry. Thus, while it is true that mathematicians rely on rigor, it is their ability to creatively connect ideas that allows them to be skillful critical thinkers and engage with GTHL in qualitatively different ways than that of the POTS.

As the average POTS will most likely not have a strong foundation and experience with higher mathematics, the way this reader engages with the book can potentially be seen as more of a learning process. They may not necessarily be able to prove the results in the way the MITA would, but they can nevertheless engage with the equations and attempt to find possible relationships and patterns, as well as be introduced to new mathematical concepts they most likely have never seen before. In this way, the POTS can be seen as a student within the landscape of GTHL, testing their knowledge of what they already know whilst also learning (or, at least engaging with) new mathematical concepts along the way. The interaction with GTHL in both its physical form and with the textual elements helps to create this complex and unusual experience for the POTS, an experience that mirrors the content and elements displayed.

4. Mathematio-Emotional Journey of GTHL

There are several different layers found within GTHL that coexist to create a unique experience for persons from both perspectives as they travel through the quadrants and embark on what we call a mathematio-emotional journey. Both the POTS and the MITA, in trying to make sense of the philosophical aspects of GTHL, inevitably are prompted to exhibit emotional responses that the ideas provoke. Chen herself provokes these responses through the textual elements that conjure a wide array of emotions such as confidence, determination, motivation, confusion, and even despair.

For the POTS, these emotions might be considered to assist them in the learning process, by eliciting the reader to embark on an exploration of knowledge that promotes thinking critically about the mathematical concepts presented in the book. Therefore, the POTS embarks on a mathematico-emotional trajectory in which the philosophical components of the book evoke emotions that guide them to think about mathematics in the ways a mathematician might. For the MITA, the emotions that are provoked reflect the work of the mathematician. As the MITA begins to appreciate the themes and ideas found in the book that reflect the world of mathematics, the textual elements mirror the experience that a mathematician undergoes in their everyday professional life.

By simulating a complex mathematics exam, Chen evokes emotion in the reader from the outset. Why she chooses to begin in this way is worth investigating. For most, taking an exam does not usually produce good feelings, especially when it is an impromptu exam. In this sense then, why would Chen choose to begin in such a way? It is almost as if she is intentionally provoking negative emotions in the POTS, by first introducing the beginning of what seems to be a very complex equation and then informing the reader: “This is a test.” However, it can also be the case that by simulating an exam environment, Chen is eliciting motivation and determination for the reader, perhaps as a way to prepare them for the journey to come. In fact, all of the text in Quadrant I seems to be generating a self-regulated learning environment in which Chen’s goal is to lay the groundwork and give the reader as little information as possible so that they are forced to rely on their own intuition and instruction to some degree and attempt to make sense of the equations and what they mean in relation to each quadrant. However, Chen does not want the reader to feel extremely discouraged just yet — by the end of Quadrant I — so that they continue in their process of self-regulated learning with a more appropriate foundation of emotions (motivation, determination, etc.) to carry on through the rest of the quadrants and complete the exam.

5. MITA: Possibilities for Engagement

We turn our attention to the content of Quadrant II (also known as the Geometry loop) to share the varied engagements of individuals representing each of the two perspectives we have identified. As mathematicians have

the ability to manipulate precise and intricate ideas on the way to formalizing abstract mathematical concepts, we can assume that they engage with Quadrant II by attempting to verify the equations presented (see Figure 4) to be true and construct connections between the mathematical concepts that they represent.

<p>Given triangle A with sides a, b, and c, and for $s = \frac{a+b+c}{2}$</p> $D = \text{Area } A = \sqrt{s(s-a)(s-b)(s-c)}.$ <p>In the hyperbolic plane, given triangle B with sides a, b, and c, and for $\alpha = \cosh(a)$, etc., and $\Delta = \sqrt{1 - \alpha^2 - \beta^2 - \gamma^2 + 2\alpha\beta\gamma}$</p> $D = \text{Area } B = \cos^{-1} \frac{\alpha + \beta + \gamma + \alpha\beta + \alpha\gamma + \beta\gamma + \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma}{(1 + \alpha)(1 + \beta)(1 + \gamma)}.$ <p>$D \neq D.$</p>
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Figure 4: Transcription of the mathematical text of the Geometry loop (Quadrant II).

This is consistent with the emotional journey Chen takes the reader through, as we will see shortly. From the first three textual elements, it seems as if from the outset Chen is chartering a sense of confidence in the reader as she states: “You will remain unwavering in your determination to succeed no matter how long it takes. You will devise many elaborate strategies, although you secretly believe that you will be able to find the solution in record time.” To emphasize this feeling, Chen introduces the first part of the equation even before presenting the textual elements, which gives triangle A with sides a , b , and c . (This text is provided in Figure 4. It is interesting to note that Chen does not provide diagrams in the Geometry loop.) Especially for the MITA, this sense of confidence remains intact with revealing of the remainder of the right-hand side of Quadrant II, as these are concepts they most certainly have seen before (equation for semi-perimeter and Heron’s (or, Hero’s) formula). A sense of confusion could potentially surface for the MITA when attempting to create a proof for Heron’s formula, and how to represent the area of a triangle using only its side lengths. We bring attention to this potential, because although the process is quite lengthy and involves tedious mathematical computations, it is not necessarily challenging for the MITA and only requires the use of the Pythagorean theorem and the equation for semi-perimeter (which was already given by Chen in the book).

Thus, even though Chen states in the fourth textual element, “But despite your best efforts, your progress will be slow and uncertain,” this potentially could refer to the second part of Quadrant II (involving hyperbolic geometry), and Chen is almost preparing the mathematician for what is to come.

The introduction to the triangle in the hyperbolic plane may present no real confusion to the mathematician, as they are compelled to think critically about the subject at hand, which allows the mathematician to formalize ideas in mathematics that they may or may not have seen before. By doing so, they are able to intuitively establish connections across mathematics instead of perceiving mathematics as isolated and independent concepts. This is most likely the goal that Chen wants the non-mathematician (i.e., the POTS) reader to achieve, to be able to understand how concepts in mathematics are interconnected in order to make sense of the mathematics they are experiencing. Each quadrant in GTHL is physically connected with one another with the unfolding of the squares. Perhaps Chen is eliciting the reader to “think like a mathematician” if they are not already doing so, through the layout of the book and the connections between the four branches of mathematics that she presents. In doing so, the POTS can begin to discover connections with the equations by simply examining the physical form of GTHL.

Continuing with the MITA analysis, as the equations and textual elements unfold, Chen states: “You will occasionally experience moments of great clarity which will be followed by intervals of confusion and doubt. As time passes, these episodes will become more frequent and intense until your resolution to continue begins to weaken” — thus revealing the last equation in Quadrant II, the area for the hyperbolic triangle. Because the equation provided by Chen is not the most conventional way to represent the area of a triangle in hyperbolic geometry, the MITA will most likely engage with the equations in the left-hand side of Quadrant II by attempting to establish the analogue of Heron’s formula, as they will begin to see the connection between both sides of the Geometry loop. We believe that this is when the sense of confusion may arise for the MITA. Deriving the area for a triangle in the hyperbolic plane in a similar form to the equation Chen presents is no easy process and requires creative manipulation of equations as well as previous knowledge of formulas such as the laws of cosine and sine (in the hyperbolic plane). Trial and error will inevitably take place, as the mathematician makes their way through the equation, tenaciously attempting to get it in a similar form.

Even when they experience moments of great clarity, as every mathematician can attest from experience, it will be followed by more confusion and doubt. This is where the emotional response switches from confusion to despair. We can think about this sense of despair in the way the mathematician Andrew Wiles described the process of doing mathematics:

Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So, each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of — and couldn't exist without — the many months of stumbling around in the dark that precede them [1, page 1].

Even though the process of engaging with GTHL is most certainly not as long as what Wiles described (although we believe it could be), the idea is the same. Chen confirms this notion throughout all of Quadrant II as she reminds the reader that along with moments of clarity, intervals of confusion and doubt may certainly follow. However, when the MITA does finally reach the answer they are looking for in regards to the area of a hyperbolic triangle, they will be filled with a strong sense of equanimity, as they complete this part of the “exam” and see from their own understanding how Euclidean properties of a triangle transform to work in a non-Euclidean plane.

6. POTS: Possibilities for Engagement

With respect to the POTS's potential mode of engagement within Quadrant II, we believe it is safe to assume she would have some kind of previous knowledge of working with properties of triangles (at least those in the Euclidean plane) and so there should be a sense of confidence that builds up with the unfolding of the right-hand side (Euclidean geometry). The process of translating the area of a triangle, $\text{Area} = \frac{1}{2} (\text{base} \times \text{height})$, to Heron's

formula (which only requires the side lengths) may not be entirely clear to the POTS and it is most likely not the route they will take to calculate the area. However, the reader may still have a sense of confidence, particularly if they are familiar with Euclidean geometry.

With respect to the mathematico-emotional journey taken by the POTS, the textual elements in Quadrant II continue to elicit feelings of determination, yet now Chen instructs the reader that their progress will be “slow and uncertain” and that while they will “experience moments of great clarity,” it will be followed by “intervals of confusion and doubt,” thus illuminating how the emotional process is dynamic and not static. Chen provokes the reader to experience these emotions in order to compel them to think critically about the material being presented. This state of confusion is most likely to be evoked in the second half of Quadrant II (the left-hand side), in which the reader, possibly for the first time, is introduced to “newer geometries incompatible with Euclid’s: mathematics thought to be true absolutely is now true only conditionally” [2, “Answer Book”, page 7]. If this is the case, when Chen states that “despite your best efforts, your progress will be slow and uncertain,” she could be referring to the reader relating the two geometries to one another. Just as the discovery of non-Euclidean geometry was ground-breaking, so it can be as well for the POTS who unfolds the rest of Quadrant II to reveal two equations that represent areas of a triangle that are seemingly inconsistent with one another.

Trying to relate these two geometries to one another can conjure feelings of both confusion and despair, especially because Chen provides no visual representation for a triangle in the hyperbolic plane. Trudeau [4] stated that the “confusion most people feel when they first encounter hyperbolic geometry swirls around the fact that they find it simply unimaginable” (page 163). The fact that Chen does not provide a visual aid only makes visualizing this “new geometry” even more challenging; therefore, understanding the area in this hyperbolic plane may seem like an impossible task. Evoking confusion and curiosity through the text allows the reader to embark on an exploration of knowledge that promotes a productive learning environment as the POTS attempts to make sense of the relations between the two triangles in Quadrant II, and imagine this potential “new geometry.” Again, Chen is encouraging the reader to “think like a mathematician,” but she is also prompting them to conjure the emotions that a mathematician feels when do-

ing mathematics. Thus, emotions play an important role in the learning process of GTHL: it is conjectured that you cannot have one without the other. In a sense, Chen is mirroring the emotions that a mathematician experiences when solving complex problems, as they often devote significant time and energy to finding a solution, no matter how frustrating it gets or how hopeless it may seem. Chen wants the reader to undergo the same experience so that they can begin to formalize ideas about the equations and concepts in the Geometry loop the way a mathematician might. Thus, Chen is not promoting negative emotions as we may have initially thought. Instead, she promotes emotions that will allow readers to think in a different way — specifically, the way a mathematician might.

7. The Purpose of GTHL: Does Perspective Make a Difference?

Chen may want the POTS to investigate different approaches of attempting to acquire inaccessible knowledge, which is represented by the mathematics found in GTHL: this alone can begin the deep investigation into what hyperbolic geometry is. By understanding the differences between these geometric domains, the POTS can begin their own investigation into what a hyperbolic triangle is, thus GTHL acts as a guide to introduce the reader, now taking on the role of a student, to a whole new world within mathematics by simply asking the question of why these two triangles do not have equal area. Hence, the focus for the POTS may not even be the areas themselves, but a tool used to open the door to “geometries incompatible with Euclid’s.” Therefore, the importance for the POTS to interact with GTHL cannot be understated, as Chen is encouraging and guiding the reader to take part in some form of self-regulated learning, as she (through her GTHL) takes on the role of the instructor who subtly guides the reader by providing them with the resources necessary to engage in higher learning. These resources include mathematical equations and concepts that promote critical thinking and textual elements that evoke emotions to assist the reader in the learning process.

It is equally important for mathematicians to engage with GTHL as they will be able to appreciate its structure in a way different from that of non-mathematicians. The way in which they connect abstract mathematical concepts allows the MITA to view GTHL as more of an artistic piece that elicits mathematical thinking instead of a tool for learning. The MITA may perceive

the process of interacting with the book and the unfolding of each quadrant as analogous to the process of doing mathematics, as they will be able to find pleasure in the process of solving or verifying equations, no matter how long it may take. By physically inspecting the ways in which concepts and patterns within GTHL are connected and the way in which new ones are evoked, the MITA can appreciate the book simply for its beauty as they would with the nature of mathematics. For the mathematician, GTHL is not only a work of art, nor is it a book with only mathematical equations or relationships; it is a perfect mixture of the two that reflects the self-contained world of mathematics.

8. Engaging a POTS and a MITA with Quadrant II

Even though we have made our own predictions on how we believe the POTS and MITA may engage with GTHL, we also wanted to observe representatives from both perspectives going through Quadrant II and how they engage with the mathematical ideas and corresponding textual elements. By doing so, we are able to create a stronger foundation for the perspective of both the POTS and the MITA and how they may take on the role of a student or a mathematical aesthete, respectively.

8.1. Experiment 1: POTS Engagement with Quadrant II

For this analysis, we observed two non-mathematicians engaging with GTHL, giving them no clear instruction except the ones given by Chen found in the textual elements in both Quadrant I and II. We instructed them to simply travel through the Geometry loop, following the order of the layout, and engage with the various elements of GTHL, in whichever way they chose. We also asked the POTS representatives to talk through the observations they made about the equations, text, and overall layout of the quadrant.

When starting to unfold the right-hand side of the quadrant, both POTS participants stated they were familiar with basic Euclidean geometrical properties (although they did not use the term “Euclidean”); however, they were not able to recognize the equations for semi-perimeter and Heron’s formula (see Figure 4). POTS 1 felt that the text in the first part seemed misleading because she had no idea what the author wanted them to do, so to her, the “solution” was unclear. POTS 2 experienced the same confusion when trying to understand what it was she was trying to do. Both POTS 1 and

POTS 2 each stated that it looked as if the author was trying to show them something through the equations and textual elements, but they did not understand what that was because they didn't "know math."

Even though she was not familiar with the area equation provided, POTS 2 was able to observe how it relates to the side lengths of the triangle that was given. She even stated that it was interesting to learn a new formula for area of a triangle because she had only known about the area formula of $\text{Area} = \frac{1}{2} (\text{base} \times \text{height})$. Receiving no real direction, POTS 2 moved on to the left-hand side — but not before noticing the "investigation" square, stating that to her, this meant she needed outside knowledge in order to understand what was going on.

Emotions were also evoked with the unfolding of the second half of the Geometry loop by both POTS participants as they were exposed to unfamiliar equations in an unfamiliar plane. They both expressed how they did not understand what a hyperbolic plane was and that the equations related to the area of a triangle in them were, as POTS 1 stated, "gibberish." POTS 2 stated that some of the symbols seemed familiar from trigonometry, yet she did not know how they were applicable in the equations presented, just that they have something to do with the area for the triangle in this foreign plane. Once the last square was revealed, with the relationship " $D \neq D$," both the POTS participants began to formulate ideas to conjecture how the two sides of the loop might relate to one another. They both stated that because the areas are not the same, clearly the triangles must be different, yet neither of them knew why or how, just that they had to be. POTS 1 stated that she was confused about how both triangles can have the same side lengths yet different areas, which led her to go back to the text in the left-hand side in hopes of getting more information. She said that not only were the equations confusing, but the text did not help her figure out what to do, just that she should be doing something. As a final question, we asked both POTS participants what they thought the overall purpose of the Geometry loop was. POTS 1 replied that it was probably to learn about areas. POTS 2 stated that she was not sure, but that it has to do with the fact that the triangles are not the same, and that if she had more time she could possibly figure out what that actually means in regards to geometry.

From our observations, it seems as if both POTS participants were just as perplexed as we are in terms of how Chen might have planned for or expected

such a reader to engage with the book. However, this could just mean that Chen wants the decision to be entirely up to the reader to determine how they should interact with the quadrants. Just as there could be many approaches to solve a mathematical equation, there could be many different approaches to engaging with GTHL. This is most likely a central theme Chen is conveying throughout each quadrant in the book.

However, one common theme that seemed to be present for both POTS participants is that they both, to some degree, took on the role of a student in learning new geometries (even if they did not realize that was what they were doing). Whether they realized it or not, each POTS was making connections between both sides of the Geometry loop regarding the triangles' areas that implicitly "evokes the surprise of newer geometries incompatible with Euclid's" ([2], "Answer Book", pages 3–4). Take, for example, how both POTS participants concluded that the triangles had to be different, even though they did not know what a triangle in the hyperbolic plane looked like. Through the equations Chen presents, POTS 1 and POTS 2 are able to begin formulating ideas about why and how the triangles are different. POTS 2 came close to this conclusion, as she stated that the triangles being different was important and it says something about triangles in geometry. Thus, the purpose of the Geometry loop can be for the reader to recognize how, through investigation, new mathematical concepts are formed and that build on others.

8.2. Experiment 2: MITA Engagement with Quadrant II

Similar to our observation of the two POTS participants' engagement with Quadrant II, we also observed a mathematician, providing no explicit instructions on how to engage with the book, except the ones given by Chen from the textual elements in GTHL found in Quadrants I and II.

Immediately after unfolding the first part of Quadrant II, the MITA was able to recognize the area of the Euclidean triangle as Heron's formula, and began his engagement by attempting to prove the provided formula to be true. After many attempts at inventive manipulation, the MITA decided to move on to the second half of the quadrant, which he believed would require deep investigation of some form. He immediately began by attempting to recall properties of hyperbolic geometry that would allow him to translate measures in the hyperbolic plane to measures in the Euclidean plane.

Again, the MITA was able to formalize ideas presented in Quadrant II, and even though he explicitly stated that he has not worked with hyperbolic geometry in a long time, he still attempted to establish connections between the two geometries. In this way, the MITA was able to think critically about the mathematical concepts presented to him in Quadrant II. As predicted, the MITA experienced a strong sense of confusion, as he engaged in trial and error in an effort to deduce some kind of relation between the two areas. As he worked, the MITA continued to wrestle with the idea that two areas were related in some way. After spending more time with the area formula in hyperbolic geometry, the MITA decided to reflect and reread the textual elements in an attempt to acquire more insight into the quadrant and analyze the text more carefully. In this way, the MITA and the POTS participants responded to Quadrant II in a similar manner, as they began to rely on the textual elements when they got stuck on the mathematics.

The MITA stated that he believes the text reflected exactly what he was feeling at certain points throughout Quadrant II, and further shared that he feels a type of “hypnosis” happening with the way the text is spelled out to mirror his emotions. He stated that the text motivated him to work harder at certain parts: “you will remain unwavering in your determination to succeed no matter how long it takes,” and, “you will devise many elaborate strategies but despite your best efforts your progress will be slow and uncertain.” These textual elements hinted to him that he would not be able to reconcile the relationship in Quadrant II, driving the MITA to want to work at it even more. However, the MITA also believes the text gives him “permission to weaken his intensity” on the hyperbolic side as he understands he is less familiar with the geometry in this plane, and the text reflects these feelings as well. Referring to the textual elements, the MITA stated:

I think the other thing that is really fascinating about this puzzle in its purpose is that when you look at the back tiles, you can read into them what you want to read into them. And so, if I need encouragement, I can collapse certain tiles and find encouragement. If I need permission to give up because I am feeling exhausted, I can find that permission to give up.

Traveling back to the mathematical components of Quadrant II, the MITA continued his analysis, by paying closer attention to the last square (or, as the MITA referred to as the last tile) that reflects the idea of the areas

not being equal in the Euclidean and hyperbolic plane. Referring to both sides of Quadrant II, he observed, “I love how these are drawn parallel and then at the end there is the D does not equal D, but my curiosity is still, just because they are not equal, is there some relationship that I could find between the two that would still be interesting?” As anticipated, the MITA attempted to formulate other relations between the two areas, as he was able to use both inductive and deductive reasoning in order to create connections between thinking with logic and order, as well as intuition and inventive manipulation, that allowed him to construct certain mathematical relations that may not be as evident to a novice.

As a final reflection, the MITA believed he needed something to make sense of the ideas found in Quadrant II, that would ground him in his abstract thinking, stemming from hyperbolic geometry. He expressed deep appreciation for the beauty that comprises GTHL, but also accepted that he would not be able to find the solution to the Geometry loop:

I think that as you unfold this book, there’s this sequence that becomes less and less certain and more taken away from the world that you are probably familiar with. . . . So, I like it because there’s this increase in uncertainty for both the novice and the expert, because the expert is trying to figure things out too. And I think the author is telling this message as things unfold if you carefully unwind them, as [the author is] trying to secretly encourage you to try various things, but also if you get frustrated at some point, you get the message that it’s okay to experience these periods of confusion and doubt. Even if they become more intense, you will occasionally “experience moments of clarity.”

The MITA, through his deep investigation of Quadrant II, was able to devise elaborate strategies which allowed him to construct ideas pertaining to Euclidean and Hyperbolic geometry. Even though he believed he was not able to find the “solution” Chen refers to, the mathematician still relished the beauty of GTHL that reflected the nature of mathematics, with which he is all too familiar. Thus, the artist’s book, as opposed to its purpose for the POTS, generates an atmosphere for the MITA to adhere to their passions and create a complicated, yet intriguing way to partake in the process of doing mathematics.

9. Extending the (Mathematico-Emotional) Journey

Our initial goal when we began this research was to conduct an analysis of GTHL quadrant by quadrant and develop responses to the fundamental questions we started with, as well to new ones we developed along the way. However, during the analysis of Quadrant II, our fruitful endeavors to understand the philosophical aspects of GTHL quickly unraveled through the investigation of the mathematical components and textual elements, which led us to pause from analyzing the remaining two quadrants and more thoroughly develop our ideas regarding how the experience with GTHL might transpire for individuals from the perspectives we identified. Consequently, for our next steps, we will continue to investigate whether our multi-perspective analysis holds for Quadrants III and IV. We also hope to develop more insight into the idea of the mathematics of GTHL serving as a “proxy for any inaccessible knowledge” (Chen, personal communication, 2020).

The notion of a proxy for inaccessible knowledge is keenly characterized by mathematician and philosopher, Alfred North Whitehead [5]. He explained how the fundamental ideas of mathematics are skipped over when the subject is taught, and instead, instruction focuses on the “technical procedure” and thus, “the unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception” [5, page 1]. Thus, for a learner such as the POTS, engaging with the equations in each quadrant of GTHL may seem like a futile task if they do not have previous knowledge of the fundamental concepts being presented. For example, it is simply not enough for the reader to know how to determine the area of a triangle in the Euclidean and hyperbolic plane, if they are not able to understand the fundamental ideas of both geometries and their relations to one another. On the other hand, this is why the MITA is able to engage with the mathematical content in such a unique way: they most likely have acquired knowledge of these abstract ideas, which, in contradiction to the experience of the POTS, is accessible.

Thus, in its multiple purposes, GTHL is an important work of art, no matter the perspective through which one may view it. For example, the work can serve as an aid for learning more about fundamental ideas in mathematics for the POTS, as a composition that exhibits the beauty of doing mathematics appreciated by the MITA, or as a perfect blend of the two for researchers who are neither mathematicians or novices (e.g., for us as the authors).

Within GTHL, there is beauty to be found within each unfolding, as well as opportunity to acquire new (no longer inaccessible) knowledge of what doing mathematics entails. Thus, we wish to underscore that doing mathematics is much more than arithmetic and computations, and in the words of Paul Lockhart [3], whose essay, interestingly, was published the same year as GTHL:

To do mathematics is to engage in an act of discovery and conjecture, intuition and inspiration; to be in a state of confusion — not because it makes no sense to you, but because you gave it sense and you still don't understand what your creation is up to; to have a breakthrough idea; to be frustrated as an artist; to be awed and overwhelmed by an almost painful beauty; to be alive, damn it. (page 16)

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